Very flat and locally very flat modules

New Pathways between Group Theory and Model Theory

A conference in memory of Rüdiger Göbel

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Jan Trlifaj (Univerzita Karlova, Praha)

Rüdiger Göbel (1940 — 2014)

I. The background: Classic structure theory of modules, and its limitations

Direct sum decompositions

A class of modules $\mathcal{C}$ is decomposable, provided that there is a cardinal κ such that each module in $\mathcal{C}$ is a direct sum of strongly $< \kappa$-presented modules from $\mathcal{C}$.

Examples

1. (Kaplansky) The class $\mathcal{P}_0$ of all projective modules is decomposable.

2. (Faith-Walker) The class $\mathcal{I}_0$ of all injective modules is decomposable iff R is a right noetherian ring.

3. (Huisgen-Zimmermann) Mod-R is decomposable iff R is a right pure-semisimple ring.
In fact, if M is a module such that $\mathrm{Prod}(M)$ is decomposable, then M is Σ-pure-injective.

Note: Krull-Schmidt type theorems hold in the cases 2. and 3.

Such examples, however, are rare in general – most classes of (large) modules are not decomposable.

Example

Assume that the ring R is not right perfect, that is, there is a strictly decreasing chain of principal left ideals

$$Ra_0 \supsetneq \cdots \supsetneq Ra_n \ldots a_0 \supsetneq Ra_{n+1}a_n \ldots a_o \supsetneq \ldots$$

Then the class $\mathcal{F}_0$ of all flat modules is not decomposable.

Example

There exist arbitrarily large indecomposable flat (= torsion-free) abelian groups.

Transfinite extensions

Let $\mathcal{A} \subseteq \text{Mod-}R$. A module M is $\mathcal{A}$-filtered (or a transfinite extension of the modules in $\mathcal{A}$), provided that there exists an increasing sequence $(M_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of M such that $M_0 = 0$, $M_\sigma = M$,

- $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and
- for each $\alpha < \sigma$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{A}$.

Notation: $M \in \text{Filt}(\mathcal{A})$. A class $\mathcal{A}$ is filtration closed if $\text{Filt}(\mathcal{A}) = \mathcal{A}$.

Eklof's Lemma

${}^\perp\mathcal{C} := \text{KerExt}^1_R(-, \mathcal{C})$ is filtration closed for each class of modules $\mathcal{C}$.

In particular, so are the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ of all modules of projective and flat dimension $\leq n$, for each $n < \omega$.

Deconstructible classes

[Eklof]

A class of modules $\mathcal{A}$ is deconstructible, provided there is a cardinal κ such that $\mathcal{A} = \mathrm{Filt}(\mathcal{A}^{<\kappa})$ where $\mathcal{A}^{<\kappa}$ denotes the class of all strongly $< \kappa$-presented modules from $\mathcal{A}$.

All decomposable classes closed under direct summands are deconstructible.

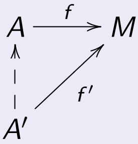
For each $n < \omega$, the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ are deconstructible.

[Eklof-T.]

More in general, for each set of modules $\mathcal{S}$, the class ${}^{\perp}(\mathcal{S}^{\perp})$ is deconstructible. Here, $\mathcal{S}^{\perp} := \mathrm{KerExt}^1_R(\mathcal{S}, -)$.

Approximations for relative homological algebra

A class of modules $\mathcal{A}$ is precovering if for each module M there is $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ factorizes through f:



The map f is an $\mathcal{A}$–precover of M.

If f is moreover right minimal (that is, f factorizes through itself only by an automorphism of A), then f is an $\mathcal{A}$–cover of M.

If $\mathcal{A}$ provides for covers for all modules, then $\mathcal{A}$ is called a covering class. Dually, $\mathcal{A}$-(pre)envelopes and enveloping classes of modules are defined.

The abundance of approximations

[Enochs], [Šťovíček]

- Each precovering class closed under direct limits is covering.
- All deconstructible classes are precovering.

In particular, the class ${}^{\perp}(\mathcal{S}^{\perp})$ is precovering for any set of modules $\mathcal{S}$.
Note: If $R \in \mathcal{S}$, then ${}^{\perp}(\mathcal{S}^{\perp})$ coincides with the class of all direct summands of $\mathcal{S}$-filtered modules.

Flat Cover Conjecture

$\mathcal{F}_0$ is deconstructible, and hence covering for any ring R (and so are the classes $\mathcal{F}_n$ for each $n > 0$).

The classes $\mathcal{P}_n$ ($n \geq 0$) are precovering. ...

Bass modules

Let R be a ring and $\mathcal{F}$ be a class of countably presented modules. $\varinjlim_\omega \mathcal{F}$ denotes the class of all Bass modules over $\mathcal{F}$, that is, the modules B that are countable direct limits of modules from $\mathcal{F}$.
W.l.o.g., such B is the direct limit of a chain

$$F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} F_i \xrightarrow{f_i} F_{i+1} \xrightarrow{f_{i+1}} \dots$$

with $F_i \in \mathcal{F}$ and $f_i \in \mathrm{Hom}_R(F_i, F_{i+1})$ for all $i < \omega$.

The classic Bass module

Let $\mathcal{F}$ be the class of all finitely generated projective modules. Then the Bass modules coincide with the countably presented flat modules.
If R is not right perfect, then a classic Bass module B arises when $F_i = R$ and f_i is the left multiplication by a_i $(i < \omega)$ where
$Ra_0 \supsetneq \dots \supsetneq Ra_n \dots a_0 \supsetneq Ra_{n+1}a_n \dots a_o \supsetneq \dots$ is strictly decreasing.
Note: B has projective dimension 1.

Flat Mittag-Leffler modules

[Raynaud-Gruson]

A module M is flat Mittag-Leffler provided the functor $M \otimes_R -$ is exact, and for each system of left R-modules $(N_i \mid i \in I)$, the canonical map $M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$ is monic.
The class of all flat Mittag-Lefler modules is denoted by $\mathcal{FM}$.

$\mathcal{P}_0 \subseteq \mathcal{FM} \subseteq \mathcal{F}_0$.
$\mathcal{FM}$ is filtration closed and closed under pure submodules.

$M \in \mathcal{FM}$, iff each countable subset of M is contained in a countably generated projective and pure submodule of M.
In particular, all countably generated modules in $\mathcal{FM}$ are projective.

Flat Mittag-Leffler modules and approximations

[Angeleri-Šaroch-T.]

Assume that R is not right perfect. Let B be a non-projective classic Bass module. Then B has no $\mathcal{FM}$-precover.

In particular, the class $\mathcal{FM}$ is not precovering, hence it is not deconstructible.

Locally free modules

Let $\mathcal{C}$ be a class of countably presented modules.
A module M is locally $\mathcal{C}$-free provided there exists a set $\mathcal{S} \subseteq \mathcal{C}$ consisting of submodules of M such that

- each countable subset of M is contained in a module from $\mathcal{S}$, and
- $\mathcal{S}$ is closed under unions of countable chains.

[Herbera-T.]

Flat Mittag-Leffler = locally $\mathcal{C}$-free,
where $\mathcal{C}$ is the class of all countably presented projective modules.

Cotorsion pairs and approximations

[Salce]

A pair of classes $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair in Mod-R if

- $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$, and
- for each module M there is an exact sequence $0 \to B \to A \to M \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$
  (so in particular, $\mathcal{A}$ is a precovering class).

Salce's Lemma

In the setting above, for each module N there is an exact sequence
$0 \to N \to B \to A \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$
(so in particular, $\mathcal{B}$ is a preenveloping class).

[Angeleri-Šaroch-T.] - The general version

Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\varinjlim \mathcal{B} = \mathcal{B}$. Then

- $\mathfrak{C}$ is complete.
- Let $\mathcal{C}$ be the class of all countably presented modules from $\mathcal{A}$, and let $\mathcal{L}$ the class of all locally $\mathcal{C}$-free modules.

Then $\mathcal{L}$ is precovering, iff all Bass modules over $\mathcal{C}$ are contained in $\mathcal{C}$, iff $\varinjlim \mathcal{A} = \mathcal{A}$.

Note: For the cotorsion pair $(\mathcal{P}_0, \text{Mod-}R)$, we recover the result on flat Mittag-Leffler modules above (the '0-tilting' case). Other cases include n-tilting cotorsion pairs, etc.

[Šaroch's Lemma]

Let $\mathcal{C}$ be any class of countably presented modules, and $\mathcal{L}$ the class of all locally $\mathcal{C}$-free modules. Let B be any Bass module over $\mathcal{C}$ such that B is not a direct summand in a module from $\mathcal{L}$. Then B has no $\mathcal{L}$-precover.

II. Motivation from algebraic geometry

Quasi-coherent sheaves as representations

Let X be a scheme and $\mathcal{O}_X$ its structure sheaf.

[Enochs-Estrada]

A quasi-coherent sheaf Q on X can be represented by an assignment

- to every affine open subscheme $U \subseteq X$, an $\mathcal{O}_X(U)$-module $Q(U)$ of sections, and
- to each pair of embedded affine open subschemes $V \subseteq U \subseteq X$, an $\mathcal{O}_X(U)$-homomorphism $f_{UV} : Q(U) \to Q(V)$ such that

  $$\mathrm{id}_{\mathcal{O}_X(V)} \otimes f_{UV} : \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} Q(U) \to \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} Q(V) \cong Q(V)$$

  is an $\mathcal{O}_X(V)$-isomorphism.

$+$ compatibility conditions for the f_{UV}.

Notation: $\mathrm{Qcoh}(X)$ = the category of all quasi-coherent sheaves on X.

Properties of the representations

Exactness

The functors $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} -$ are exact, i.e.,
the $\mathcal{O}_X(U)$-modules $\mathcal{O}_X(V)$ are flat.

The affine case [Grothendieck]

If $X = \mathrm{Spec}(R)$ for a commutative ring R, then $\mathrm{Qcoh}(X) \simeq \text{Mod-}R$.

Non-uniqueness of the representations

Not all affine open subschemes are needed: a set of them, $\mathcal{S}$, covering both X, and all $U \cap V$ where $U, V \in \mathcal{S}$, will do.

Extending properties of modules to quasi-coherent sheaves

Examples

If each module of sections is

- projective,
- (restricted) flat Mittag-Leffler,
- flat,

then the quasi-coherent sheaf Q is called

- an infinite dimensional vector bundle,
- (restricted) Drinfeld vector bundle,
- flat quasi-coherent sheaf.

[Raynaud-Gruson], [Estrada-Guil-T.]

The notions above are local, i.e., independent of the representation (choice of the affine open covering $\mathcal{S}$ of the scheme X).

Computing cohomology of quasi-coherent sheaves

Hovey's Strategy

- Complete cotorsion pairs of modules (or qc-sheaves on schemes) give rise to complete cotorsion pairs for complexes of modules (qc-sheaves),
- these in turn yield model category structures on the categories of complexes,
- and hence ways of computing sheaf cohomology ($=$ morphisms in the corresponding unbounded derived categories of qc-sheaves).

The dual setting: contraherent cosheaves

Definition (Positselski)

Let X be a scheme and $\mathcal{O}_X$ its structure sheaf.
A contraherent cosheaf P on X can be represented by an assignment

- to every affine open subscheme $U \subseteq X$, of an $\mathcal{O}_X(U)$-module $P(U)$ of cosections, and
- to each pair of embedded affine open subschemes $V \subseteq U \subseteq X$, an $\mathcal{O}_X(U)$-homomorphism $g_{VU} : P(V) \to P(U)$ such that

$$\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), g_{VU}) : P(V) \to \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), P(U))$$

is an $\mathcal{O}_X(V)$-isomorphism.

$+$ compatibility conditions for the g_{VU}.

A drawback, and a remedy

The drawback

The $\mathcal{O}_X(U)$-module $\mathcal{O}_X(V)$ is only flat, but not projective in general, so the Hom-functor above is not exact.

The remedy

Exactness is forced by an extra condition on the contraherent cosheaf P:

$$\mathrm{Ext}^1_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), P(U)) = 0.$$

Moreover, the $\mathcal{O}_X(U)$-modules $\mathcal{O}_X(V)$ are very flat ...

Very flat modules

Definition

Let $\mathcal{L} = \{R[s^{-1}] \mid s \in R\}$, where $R[s^{-1}]$ denotes the localization of R at the multiplicative set $\{1, s, s^2, ...\}$.

$\mathcal{CA} := \mathcal{L}^{\perp}$ is the class of all contraadjusted modules, and

$\mathcal{VF} := {}^{\perp}(\mathcal{L}^{\perp})$ the class of all very flat modules.

Lemma (Positselski)

Let $R \to S$ be a homomorphism of commutative rings such that the induced morphism of affine schemes $Spec(S) \to Spec(R)$ is an open embedding. Then S is a very flat R-module.

III. Structure and approximation properties of very flat and locally very flat modules

Basic properties

- $\mathcal{P}_0 \subseteq \mathcal{VF} \subseteq \mathcal{F}_0 \cap \mathcal{P}_1$.
- $(\mathcal{VF}, \mathcal{CA})$ is a complete cotorsion pair.
- $\mathcal{VF} = \mathrm{Filt}(\mathcal{VF}^{\leq\omega})$.

Definition

Denote by $\mathcal{LV}$ the class of all locally very flat modules,
i.e., the $\mathcal{C}$-free modules where $\mathcal{C} = \mathcal{VF}^{\leq\omega}$.

Since $\mathcal{P}_0 \subseteq \mathcal{VF}$, we have $\mathcal{FM} \subseteq \mathcal{LV} \subseteq \mathcal{F}_0$. Also $\mathcal{EC} \subseteq \mathcal{CA}$.
If R is a domain, then $\mathcal{DI} \subseteq \mathcal{CA}$.

Example: the case of Dedekind domains

Lemma (Slávik-T.)

Let R be a Dedekind domain and M be a module.

- $\mathcal{VF} = Filt(\mathcal{T})$, *where* $\mathcal{T}$ = *the set of all submodules of the modules in* $\mathcal{S}$.
- *If M is a non-zero module of finite rank, then* $M \in \mathcal{VF}$, *iff there exists* $0 \neq s \in R$ *such that* $M \otimes_R R[s^{-1}]$ *is a non-zero projective* $R[s^{-1}]$*-module.*
- *('Pontryagin Criterion')* $M \in \mathcal{LV}$, *iff each finite subset of M is contained in a countably generated very flat pure submodule of M, iff each finite rank submodule of M is very flat.*

• The only-if part of the second claim holds whenever R is a commutative ring whose classical quotient ring is artinian (e.g., a domain).
• Let R be a noetherian domain, M a very flat of finite rank n, and F its free submodule of rank n, then the module M/F has only finitely many associated primes of height 1.

Locally very flat modules and precovers

Theorem (Slávik-T.)

Let R be a noetherian domain. Then the following conditions are equivalent:

- *$Spec(R)$ is finite,*
- *$\mathcal{LV}$ is a precovering class,*
- *$\mathcal{VF}$ is a covering class,*
- *$\mathcal{CA}$ is an enveloping class.*

In this case, R has Krull dimension 1.

References

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3. A.Slávik, J.T.: *Very flat, locally very flat, and contraadjusted modules*, preprint, arXiv:1601.00783v1.