TOPICS IN HOMOLOGICAL ALGEBRA

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Introduction

According to [6], homological algebra (HA) is primarily a tool for proving non-constructive existence theorems in algebra, and for quantifying obstructions for validity of various algebraic properties. For example, by employing the Ext and Tor functors, one can measure the non-exactness of the Hom and tensor product functors, the Tor functor measures the lack of torsionfreeness, etc. The extra feature of HA that we will stress here is the possibility to prove directly structural results for infinitely generated modules without dealing with the finitely generated ones (e.g., in the classification of tilting modules over various commutative rings that is not affected by the fact that all finitely generated tilting modules are trivial).

Since this series is part of a *Non-commutative Algebra Program*, our selection from the rich supply of results developed within, or with substantial influence, of HA is aimed at applications in module and representation theory over general (not necessarily commutative) rings:
(a) First, we present classic basics of HA for categories of modules (these results easily extend to corresponding relative versions – see [2]); then
(b) we introduce a more recent branch of HA called the set-theoretic homological algebra, which provides powerful tools for investigating the structure and approximation properties of modules [4] (as well as objects of more general Grothendieck categories), and finally
(c) we give some applications, notably to (infinite dimensional) tilting theory, and finish by recent results combining Mittag-Leffler conditions with tilting that yield bounds for the approximation theory.

1. Basic concepts

1(i) The Hom and $\otimes$ bifunctors

• The setting: R associative (possibly non-commutative) ring with unit, Mod–R the category of all (possibly infinitely generated) right R-modules, morphisms are written as acting on the opposite side from the scalars.

This setting includes includes the categories of abelian groups, linear spaces, representations of groups, representations of Lie algebras, representations of quivers, quasi-coherent sheaves over affine schemes, et al.

The basic **Hom-bifunctor**: $\mathrm{Hom}_R(-,-)$ from Mod–$R \times$ Mod–R to Mod–$\mathbb{Z}$: additive, and contravariant (covariant) in the first (second) variable. Variants for bimodules.

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(Unbounded) **chain complex** of modules C (the differential δ_n maps C_n to C_{n-1}), long exact sequence, short exact sequence. The long exact sequence as a splicing of short exact ones. Dually: (unbounded) **cochain complexes** C (where δ_n maps C_n to C_{n+1}).

Both the covariant $F = \mathrm{Hom}_R(M, -)$ functor, and the contravariant $G = \mathrm{Hom}_R(-, N)$, are left exact.

• Definition: M (N) is a **projective** (**injective**) module provided that the functor F (G) is exact.

Each free module is projective. Projectives = direct summands of free modules. Each projective module is a direct sum of countably generated projective modules (Kaplansky). Each module is a homomorphic image of a free one. Corollary: Projective (free) resolutions exist.

The character module **duality** $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ from Mod–R to R–Mod (other dualities available in special cases: for R commutative: the injective cogenerator duality $\mathrm{Hom}_R(-, W)$, for R fin.dim. k-algebra: the k-duality $\mathrm{Hom}_k(-, k)$, etc.). The dual of a projective module is injective. Each module embeds into an injective one; injective hull; (minimal) injective coresolution.

For non-right noetherian rings: no classification of injectives possible (Faith-Walker). For right noetherian: direct sums of hulls of indecomposable cyclics suffice, for comm. noe.: even direct sums of $E(R/p)$ for $p \in \mathrm{Spec}(R)$ (Matlis).

• The **tensor product** bifunctor $- \otimes_R -$: Mod–$R \otimes_R R$–Mod $\to$ Mod–$\mathbb{Z}$ (defined by the universal property for bilinear maps) is additive, covariant and right exact in each variable. Variants for bimodules. **Flat modules** (making $\otimes$ exact); e.g., the projectives.

The tensor product and the Hom-functor are **adjoint**, that is, for each bimodule $N \in R$–Mod–S, the induced functors $T = - \otimes_R N$: Mod–$R \to$ Mod–S and $H = \mathrm{Hom}_S(N, -)$: Mod–$S \to$ Mod–R have the property that $\mathrm{Hom}_R(-, H(-))$ and $\mathrm{Hom}_S(T(-), -)$ are naturally isomorphic (as bifunctors from Mod–$R \times$ Mod–$S \to$ Mod–$\mathbb{Z}$), cf. [2, 2.16]).

For $M \in$ Mod–R and $P \in$ Mod–S, the isomorphism of $\mathrm{Hom}_R(M, \mathrm{Hom}_S(N, P))$ onto $\mathrm{Hom}_S(M \otimes_R N, P)$ is given by $f \mapsto ((m \otimes_R n) \mapsto f(m)(n))$.

1(ii) Forming the Ext groups and their long exact sequences

• Morphism of complexes, the category C(Mod–R) of (unbounded) complexes of modules. Subcomplex, quotient complex.

The nth **homology module** $H_n(C)$ of a chain complex C: $H_n(C) = Z_n(C)/B_n(C)$, where $Z_n(C) = \mathrm{Ker}(\delta_n)$ is the nth **cycle** and $B_n(C) = \mathrm{Im}(\delta_{n+1})$ is the nth **boundary**. Note: C is exact, iff $H_n(C) = 0$ for each n. For cochain complexes, the term **cohomology module** is used.

Each morphism of (co)chain complexes $f : C \to D$ induces for each n a module homomorphism $H_n(f) : H_n(C) \to H_n(D)$. Moreover, $H_n : C(\text{Mod–}R) \to$ Mod–R is an additive covariant functor (**the nth (co)homology functor**), cf. [2, 1.5.4].

A technical tool in Mod–R: the **Snake lemma** provides for the **connecting homomorphism**, see [2, 1.2.16]. Note: There is also a version of the lemma for complexes of modules, see [3, 2.3].

• **Long exact sequences of (co)homologies**: each short exact sequence of complexes $0 \to C' \to C \to C'' \to 0$ induces a long exact sequence of their (co)homologies (to go from H_n to H_{n+1}, use the connecting homomorphism, cf. [2, p. 30, 1.5.7]).

- The **Ext-groups**: Let I be an injective coresolution (cochain complex) of the module N and let I^* be its **deleted** complex. Apply $\mathrm{Hom}_R(M,-)$ to I^* to obtain the cochain complex $\mathrm{Hom}_R(M,I^*)$ of abelian groups. Its nth cohomology group is denoted by $\mathrm{Ext}^n_R(M,N)$.

Computations: $\mathrm{Ext}^0_R(M,N) \cong \mathrm{Hom}_R(M,N)$,

$\mathrm{Ext}^1_R(M,N) \cong \mathrm{Hom}_R(M,\Omega^{-1}(N))/\pi_N(\mathrm{Hom}_R(M,I_0)$ where $\Omega^{-1}(N) = I_0/N$ is the 1**st cosyzygy** of N in I.

- Let $f,g : C \to C'$ be morphisms of chain complexes. Then $f \sim g$ (f is **homotopic** to g) provided there exists ('backward diagonal') morphisms $s_n : C_n \to C'_{n+1}$ such that $f_n - g_n = \delta'_{n+1}s_n + s_{n-1}\delta_n$. ($s$ is called the **chain homotopy** between f and g). Similarly, **cochain homotopy** is defined for cochain complexes.

$\sim$ is an equivalence, invariant under composition with morphism of chain complexes (both from left and right), and sums. In fact, $f \sim g$ iff $f - g \sim 0$ (i.e., $f - g$ is **null homotopic**).

Easy: Homotopic morphisms induce the same maps at homology modules, i.e., $f \sim g$ implies $H_n(f) = H_n(g)$ for each n, see [2, 1.5.13].

- **Comparison Lemma**: Let I and J be injective coresolutions of the modules P and Q, respectively. Then each $\varphi \in \mathrm{Hom}_R(P,Q)$ extends to a chain map between the deleted complexes, $f : I^* \to J^*$, which is unique up to cochain homotopy (i.e., $f - f' \sim 0$ for any other extension $f' : I^* \to J^*$).

Using this for $P = Q = N$, and applying $\mathrm{Hom}_R(M,-)$, we infer that the definition of $\mathrm{Ext}^n_R(M,N)$ does not depend on a particular choice of the injective coresolution of N (if $f : I^* \to J^*$ and $g : J^* \to I^*$ are cochain maps induced by id_N, then $\mathrm{Hom}_R(M,g) \circ \mathrm{Hom}_R(M,f) = \mathrm{Hom}_R(M,gf) \sim \mathrm{Hom}_R(M,\mathrm{id}_{I^*})$ and $\mathrm{Hom}_R(M,f) \circ \mathrm{Hom}_R(M,g) = \mathrm{Hom}_R(M,fg) \sim \mathrm{Hom}_R(M,\mathrm{id}_{J^*})$, so for each $n \geq 0$, $H_n(\mathrm{Hom}_R(M,g)) \circ H_n(\mathrm{Hom}_R(M,f)) = \mathrm{id}_{\mathrm{Hom}_R(M,I^*)}$ and $H_n(\mathrm{Hom}_R(M,f)) \circ H_n(\mathrm{Hom}_R(M,g)) = \mathrm{id}_{\mathrm{Hom}_R(M,J^*)}$, and the nth cohomology groups are isomorphic).

The nth cohomology functor $\mathrm{Ext}^n_R(M,-)$ is called the **nth right derived functor** of $\mathrm{Hom}_R(M,-)$.

Note: $\mathrm{Ext}^n_R(M,N) = 0$ whenever N is injective and $n > 0$.

- **Horseshoe Lemma**: Given a short exact sequence of modules $\mathcal{E} : 0 \to N' \to N \to N'' \to 0$ and injective coresolutions I' and I'' of N' and N'', resp., there is an injective coresolution I of N (with $I_n = I'_n \oplus I''_n$ for each $n \geq 0$) such that $0 \to I' \to I \to I'' \to 0$ is a short exact sequence of complexes expanding $\mathcal{E}$ (Proof by induction on the short exact 'columns': the first column $= \mathcal{E}$, second column $=$ the cokernel ($=$ 1st cosyzygy) exact sequence, etc.)

The dual Horseshoe lemma for projective resolutions, [2, 8.2.1].

Applying the Horseshoe Lemma and the long exact sequence of cohomologies, we obtain the **long exact sequence for Ext** measuring the non-right exactness of Hom: $0 \to \mathrm{Hom}_R(M,N') \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N'') \to \mathrm{Ext}^1_R(M,N') \to \mathrm{Ext}^1_R(M,N) \to \mathrm{Ext}^1_R(M,N'') \to \mathrm{Ext}^2_R(M,N') \to \dots$.

Convenient for computations of Ext: since $\mathrm{Ext}^i_R(M,I) = 0$ for each injective module I and $i > 0$, computation of the higher Ext's reduces to the first one by **dimension shifting** in the second component, using the cosyzygy modules:

$\mathrm{Ext}^{n+1}_R(M,N) \cong \mathrm{Ext}^n_R(M,\Omega^{-1}(N)) \cong \dots \cong \mathrm{Ext}^1_R(M,\Omega^{-n}(N))$ for $n \geq 1$.

• The **dual approach** for computing $\mathrm{Ext}^n_R(M,N)$: Let P be a projective resolution (chain complex) of the module M and let P^* be its deleted complex. Apply $\mathrm{Hom}_R(-,N)$ to P^* to obtain the cochain complex $\mathrm{Hom}_R(P^*,N)$ of abelian groups. Its nth cohomology group is also denoted by $\mathrm{Ext}^n_R(M,N)$.

A dual version of the Comparison Lemma shows that this group does not depend on the choice of P. $\mathrm{Ext}^n_R(-,N)$ is called the n**th right derived functor** of $\mathrm{Hom}_R(-,N)$.

Also, $\mathrm{Ext}^n_R(M,N)=0$ whenever M is projective and $n>0$; dimension shifting in the first component using syzygy modules reduces computation of Ext^n for $n>1$ to Ext^1.

The dual Horshoe Lemma and the long exact sequence of cohomologies yields the long exact sequence $0\to \mathrm{Hom}_R(M',N)\to \mathrm{Hom}_R(M,N)\to \mathrm{Hom}_R(M'',N)\to \mathrm{Ext}^1_R(M',N)\to \mathrm{Ext}^1_R(M,N)\to \mathrm{Ext}^1_R(M'',N)\to \mathrm{Ext}^2_R(N',N)\to \dots$.

• **The balance**: In order to show that the two definitions of $\mathrm{Ext}^n_R(M,N)$ above yield the same notion, we fix a deleted projective resolution P^* of M with the differentials δ_i, and deleted injective coresolution I^* of N with the differentials ϵ_j. We form a commutative first quadrant diagram with exact rows and columns, consisting of the groups $\mathrm{Hom}_R(P_i,I_j)$ for $i,j\geq 0$, the horizontal maps $\mathrm{Hom}_R(\delta_{i+1},I_j)$, and vertical maps $\mathrm{Hom}_R(P_i,\epsilon_j)$. Let $C_j=\mathrm{Ker}(\mathrm{Hom}_R(\delta_1,I_j))$ and $D_i=\mathrm{Ker}(\mathrm{Hom}_R(P_i,\epsilon_0))$.

Then $C_j\cong \mathrm{Hom}_R(M,I_j)$ and $D_i\cong \mathrm{Hom}_R(P_i,N)$, and we can extend the diagram by adding one column to the left, formed by the complex $0\to C_0\to C_1\to\dots$ with the differentials $\mathrm{Hom}_R(M,\epsilon_j)$, and one bottom row formed by the complex $0\to D_0\to D_1\to\dots$ with the differentials $\mathrm{Hom}_R(\delta_{i+1},N)$.

The extended diagram is still commutative, the nth cohomology group of the complex added to the left is $A_n=\mathrm{Ext}^n_R(M,N)$ computed in the first way, while the nth cohomology group of the bottom complex is $B_n=\mathrm{Ext}^n_R(M,N)$ computed in the second (dual) way. Using exactness of the rows and columns of the original diagram, we can chase it diagonally and define maps $\alpha_n:A_n\to B_n$ and $\beta_n:B_n\to A_n$ so that $\alpha_n\beta_n=\mathrm{id}$ and $\beta_n\alpha_n=\mathrm{id}$, giving the desired isomorphisms (cf. [2, 1.4.16 and 8.2.14]).

• **The Baer/Yoneda Ext**. The group $\mathrm{Ext}^1_R(M,N)$ can alternatively be defined as the group of all equivalence classes of extensions (= short exact sequences) of the form $0\to N\to X\to M\to 0$ (Hint: use a pushout to form such an extension from a projective presentation of M). The zero element in $\mathrm{Ext}^1_R(M,N)$ then corresponds to the equivalence class of the split exact sequence $0\to N\to N\oplus M\to M\to 0$, and addition in $\mathrm{Ext}^1_R(M,N)$ to the Baer sum of extensions.

Generalization for $n\geq 1$: the Yoneda Ext. Note: This construction is available to define Ext^n even for abelian categories without projective or injective objects.

• **The Tor bifunctor**. The $\mathrm{Tor}^R_n(-,N)$ functor is defined as the nth left derived functor of $-\otimes_R N$, and $\mathrm{Tor}^R_n(M,-)$ of $M\otimes_R -$, using projective resolutions and the corresponding variants of the Comparison and Horseshoe Lemmas. The long exact sequence for Tor follows, and measures the non-left exactness of $\otimes$. Also the balance holds true for $\mathrm{Tor}^R_n(M,N)$, [2, p.183, Ex.17].

The adjointness of Hom and $\otimes$ yields **duality formulas** for Ext^n and Tor_n:

$\mathrm{Ext}^n_R(M,\mathrm{Hom}_S(N,I))\cong \mathrm{Hom}_S(\mathrm{Tor}^R_n(M,N),I)$ where $M\in\text{Mod–}R$, $N\in R-\text{Mod–}S$, and $I\in\text{Mod–}S$ is injective, cf. [2, 2.16(b)].

• **The mapping cone**. Let $f : C \to D$ be a morphism of chain complexes. The complex $C(f)$ with n-th term $D_n \oplus C_{n-1}$ and differential $\delta_n(y,x) = (d_n(y)+(-1)^{n-1}f_{n-1}(x), c_{n-1}(x))$ is the **mapping cone** of f. A variant definition: $\delta_n(y,x) = (d_n(y) + f_{n-1}(x), -c_{n-1}(x))$.

There is a short exact sequence of complexes of modules $0 \to D \to C(f) \to S(C) \to 0$ where $S(C)$ is the **suspension** of C (i.e., the complex whose nth term is C_{n-1} and nth differential is c_{n-1}, or $-c_{n-1}$ for the variant definition).

The role of the mapping cone: $C(f)$ is exact, iff f is a homology isomorphism (i.e., $H_n(f)$ is an isomorphism for each $n < \omega$). Proof: Apply the Snake lemma to the complex above to get a long exact sequence of homology complexes, and observe that the connecting homomorphisms are (variants) of the mapping cone and its iterated suspensions.

1(iii) Direct and inverse limits of modules.

• Let $(I, \leq)$ be an upper directed poset, $\mathcal{M} = (M_i, f_{ji} \mid i \leq j \in I)$, a **direct system** of modules. The colimit $(M, f_i \mid i \in I)$ of $\mathcal{M}$ in Mod–R is called the **direct limit** of $\mathcal{M}$ and denoted by $\varinjlim M_i$.

Special cases: If all f_{ji} are monic, then so are the f_i, and M is the **directed union** of the $f_i(M_i)$. Typical example: M expressed as the directed union of (some of) its 'small' submodules (e.g., the finitely generated ones). Note: direct limits of monomorphisms include arbitrary direct sums as a particular case.

If $\{m_\alpha \mid \alpha < \kappa\}$ is a set of generators of M, then M is the directed union of the chain $(M_\alpha \mid \alpha < \kappa)$ where $M_\alpha = \sum_{\beta<\alpha} m_\beta R$.

Important case: the **$\mathcal{C}$-filtration** of a module M, see [4, 6.1]. Example: the Prüfer p-group $\mathbb{Z}_{p^\infty}$ is $\{\mathbb{Z}_p\}$-filtered.

• The dual setting: an **inverse system** $\mathcal{I}$ of left R-modules, the **inverse limit** of $\mathcal{I}$. Example: the I-adic completion $\hat{M} = \varprojlim M/MI^n$ (where I is an ideal of a commutative ring R such that $\bigcap_n I^n = 0$).

Special case: inverse system of epimorphisms, still more special: **$\mathcal{C}$-cofiltration**, [4, 6.34]. Note: inverse limits of epimorphisms include direct products as a particular case.

If M is $\mathcal{C}$-filtered, then M^* is $\mathcal{C}^*$-cofiltered. Example: the p-adic group $\mathbb{J}_p$ is $\{\mathbb{Z}_p\}$-cofiltered, see [4, 6.35].

• **Commutativity formulas** (cf. [4]): Hom always commutes with inverse limits, i.e., the canonical homomorphism $\mathrm{Hom}_R(M, \varprojlim N_i) \to \varprojlim \mathrm{Hom}_R(M, N_i)$ is an isomorphism. If M is finitely presented, then $\varinjlim \mathrm{Hom}_R(M, N_i) \cong \mathrm{Hom}_R(M, \varinjlim N_i)$.

If $n > 0$ and M is an FP_{n+1} module, then $\mathrm{Ext}^n_R(M, \varinjlim N_i) \cong \varinjlim \mathrm{Ext}^n_R(M, N_i)$. However, $\mathrm{Ext}^n_R(M, -)$ commutes with inverse limits iff $\mathrm{Ext}^n_R(M, -) = 0$, i.e., iff M has projective dimension $\leq n$.

Also $\mathrm{Hom}_R(\varinjlim M_i, N) \cong \varprojlim \mathrm{Hom}_R(M_i, N)$ canonically. If $n > 0$ and N is pure-injective, then $\mathrm{Ext}^n_R(\varinjlim M_i, N) \cong \varprojlim \mathrm{Ext}^n_R(M_i, N)$. For $n = 1$, this condition characterizes the pure-injectivity of N (Auslander).

2. Approximations and relative homological algebra

2(i) Basics of the approximation theory.

• **Preenvelopes** (**precovers**) as left (right) approximations. Minimal versions, special versions.

Wakamatsu Lemma: minimal approximations are special, cf. [4, 5.13].

Basic examples: projective (pre)covers, injective hulls.

Relative homological algebra uses general precovering/preenveloping classes in place of projectives/injectives in order to define relative (co)homology. This is the main topic of [2]. Here, we rather follow [4] and focus on the approximation theory and its applications to the structure theory of infinitely generated modules.

Perps (= kernels of Ext^1): $\mathcal{C}^\perp$, ${}^\perp\mathcal{C}$. A **cotorsion pair** $(\mathcal{A}, \mathcal{B})$ is defined by $\mathcal{A} = {}^\perp\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$. Hereditary cotorsion pairs (also higher Ext's vanish).

• **Salce's Lemma**: If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, then $\mathcal{A}$ is special precovering, iff $\mathcal{B}$ is special preenveloping, cf. [4, 5.20]. Such cotorsion pair is called **complete**. Comment: cotorsion pairs are formal analogs of torsion pairs, but more important is their relation to approximations; e.g., Salce's Lemma substitutes for the lack of explicit dualities within Mod–R.

2(ii) Transfinite extensions and deconstructibility.

• Recall: $\mathcal{C}$-filtrations (= transfinite extensions of the modules in $\mathcal{C}$); $\mathrm{Filt}(\mathcal{C})$.

A class $\mathcal{D}$ is **deconstructible** provided that $\mathcal{D} = \mathrm{Filt}(\mathcal{S})$ for a set $\mathcal{S}$. Implies: closure under extensions and direct sums.

• **Eklof's Lemma**: ${}^\perp\mathcal{C}$ is closed under transfinite extensions for each class $\mathcal{C}$, cf. [4, 6.2]. Note: but (consistently with ZFC), the class ${}^\perp\mathbb{Z}$ is not deconstructible (Whitehead groups).
Open problem: Is there a class of the form ${}^\perp\mathcal{C}$ which is not deconstructible (provably in ZFC)?

Examples of deconstructible classes: $\mathcal{P}_n$ (Proof: zigzag in projective resolutions), $\mathcal{F}_n$ (Proof: using purification), torsion-free modules over a domain, etc., see [4, §8.1].

• **The Enochs-Šťovíček Lemma**: Each deconstructible class is precovering, cf. [4, 7.2]. Note: often special precovering (see below).

2(iii) Cofiltrations and the Lukas lemma.

• Recall: $\mathcal{C}$-cofiltrations. Lukas Lemma = dual Eklof Lemma (with a dual proof! See [4, 6.37].)

Corollary: Countable inverse limits of epimorphisms of injective modules have injective dimension ≤ 1, [4, 6.38].

Compare with (Bergman): Each module is an inverse limit of injective modules. Moreover, each module over a right noetherian ring is an inverse limit of an inverse system of epimorphisms of injective modules. The latter inverse system is uncountable, based on an Aronszajn tree. Note: the Aronszajn tree makes it possible to construct an uncountable inverse system of epimorphisms whose inverse limit is 0, cf. [4, 6.32].

3. Set-theoretic homological algebra

3(i) Quillen's small object argument.

• For each module M and each set $\mathcal{S}$ of modules, there is a short exact sequence $0 \to M \to P \to N \to 0$ such that $P \in \mathcal{S}^{\perp}$ and N is $\mathcal{S}$-filtered, [4, 6.11].

Corollary: The cotorsion pair generated by any set $\mathcal{S}$ is complete. If $R \in \mathcal{S}$, then its left class $\mathcal{A}$ consists of direct summands of $\mathcal{S}$-filtered modules, [4, 6.14]. Moreover, $\mathcal{A}$ is deconstructible. Example: FCC - flat covers and cotorsion envelopes.

3(ii) Bongartz Lemma and its dual.

• W.l.o.g., $\mathcal{S} = \{T\}$ for a single module T. **Bongartz Lemma** is a special case of the above when $\operatorname{Ext}^1_R(T, T^{(\kappa)}) = 0$ for all κ. Then even $N \cong T^{(\lambda)}$ for some λ, [4, 6.15].

Dual Bongartz Lemma: If $\operatorname{Ext}^1_R(C^{\kappa}, C) = 0$ for all κ, then for each module M there is a short exact sequence $0 \to N \to P \to M \to 0$ such that $P \in {}^{\perp}C$ and $N \cong C^{\lambda}$ for some λ, [4, 6.44].

Warning: There is no dual to the small object argument (by the Eklof-Shelah's consistency result above, see [4, p.153]).

3(iii) The Hill Lemma.

• The point: expands a single $\mathcal{C}$-filtration $\mathcal{M}$ of a module M into a family $\mathcal{H}$ of $\mathcal{C}$-filtered submodules that forms a complete distributive sublattice of the modular lattice $L(M)$.

More precisely (see [4, 7.10]): assume that each module in $\mathcal{C}$ is $\leq \kappa$-presented for κ regular uncountable. Then $\mathcal{H}$ can be constructed to satisfy:
(H1) (expansion) $\mathcal{M} \subseteq \mathcal{H}$;
(H2) (distributivity) $\mathcal{H}$ is a complete distributive sublattice of $L(M)$;
(H3) ($\mathcal{C}$-filtration) If $N \subseteq P$ are both in $\mathcal{H}$, then P/N is $\mathcal{C}$-filtered (in particular, for each $N \in \mathcal{H}$, both N and M/N are $\mathcal{C}$-filtered).
(H4) (density) If $N \in \mathcal{H}$, and $X \subseteq M$ has cardinality $< \kappa$, then there exists $N \cup X \subseteq P \in \mathcal{H}$ such that P/N is $< \kappa$-presented.

Idea of proof: Let $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ be a $\mathcal{C}$-filtration of M and fix $A_\alpha < \kappa$-generated such that $M_\alpha + A_\alpha = M_{\alpha+1}$. Take

$$\mathcal{H} = \{\sum_{\alpha \in S} M_\alpha \mid S \text{ a closed subset of } \sigma\}.$$

Here, **closed** means that $M_\alpha \cap A_\alpha \subseteq \sum_{\beta < \alpha, \beta \in S} M_\beta$, for each $\alpha \in S$.

The point: $M_\alpha \cap \sum_{\alpha \in S} M_\alpha = \sum_{\beta \in S \cap \alpha} M_\beta$ ('from submodules to subsets').

Applications of the Hill lemma (see [4, Chapter 7]): replacing a particular $\mathcal{C}$-filtration by a 'better' one, Kaplansky's theorem for cotorsion pairs, existence of $\mathcal{C}$-socle sequences, Shelah's singular compactness for $\mathcal{C}$-filtered modules, locality of Drinfeld vector bundles, etc.

4. Some applications to representation/module theory

4(i) Infinite dimensional tilting theory.

• A module T is n**-tilting** if
(T1) T has projective dimension $\leq n$;
(T2) $\operatorname{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all κ and all $i > 0$;
(T3) There is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ with $T_i \in \operatorname{Add} T$ for all $i \leq n$.

The class $\mathcal{T} = \bigcap_{i>0} \operatorname{Ker}\operatorname{Ext}^i_R(T, -)$ is the n**-tilting class** induced by T. Two tilting modules inducing the same tilting class are called equivalent.

• (Angeleri-Coelho) If $\mathcal{T}$ is a tilting class, then the inducing tilting module T can be obtained from an iteration of special $\mathcal{T}$-preenvelopes of the regular module R (a la (T3)).

• (Miyashita-Bazzoni) n-tilting modules induce n-tuples of category equivalences between subcategories Mod–R and Mod–S where $S = \operatorname{End} T$ generalizing the Morita equivalence.

Here, T need not be finitely generated - this is essential when R is commutative (because there, all finitely generated tilting modules are projective, [4, 13.2]).

If T is 1-tilting, then $\mathcal{T}$ is a special preenveloping torsion class. Conversely, each special preenveloping torsion class is 1-tilting, [4, 14.4].

Tilting classes and/or modules have been classified over many rings (tame hereditary algebras, Prüfer domains, commutative noetherian rings, ...), see [4, Part III]. A more involved application of set-theoretic homological algebra yields

• **Finite type of tilting modules**: For each n-tilting module T there is a set $\mathcal{S}$ of strongly finitely presented modules of projective dimension $\leq n$ such that $\mathcal{T} = \mathcal{S}^\perp$. In particular, the tilting class $\mathcal{T}$ is axiomatizable, [4, 13.46].

4(ii) The structure of Mittag-Leffler and locally T-projective modules.

• (Raynaud-Gruson) A module M is **Mittag-Leffler** provided that the canonical map $M \otimes_R \prod_i N_i \to \prod_i M \otimes_R N_i$ is monic for each family $(N_i \mid i \in I)$ of left R-modules.

Equivalent conditions: $\aleph_1$-pure-projective, a 'ML-limit' of finitely presented modules, etc. [4, 3.14]. In particular, flat Mittag-Leffler = locally R-projective (= $\aleph_1$-projective). Applying set-theoretic homological algebra, one arrives at a barrier:

• **Complexity of flat Mittag-Leffler modules**: If R is not right perfect, then the class $\mathcal{F}$ of all flat Mittag-Leffler modules is closed under transfinite extensions, but it is not precovering, and hence not deconstructible.

$\mathcal{F}$ is the class of all 'locally T-projective' modules for the trivial tilting module $T = R$. The non-precoverin, and hence non-deconstructibility, extends to any class of locally T-projective modules when T is 'non-$\sum$-pure split', cf. [S] and [ASR]. In this way, the ML-theory yields further boundaries of the approximation theory of modules.

References

[1] Angeleri Hügel, L., Šaroch J., Trlifaj J.: **Tilting approximations and Mittag-Leffler conditions - the applications**, to appear in Israel J. Math., arXiv:1612.01140v1.
[2] Enochs, E.E., Jenda, O.M.G.: **Relative Homological Algebra**, 2nd ed., GEM **30**, W. de Gruyter, Berlin-Boston 2011.
[3] Enochs, E.E., Jenda, O.M.G.: **Relative Homological Algebra**, Vol. 2, GEM **54**, W. de Gruyter, Berlin-Boston 2011.
[4] Göbel, R., Trlifaj, J.: **Approximations and Endomorphism Algebras of Modules**, 2nd rev. ext. ed., Vol.1, GEM **41**, W. de Gruyter, Berlin-Boston 2012.
[5] Šaroch J.: **Tilting approximations and Mittag-Leffler conditions - the tools**, to appear in Israel J. Math., arXiv:1612.01138v1.
[6] Weibel, C.: **An Introduction to Homological Algebra**, CSAM **38**, Cambridge Univ. Press, Cambridge 1994.

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