Complex Variables, 1985, Vol. 5, pp. 53–70
0278-1077/85/0501-$20.00

Generalized Hypercomplex Analysis and Its Integral Formulas

J. BUREŠ

Department of Mathematics, Faculty of Mathematics and Physics of Charles University, Sokolovska 83, Prague 8, Czechoslovakia

and

V. SOUČEK

Communicated by R. Delanghe

(*Received January 10, 1985*)

The homological version of the Cauchy integral formula is formulated in the paper for solutions of corresponding equations in complexified hypercomplex analysis. Many different cases are treated in unified manner, including some higher order operators. The notion of index of n-cycles is defined in this complexified situation and its properties are studied.

AMS (MOS): 30G35

1. INTRODUCTION

The purpose of the paper is to present a reasonably general Cauchy integral formula in hypercomplex analysis. There is a lot of different generalizations of Cauchy–Riemann equations for holomorphic functions and there is a lot of corresponding integral formulas. The version presented here generalizes many of them.

The generalization goes in two directions. First, Cauchy integral formula is presented in homological form. Second, the presented version of the formula treat many different cases in an unified manner.

The homological formulation of the Cauchy integral formula is much more important in higher dimensions than in complex analysis. Such a formulation is achieved here by a suitable definition of index of a point with respect to a cycle. In the real case (cycles in $\mathbb{R}_{n+1}$) the notion of the index for n-cycles in [8] and the index (or winding numb...) for general cycles in $\mathbb{R}_{n+1}$ was defined using Clifford-valued differential forms in [18]. The notion of the index in the complex case was studied in [3] in dimension 4. These results are generalized here (in complex case) to any even dimension (see Section 3).

Integral formula described here generalizes integral formulas in complex Clifford analysis ([2], [17]) for regular functions in the sense of Fueter of a complex quaternionic variable ([20]), for solutions of complex Laplace equation ([4], [17]), for massless fields of arbitrary spin ([19], [20]), for quaternionic formulation of classical electrodynamics ([10]) and for k-monogenic functions ([5]).

After the description of the form of considered differential operators (Section 2), the notion of index is defined and its basic properties are proved (Section 3).

The Cauchy integral formula for the first order operators is described in Section 4, while the corresponding formula for higher order operators is proved in Section 5.

Finally, the application of theorems of Sections 4, 5 on many examples is discussed in Section 6.

2. HYPERCOMPLEX DIFFERENTIAL OPERATORS

To fix the notation, let us agree that we shall denote by $\mathscr{C}_n$ the Clifford algebra of the Euclidean space $\mathbb{R}_n$ with negative definite form on it. If $e_1, \ldots, e_n$ is an orthonormal basis in $\mathbb{R}_n$ then the algebra $\mathscr{C}_n$ is generated by $e_1, \ldots, e_n$ and we have the usual relations $e_i^2 = -e_0$, $e_ie_j + e_je_i = 0$, $i,j = 1, \ldots, n$, where e_0 is the identity in $\mathscr{C}_n$.

We shall consider mainly the complexified Clifford algebra $\mathscr{C}_n^{\mathbb{C}}$ which is defined to be the tensor product $\mathscr{C}_n^{\mathbb{C}} := \mathscr{C}_n \otimes \mathbb{C}$ and we shall imbed $\mathbb{C}_n$ into $\mathscr{C}_n^{\mathbb{C}}$ in the standard way. Moreover, we shall denote by $\mathbb{C}_{n+1}$ the subspace of $\mathscr{C}_n^{\mathbb{C}}$, generated by $e_0, \ldots, e_n$.

The set

$$\{e_A = e_{i_1} \ldots e_{i_r} \mid A = (i_1, \ldots, i_r), 1 \leq i_1 < \cdots < i_r \leq n\}$$

is the basis of $\mathscr{C}_n^{\mathbb{C}}$ (as a vector space). The conjugation in $\mathbb{C}_{n+1}$ is given by

$$Q^+ = \left(\sum_0^n Q_ie_i\right)^+ = Q_0e_0 - \sum_1^n Q_ie_i$$

the norm by

$$|Q|^2 = \sum_0^n Q_i^2 .$$

The basic differential operators in Clifford analysis (see [2]) are

$$\partial = \sum_0^n e_i \frac{\partial}{\partial Q_i} \quad \text{and} \quad \partial^+ = e_0 \frac{\partial}{\partial Q_0} - \sum_1^n e_i \frac{\partial}{\partial Q_i}.$$

Let us denote further the complex light cone by

$$\mathbb{C}N_P = \{Q \in \mathbb{C}_{n+1} \mid |Q - P|^2 = 0\}.$$

The ring of holomorphic functions on $\Omega \subset \mathbb{C}_{m+1}$ with values in a complex vector space V will be denoted by $\mathcal{O}(\Omega, V)$.

Let us agree that all maps $f: \Omega \rightarrow V$ will be automatically supposed to be holomorphic. Let us denote $\mathcal{L}(V_1, V_2)$ the space of all (complex) linear maps from a vector space V_1 to a vector space V_2.

For further use we shall pick up the special $\mathscr{C}_n^{\mathbb{C}}$-valued differential forms on $\mathbb{C}_{n+1}$.

$$DQ = \sum_0^n (-1)^i e_i \, d\hat{Q}_i \qquad D^+Q = e_0 \, d\hat{Q}_0 + \sum_1^n (-1)^{j+1} e_j \, d\hat{Q}_j$$

and

$$\Omega = dQ_0 \wedge \cdots \wedge dQ_n$$

where

$$d\hat{Q}_i = dQ_0 \wedge \cdots \wedge dQ_{i-1} \wedge dQ_{i+1} \wedge \cdots \wedge dQ_n$$

Because of noncommutativity, we have to write sometimes differential operators on the right-hand side (e.g. $f\partial$), hence we would like to state explicitly that "the differential operators are always acting in both directions".

Let us now define two types of differential operators, which will be studied in the paper.

Definition 1 Let V_1, V_2 be two complex vector spaces, let us consider a linear mapping $\phi: \mathbb{C}_{n+1} \rightarrow \mathcal{L}(V_1, V_2)$, let $\Omega \subset \mathbb{C}_{n+1}$ be a domain.

We shall define an operator

$$\mathcal{D}_\phi: \mathcal{O}(\Omega, V_1) \rightarrow \mathcal{O}(\Omega, V_2)$$

by

$$\mathcal{D}_\phi(f) = \sum_0^n \phi(e_i) \frac{\partial f}{\partial Q_i}, \qquad f \in \mathcal{O}(\Omega, V_1).$$

Definition 2 Let V be a complex vector space and let

$$\phi_j: \mathbb{C}_{n+1} \to \mathrm{End}(V)$$

be a linear mapping. Then the operator $\mathscr{D}_{\phi_1,\ldots,\phi_n}$ is defined to be the composition $\mathscr{D}_{\phi_1} \circ \cdots \circ \mathscr{D}_{\phi_n}$.

Remark It would be possible to consider vector spaces $V_1, \ldots, V_n$ in the Definition 2 as well, but there is no reasonable example motivating such generalization.

The basic motivation for these two definitions are the following examples.

Example 1 (Spinor fields) The spaces V_1 and V_2 will be the basic spinor representations in this example. These representation spaces are usually realized as subspaces of $\mathscr{C}_n^{\mathbb{C}}$ (we shall always suppose it). To describe them in more detail, we shall discuss odd and even dimensional cases separately.

(a) Consider $n = 2k$, then any minimal left ideal V in $\mathscr{C}_n^{\mathbb{C}}$ is an irreducible $\mathscr{C}_n^{\mathbb{C}}$-module. Because $\mathscr{C}_n^{\mathbb{C}}$ is isomorphic with the matrix algebra $\mathbb{C}(2^k)$, we can take in this realization any column in $\mathbb{C}(2^k)$ for V, for example. Now, if $(\mathscr{C}_{n+1}^{\mathbb{C}})_+$ denotes the even part of $\mathscr{C}_{n+1}^{\mathbb{C}}$, we have the inclusion $\mathrm{Spin}(n+1) \subset (\mathscr{C}_{n+1}^{\mathbb{C}})_+$ and the isomorphism $(\mathscr{C}_{n+1}^{\mathbb{C}})_+ \simeq \mathscr{C}_n^{\mathbb{C}}$ and V is the basic $\mathrm{Spin}(n+1)$ representation (see [9], p. 185). Moreover it is clear from the definition of V that the subspace $V \subset \mathscr{C}_n^{\mathbb{C}}$ is preserved under the left multiplication by elements of $\mathscr{C}_n^{\mathbb{C}}$.

(b) Let $n = 2k - 1$. The algebra $\mathscr{C}_{n+1}^{\mathbb{C}}$ has the natural $\mathbb{Z}_2$-gradation $\mathscr{C}_{n+1}^{\mathbb{C}} = (\mathscr{C}_{n+1}^{\mathbb{C}})_+ \oplus (\mathscr{C}_{n+1}^{\mathbb{C}})_-$ (the decomposition into the even and odd part). Consider any minimal $\mathbb{Z}_2$-graded left ideal in $\mathscr{C}_{n+1}^{\mathbb{C}}$ (i.e. we suppose that $V = V_+ \oplus V_-$, where $V_+ = V \cap (\mathscr{C}_{n+1}^{\mathbb{C}})_+$, $V_- = V \cap (\mathscr{C}_{n+1}^{\mathbb{C}})_-$). Because $\mathrm{Spin}(n+1) \subset (\mathscr{C}_{n+1}^{\mathbb{C}})_+$ it is clear that V_+ and V_- are $\mathrm{Spin}(n+1)$-modules and that the multiplication by the generators $e_0, \ldots, e_n$ of $\mathscr{C}_{n+1}^{\mathbb{C}}$ maps V_+ into V_- and vice versa. We have $V = V_+ \oplus V_-$ and $V_\pm$ are the basic spinor representations of $\mathrm{Spin}(n+1)$.

Now let us denote $V_1 = V_+$, $V_2 = V_-$ for $n = 2k - 1$ and $V_1 = V_2 = V$ for $n = 2k$. In the case $n = 2k - 1$ the map ϕ is defined using the multiplication by $e_i \in \mathscr{C}_{n+1}^{\mathbb{C}}$, $i = 0, \ldots, n$ from the left. In the case $n = 2k$ we shall define ϕ using the left multiplication by $e_1, \ldots, e_n \in \mathscr{C}_n^{\mathbb{C}}$ and $\phi(e_0) = \textit{identity}$. Then $\mathscr{D}_\phi$ is the (complex) Dirac operator.

Using conjugation and the right multiplication we can define another three similar versions of the Dirac operator.

Example 2 (Regular functions in Clifford analysis) Suppose that $V_1 = V_2 = \mathscr{C}_n^{\mathbb{C}}$, then we can define the mapping ϕ by the left multiplication, i.e. $\phi(e_i)(a) = e_i \cdot a$, $i = 0, \ldots, n$, $a \in \mathscr{C}_n^{\mathbb{C}}$. Then the operator $\mathscr{D}_\phi$ coincide with the operator used in complex Clifford analysis ([4], [17]). There are again three other versions (using conjugation and right multiplication) of such operator. Note that the corresponding equation is reducible, i.e. in a suitable coordinates can be split into several independent pieces, each of which being isomorphic to the equation described in Example 1.

Example 3 (Massless fields with spin $m/2$) The dimension $n = 3$ is the special case, where the connection with mathematical physics is very strong. For $n = 3$ the space $\mathbb{C}_4$ can be identified with the complex Minkowski space $\mathbb{C}M$, the equation from Example 1 coincides (for $n = 3$) with the Weyl equation for massless fields of spin $\frac{1}{2}$. In the spinor notation ([14]), the operator from Example 1 can be identified with the operator $\nabla_{AA'}\phi^A$, studied in ([1]).

To include the massless field equation for arbitrary spin, we have to take $V_1 = S^m V_+$, $V_2 = V_- \otimes S^{m-1} V_+$ (S^m denoting symmetric tensor product) and to define the map ϕ by

$$\phi(e_i)(v_1 \odot \cdots \odot v_m) = e_i v_1 \otimes v_2 \odot \cdots \odot v_m, \qquad i = 0, \ldots, n$$

(see [7]). The corresponding operator looks in spinor language like $\nabla_{AA'}\phi^{A \cdots E}$ ($\phi^{A \cdots E}$ has m indices).

Example 4 (Complex Fueter equation) The algebra $\mathscr{C}_3^{\mathbb{C}}$ being isomorphic with the algebra $\mathbb{CH} \oplus \mathbb{CH}$ ($\mathbb{CH}$ denoting the space of complex quaternions), we can consider the space $\mathbb{CH}$ imbedded in $\mathscr{C}_3^{\mathbb{C}}$ (e.g. identifying it with the first factor). Then it is possible to take $V_1 = V_2 = \mathbb{CH}$ and to define the map ϕ by the left multiplication. The corresponding equation $\mathscr{D}_\phi f = 0$ is just complexified Fueter equation studied in ([20], [11]). At the same time it is the equation used by Imaeda in the description of classical electrodynamics by means of complex quaternions ([10]). The equation can be identified then with Maxwell equations (on complex Minkowski space).

Example 5 (Complex Laplace equation) Let us t $V_1 = V_2 = \mathscr{C}_n^{\mathbb{C}}$ and let us define operators ϕ_1, ϕ_2 by $\phi_1(e_i)(a) = e_i \cdot a$, $\phi_2(e_i)(a) = e_i^+ \cdot a$, $a \in \mathscr{C}_n^{\mathbb{C}}$, $i = 0, \ldots, n$.

Then the operator $\mathscr{D}_{\phi_2,\phi_1}$ can be restricted to complex-valued functions (identifying $\mathbb{C}$ with the subspace of $\mathscr{C}_n^{\mathbb{C}}$, generated by e_0) and coincide then with the complex Laplace operator on $\mathbb{C}_{n+1}$ (see [4]). It means simply that

$$\Delta = \partial^+\partial = \left(e_0 \frac{\partial}{\partial Q_0} - \sum_1^n e_i \frac{\partial}{\partial Q_i}\right)\left(e_0 \frac{\partial}{\partial Q_0} + \sum_1^n e_i \frac{\partial}{\partial Q_i}\right).$$

Example 6 (k-monogenic functions) Take ϕ as in Example 2, i.e. $\mathscr{D}_\phi = \partial$, $V_1 = V_2 = \mathscr{C}_n^{\mathbb{C}}$. Then the operator $\mathscr{D}_{\underbrace{\phi,\ldots,\phi}_{k\text{-times}}}$ is the complexification of the operator, defining (left) k-monogenic functions (see [5]).

Example 7 All operators, described in previous examples, can be restricted to real subspaces of $\mathbb{C}_{n+1}$ and their values can be (possibly) restricted to real subspaces of V_1. Under such restriction we shall obtain further interesting examples.

(a) The mappings in Example 2 can be restricted to $\mathbb{R}_{n+1} \subset \mathbb{C}_{n+1}$ and their values can be considered to be in $\mathscr{C}_n \subset \mathscr{C}_n^{\mathbb{C}}$. The corresponding equation coincides with the equation for regular functions in Clifford analysis. The comprehensive study of regular functions were published in ([2]).

(b) In the case $n = 3$ we can restrict spinor fields, described in Example 3 to Minkowski subspace $M \subset \mathbb{C}M$. Then the corresponding equation is (really physical) Weyl equation (the case of spin $\frac{1}{2}$) (see [21]), other equations are massless field equations on Minkowski space (they include Maxwell equation for spin 1 and linearized gravitation for spin 2) (see [1]).

(c) Identifying $\mathbb{C}\mathbb{H}$ with $\mathbb{C}_4$ (see [20]), we can restrict maps from Example 4 to the real subspaces $\mathbb{H} \subset \mathbb{C}\mathbb{H}$ and to take $V_1 = \mathbb{H} \subset \mathbb{C}M$. Then the operator $\mathscr{D}_\phi$ from Example 4 is just the Fueter operator ([6]).

(d) The restriction of maps from Example 6 to $\mathbb{R}_{n+1} \subset \mathbb{C}_{n+1}$ with values in $\mathscr{C}_n \subset \mathscr{C}_n^{\mathbb{C}}$, we shall obtain k-monogenic functions from [5].

3. THE INDEX

We have to restrict ourselves to the case n odd, $n = 2k - 1$. The standard kernel, used in hypercomplex analysis looks like

$$\frac{Q^+}{|Q|^{2k}} \quad \text{and its singularity set is} \quad \mathbb{C}N_0 = \{Q \in \mathbb{C}_{n+1} \mid |Q|^2 = 0\}.$$

The integration in the Cauchy formula has to be done with respect to n-cycle Γ, which has the empty intersection with $\mathbb{C}N$.

To have a reasonable generalization of the index, we have first to study the group $H_n(\mathbb{C}_{n+1} \backslash \mathbb{C}N_0, \mathbb{Z})$.

THEOREM 1 *Let n be any positive integer. Then*

$$H_n(\mathbb{C}_{n+1} \backslash \mathbb{C}N_P, \mathbb{Z}) \simeq \mathbb{Z}$$

and the sphere $S_n = \{P + Q \mid Q \in \mathbb{R}_{n+1} \subset \mathbb{C}_{n+1}, \sum_0^n Q_i^2 = 1\}$ is the generator of this group.

Proof Suppose that $P = 0$.

Clearly $S_n \subset \mathbb{C}_{n+1} \backslash \mathbb{C}N_0$. It is sufficient to prove that the inclusion $\iota: S_n \longrightarrow \mathbb{C}_{n+1} \backslash \mathbb{C}N_0$ induces the isomorphism

$$\iota_*: H_n(S_n, \mathbb{Z}) \longrightarrow H_n(\mathbb{C}_{n+1} \backslash \mathbb{C}N_0, \mathbb{Z}).$$

Let us define further $S_{2n+1} = \{Q = x + iy \in \mathbb{C}_{n+1} | \sum_0^n x_i^2 + \sum_0^n y_i^2 = 1\}$ and denote

$$E = S_{2n+1} \cap (\mathbb{C}_{n+1} \backslash \mathbb{C}N_0).$$

The principal tool for the proof is the fibration

$$p: E \longrightarrow S_1$$
$$Q \in E \longmapsto \frac{\sum_0^n Q_i^2}{|\sum_0^n Q_i^2|}$$

where $|\sum_0^n Q_i^2|$ means the absolute value of $\sum_0^n Q_i^2 \in \mathbb{C}$.

Denote $F_0 = p^{-1}(1)$, then

$$F_0 = \Big\{[Q_0, \ldots, Q_n] \mid Q_j = x_j + iy_j \in \mathbb{C}_{n+1},$$
$$\sum_0^n x_i y_i = 0, \sum_0^n x_i^2 - \sum_0^n y_i^2 > 0, \sum_0^n x_i^2 + \sum_0^n y_i^2 = 1\Big\}.$$

So we have the chain of inclusions

$$S_n \subset F_0 \subset E \subset \mathbb{C}_{n+1} \setminus \mathbb{C}N_0$$

and it is sufficient to prove that each of these inclusions induces the isomorphism of the corresponding n-dimensional group of homology.

(1) The set E is the deformational retract of $\mathbb{C}_{n+1} \setminus \mathbb{C}N_0$ (e.g. using the projection along rays from the origin onto S_{2n+1}). The standard theorem in algebraic topology gives us that ι_* is the isomorphism.

(2) The Wang sequence ([13], Lemma 8.4) gives us the exact sequence

$$\cdots \longrightarrow H_n(F_0, \mathbb{Z}) \overset{w}{\longrightarrow} H_n(F_0, \mathbb{Z}) \longrightarrow H_n(E, \mathbb{Z}) \longrightarrow H_{n-1}(F_0, \mathbb{Z}) \longrightarrow \cdots$$

Theorem 9.1 and Lemma 8.1 from [13] tells us that $w = 0$, moreover, Theorem 6.3 ([13]) asserts that $H_{n-1}(F_0, \mathbb{Z}) = 0$. It follows that

$$\iota_*: H_n(F_0, \mathbb{Z}) \longrightarrow H_n(E, \mathbb{Z})$$

is the isomorphism.

(3) The set S_n is the deformational retract of F_0. The deformation can be described explicitly:

Take $t \in \langle 0, 1 \rangle$, denote $\alpha = \sqrt{\sum_0^n x_i^2}$, $\beta = \sqrt{\sum_0^n y_i^2}$. Then

$$F_0 = \left\{ Q = x + iy \in \mathbb{C}_{n+1} \,\middle|\, \sum_0^n x_i y_i = 0, \alpha^2 - \beta^2 > 0, \alpha^2 + \beta^2 = 1 \right\}.$$

Let us define the map $\theta_t: F_0 \longrightarrow F_0$, $t \in \langle 0, 1 \rangle$ by

$$\theta_t(Q) = [sx_0 + ity_0, \ldots, sx_n + ity_n]$$

where

$$s = s(t) = \sqrt{\frac{1 - t^2\beta^2}{\alpha^2}}.$$

It is easy to verify that θ_t is the needed deformation. Hence ι_* is again the isomorphism.

Let us discuss again only the case $n = 2k - 1$.

We shall give a definition of the index of a point P with respect to a n-dimensional cycle Γ in a similar way as it is done in complex analysis.

Definition 3 Let Γ be a n-dimensional cycle in $\mathbb{C}_{n+1} \setminus \mathbb{C}N_P$, $P \in \mathbb{C}_{n+1}$. Then we shall define the index of P with respect to Γ by

$$\operatorname{Ind}_\Gamma P = \frac{1}{\omega_n} \int_\Gamma \frac{(Q-P)^+}{|Q-P|^{n+1}} DQ$$

where ω_n is the area of the unit sphere in $\mathbb{R}_{n+1}$.

Example Take $S_n \subset \mathbb{C}_{n+1} \backslash \mathbb{C}N_0$. Then $\operatorname{Ind}_{S_n} 0 = 1$. This fact can be either verified by the direct computation in spherical coordinates, or deduced from the Cauchy integral formula for regular function in Clifford analysis ([2], take $f = 1$).

THEOREM 2 *Let Γ be a n-dimensional cycle in $\mathbb{C}_{n+1} \backslash \mathbb{C}N_0$. Let us denote $\Omega = \mathbb{C}_{n+1} \backslash \bigcup_{P \in \Gamma} \mathbb{C}N_P$. Then*

(1) $\operatorname{Ind}_\Gamma P \in \mathbb{Z}$ *for all* $P \in \Omega$.
(2) *The function* $\operatorname{Ind}_\Gamma P$ (Γ *fixed*) *is locally constant on* Ω.

Proof The sphere S_n is the generator of $H_n(\mathbb{C}_{n+1} \backslash \mathbb{C}N_0, \mathbb{Z})$. Hence there exists a positive integer m such that $\Gamma \sim m \cdot S_n$. The form under the integral sign is closed (for details see next paragraph), hence $\operatorname{Ind}_\Gamma P = \operatorname{Ind}_{m \cdot S_n} P = m$.

The function $\operatorname{Ind}_\Gamma P$ is continuous and integer-valued, hence locally constant.

Remark The information gained by Theorems 1 and 2 can be used also in another way. Suppose that it can be computed that $\operatorname{Ind}_\Gamma 0 = 1$, then $\Gamma \sim S_n$ in $\mathbb{C}_{n+1} \backslash \mathbb{C}N_0$. This fact can be used, for example, for the simpler proof of the fact that the two contours of integration, considered in [4, §3], are homological one to another.

4. CAUCHY INTEGRAL FORMULA FOR $\mathscr{S}_{\phi_1}$

The equation described in Definition 1 can be written without any further assumption, but in order to have a Cauchy type integral formula for corresponding solutions, some further conditions have to be imposed on the map ϕ. Let us try to find a reasonable set of such assumptions.

Let us write (to have a model) the Cauchy integral formula from complex Clifford analysis:

If $f: \mathbb{C}_{n+1} \to \mathscr{C}_n^{\mathbb{C}}$ satisfies the equation $\partial f = 0$, then

$$f(P) = \frac{1}{\omega_n} \int_{S_n} \frac{(Q-P)^+}{|Q-P|^{n+1}} DQ f(Q).$$

First problem to be solved on the way to a generali...ion of the formula for $\mathscr{D}_\phi$ can be immediately seen. Substituting $\phi(e_i)$ instead of e_i into DQ, the result of the action of $\phi(e_i)$ on $f(Q) \in V_1$ belongs to V_2. But there is no chance now to do the same for $(Q - P)$, because $\phi(e_i)$ are not defined on V_2. Moreover, the result of such action has to be in V_1, because it is the only chance how to recover—after the integration being done—the value $f(P) \in V_1$.

This suggests the following possibility.

Let us suppose that there exists a linear mapping $\tilde{\phi}$: $\mathbb{C}_{n+1} \to \mathscr{L}(V_2, V_1)$, hence together we have a map $\bar{\phi}$: $V_1 \oplus V_2 \to V_1 \oplus V_2$ given by

$$\bar{\phi}([v_1, v_2]) = [\tilde{\phi}(v_2), \phi(v_1)].$$

The standard proof of the formula (1) is based on the following facts:

(1) The form $\dfrac{(Q-P)^+}{|Q-P|^{n+1}} DQf(Q)$ is closed.

(2) The integral $\displaystyle\int_{S_n} \frac{(Q-P)^+}{|Q-P|^{n+1}} DQ$ has to be invertible.

Let us try to ensure the similar conditions for $\bar{\phi}$, too.

First, a few words have to be added about the notation. Let ϕ be the map from $\mathbb{C}_{n+1}$ into $\mathscr{L}(V_1, V_2)$. The symbol ϕ $(Q^+/|Q|^{n+1})$ will be used for the expression, where e_i were substituted by $\phi(e_i)$. The result is clearly the mapping from V_1 into V_2, depending on Q. And by $\phi(DQ)$ we shall denote the $\mathscr{L}(V_1, V_2)$-valued differential form, which will be obtained when e_i are substituted by $\phi(e_i)$. Hence if f is a map from $\mathbb{C}_{n+1}$ to V_1, $\phi: \mathbb{C}_{n+1} \to \mathscr{L}(V_1, V_2)$, $\tilde{\phi}: \mathbb{C}_{n+1} \to \mathscr{L}(V_2, V_1)$, then the expression like

$$\bar{\phi}\left(\frac{(Q-P)^+}{|Q-P|^{n+1}}\right)\bar{\phi}(DQ)f(Q)$$

has a good sense and the result is a V_1-valued differential form.

Now, the short computation gives us the result:

$$d\left[\bar{\phi}\left(\frac{(Q-P)^+}{|Q-P|^{n+1}}\right)\bar{\phi}(DQ)f(Q)\right]$$

$$= \left[\frac{|Q|^2 - (n+1)Q_0^2}{|Q|^{n+3}}\right]\bar{\phi}(e_0)\bar{\phi}(e_0)(f)$$

$$- \sum_{1}^{n} \left[\frac{|Q|^2 - (n+1)Q_i^2}{|Q|^{n+3}} \right] \bar{\phi}(e_i)\bar{\phi}(e_i)(f)$$

$$+ \sum_{1}^{n} \frac{-(n+1)Q_0 Q_i}{|Q|^{n+3}} [-\bar{\phi}(e_i)\bar{\phi}(e_0) + \bar{\phi}(e_0)\bar{\phi}(e_i)](f)$$

$$- \sum_{i<j} \frac{-(n+1)Q_i Q_j}{|Q|^{n+3}} [\bar{\phi}(e_i)\bar{\phi}(e_j) + \bar{\phi}(e_j)\bar{\phi}(e_i)](f).$$

Hence the reasonable sufficient condition for (1) being satisfied can be written as:

(i) $\bar{\phi}(e_i)\bar{\phi}(e_i) = -\bar{\phi}(e_0)\bar{\phi}(e_0)$ for all $i = 1, \ldots, n$

(ii) $\bar{\phi}(e_i)\bar{\phi}(e_0) = \bar{\phi}(e_0)\bar{\phi}(e_i)$ for all $i = 1, \ldots, n$

(iii) $\bar{\phi}(e_i)\bar{\phi}(e_j) + \bar{\phi}(e_j)\bar{\phi}(e_i) = 0.$

Now, the integral $\int_\Gamma (Q-P)^+/|Q-P|^{n+1}\, DQ$ can be computed componentwise and the integral of the components at $e_0 \cdot e_k$ and $e_i \cdot e_k$ have to vanish (because the index is an integer).

The same computation shows that, under the assumptions (i)–(iii), we have

$$\int_\Gamma \bar{\phi}\left(\frac{(Q-P)^+}{|Q-P|^{n+1}}\right)\bar{\phi}(DQ) = \operatorname{Ind}_\Gamma P \cdot \bar{\phi}(e_0)\bar{\phi}(e_0)$$

hence we have to suppose that $\bar{\phi}(e_0)\bar{\phi}(e_0)$ is the isomorphism. For simplicity we shall suppose that

(iv) $\bar{\phi}(e_0)\bar{\phi}(e_0)$ is the identity.

Only elements of the order 0 and 2 of the algebra $\mathscr{C}_n^{\mathbb{C}}$ were involved in the discussion, hence the subspaces V_1, V_2 are invariant subspaces with respect to them and we can consider a weaker condition, namely that the relations (i), . . . , (iv) are satisfied only on V_1, for example. This will be just the case in the example of massless fields of higher spin.

Before stating the Cauchy integral formula, there is one more thing to be discussed. In complex analysis it is sufficient to suppose in the Cauchy integral formula in homological form that the domain Ω is arbitrary and that the contour of integration Γ is homologically trivial. The same is true for the Cauchy integral formula in (real) Clifford analysis. The reason for it is that the contour Γ (being homologically trivial in Ω) is homological (in $\Omega \backslash \{P\}$) to a small sphere S_n around P and it is sufficient to let the radius of the sphere go to zero. The situation is qui: ifferent after the complexification. It follows from fact that, if the contour Γ is homologically trivial in $\Omega \subset \mathbb{C}_{n+1}$, it is no more true that Γ is homological (in $\Omega \backslash \mathbb{C}N_P$) to a small sphere around P. (To have an idea why let us consider Ω consisting of two layers, one containing P and another containing Γ, then Γ is homologically trivial, but is clearly not homological to a small sphere around P. The example can be made a little bit complicated to have Ω connected.)

Hence a restriction on Ω has to be imposed. The idea is that once Γ is (during the deformation to a point) near the cone $\mathbb{C}N_P$ (which prevents a further deformation in $\Omega \backslash \mathbb{C}N_P$), we must have the possibility to follow rays on $\mathbb{C}N_P$ towards the point P. There is a simple condition which allows to do it.

Definition 4 We shall say that a domain $\Omega \subset \mathbb{C}_{n+1}$ is null-convex, if for all $P, Q \in \Omega$, $|P - Q|^2 = 0$, the whole segment $\overline{PQ}$ belongs to Ω.

We are now able to state the Cauchy integral formula:

THEOREM 3 *Let $\phi: \mathbb{C}_{n+1} \rightarrow \mathscr{L}(V_1, V_2)$ be a linear map, for which there is a linear map $\tilde{\phi}: \mathbb{C}_{n+1} \rightarrow \mathscr{L}(V_2, V_1)$ satisfying* (i)–(iv) *on V_1. Let $\Omega \subset \mathbb{C}_{n+1}$ be a null-convex domain. Let Γ be a n-dimensional cycle in $\Omega \backslash \mathbb{C}N_P$, which is homologically trivial in Ω. Then*

$$f(P)\,\mathrm{Ind}_\Gamma P = \frac{1}{x_n}\int_\Gamma \tilde{\phi}\left(\frac{(Q-P)^+}{|Q-P|^{n+1}}\right)\phi(DQ)f(Q), \qquad \text{if} \quad \mathscr{D}_\phi f = 0. \quad (1)$$

Proof Take $P = 0$. Let S_n be a sphere around 0 in $\mathbb{R}_{n+1} \subset \mathbb{C}_{n+1}$ with a sufficiently small radius. We know already that the form under the integral sign is closed. Hence the proof can be divided into two parts:

(a) There exists an integer k such that $\Gamma \sim k \cdot S_n$ in $\Omega \backslash \mathbb{C}N_P$.
(b) The theorem holds for $\Gamma = S_n$.

To prove the first point, we can denote $A = \mathbb{C}_{n+1} \backslash N_P$, $B = \Omega$ then $A \cap B = \Omega \backslash \mathbb{C}N_P$ and $A \cup B$ is star-shaped (because Ω is null-convex), hence homologically trivial. Mayer–Vietoris sequence ([12]) tells us that

$$\iota_*: H_n(A \cap B, \mathbb{Z}) \rightarrow H_n(A, \mathbb{Z}) \oplus H_n(B, \mathbb{Z})$$

is the isomorphism. But there exists an integer k such that $\Gamma \sim k \cdot S_n$ in A and $\Gamma \sim 0$ in B. Hence $\Gamma \sim k \cdot S_n$ in $A \cap B$.

To prove the second point, we can substitute $f(0)$ instead of $f(Q)$ (for a radius of the sphere approaching zero), hence using (iv) we have

$$\frac{1}{x_n}\int_{S_n}\bar{\phi}\left(\frac{Q^+}{|Q|^{n+1}}\right)\phi(DQ)f(0)=\bar{\phi}(e_0)\bar{\phi}(e_0)f(0)\,\mathrm{Ind}_{S_n}0=f(0), \qquad x_n=\omega_n$$

5. THE CAUCHY INTEGRAL FORMULA FOR $\mathscr{D}_{\phi_1,\ldots,\phi_m}$

Let us now formulate and prove the Cauchy integral formula for solutions of higher order differential operators, defined in Definition 2. Suppose hence that V is a complex vector space and let $\phi_j: \mathbb{C}_{n+1} \to \mathscr{L}(V, V)$, $j = 1, \ldots, m$ be linear maps.

To find an integral formula for solution of $\mathscr{D}_{\phi_1,\ldots,\phi_m} f = 0$ we shall introduce the differential form

$$\begin{aligned}\omega &= [\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_{m-1}]\phi_m(DQ)f(Q)\\ &\quad - [\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_{m-2}]\phi_{m-1}(DQ)[\mathscr{D}_m f(Q)]\\ &\quad + \cdots \pm \phi_1(g)\phi_1(DQ)[\mathscr{D}_2 \cdots \mathscr{D}_m f(Q)]\end{aligned}$$

where $\mathscr{D}_i = \mathscr{D}_{\phi_i}$, $i = 1, \ldots, m$.

To imitate again the standard procedure for the proof of a formula, we have first to find a condition for ω to be closed. But

$$\begin{aligned}d\omega &= [\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_m]f + [\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_{m-1}][\mathscr{D}_m f]\\ &\quad - [\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_{m-1}][\mathscr{D}_m f] \pm \phi_1(g)[\mathscr{D}_1 \cdots \mathscr{D}_m f]\\ &= [\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_m]f + \phi_1(g)[\mathscr{D}_1 \cdots \mathscr{D}_m f]\end{aligned}$$

hence we need only to suppose that

(1) $\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_m = \mathscr{D}_1 \cdots \mathscr{D}_m f = 0.$

Second, the formula has to be true for constants, hence we have to suppose

(2) $\displaystyle\int_{S_n} [\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_{m-1}]\phi_m(DQ) = 1.$

Finally, to be sure that under the limiting procedure all unwanted terms will approach zero, we shall suppose that

(3) (a) $[\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_{m-1}]|Q|^{n+1} \longrightarrow 0$ for $|Q| \to \infty$, $Q \in \mathbb{R}_{n+1}$.
$\lim_{\substack{|Q| \to 0 \\ Q \in \mathbb{R}_{n+1}}} \{[\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_{m-1}]|Q|^{n+1}\}\mathscr{D}_n$ exists.

(b) For all $j = 1, \dots, n-2$ the function $[\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_j]|Q|^{n+1}$ approach zero together with its first derivatives, when $|Q| \to 0$, $Q \in \mathbb{R}_{n+1}$.

There are more possibilities how to find a mapping g satisfying all these conditions. Let us show only two examples.

Example 1 Let $V = \mathscr{C}_n^{\mathbb{C}}$. Let $\phi_1 = \cdots = \phi_m$ is given by the inclusion $\mathbb{C}_{n+1} \subset \mathscr{C}_n^{\mathbb{C}}$. Then $\mathscr{D}_1 = \cdots = \mathscr{D}_m = \partial$, $\phi_m(DQ) = DQ$ and we can find $g: \mathbb{C}_{n+1} \longrightarrow \mathscr{C}_n^{\mathbb{C}}$ such that

$$g\mathscr{D}_1 \cdots \mathscr{D}_{m-1} = \frac{1}{x_n} \frac{Q^+}{|Q|^{n+1}}$$

we can take, for example

$$g = \frac{1}{x_n} \frac{Q^+ Q_0^{m-1}}{|Q|^{n+1}(n-1)!}.$$

The standard results from hypercomplex analysis ([5]) gives us that such a function g satisfies the conditions (1), (2), the condition (3) being trivially satisfied in this case.

Example 2 Let $\phi_1(e_i) = e_i^+$, $\phi_2(e_i) = e_i$ (acting on $V = \mathscr{C}_n^{\mathbb{C}}$ on the left), then $\mathscr{D}_1 = \partial^+$, $\mathscr{D}_2 = \partial$, $\mathscr{D}_{\phi_1, \phi_2} = \partial^+ \partial = \Delta$ is the Laplace operator and we can take

$$g = \frac{1}{x_n} \frac{1}{|Q|^{n-1}} \frac{1}{1-n}.$$

Then

$$g\partial^+ = \frac{1}{x_n} \frac{Q^+}{|Q|^{n+1}}$$

and the conditions (1)–(3) are again satisfied.

THEOREM 4 *Let V be a vector space and $\phi_1, \dots, \phi_m$ linear maps from $\mathbb{C}_{n+1}$ to $\mathscr{L}(V, V)$ satisfying (1)–(3). Suppose that $\Omega \subset \mathbb{C}_{n+1}$ is a null-convex domain and f a map from Ω to V, satisfying the equation $\mathscr{D}_{\phi_1, \dots, \phi_m} f = 0$.*

Take $P \in \Omega$ and a n-dimensional cycle Γ in $\Omega \backslash \mathbb{C}N_P$, which is homologically trivial in Ω. Then

$$f(P)\cdot \mathrm{Ind}_\Gamma P = \int_\Gamma \omega$$

where

$$\begin{aligned}\omega = {} & [\phi_1(g)(Q-P)\mathscr{D}_1 \cdots \mathscr{D}_{m-1}]\phi_m(DQ)f(Q) \\ & - [\phi_1(g)(Q-P)\mathscr{D}_1 \cdots \mathscr{D}_{m-2}]\phi_{m-1}(DQ) \\ & \times [D_m f(Q)] + \cdots \pm \phi_1(g)(Q-P)\phi_1(DQ)[\mathscr{D}_1 \cdots \mathscr{D}_m f(Q)].\end{aligned}$$

Proof As in the proof of Theorem 3, it is sufficient to prove the theorem for $P = 0$ and for $\Gamma = S_n$.

If ρ is the radius of the sphere S_n, then the right-hand side can be written as

$$\begin{aligned} & \frac{1}{\rho^{n+1}} \int_{S_n} [[\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_{m-1}]|Q|^{n+1}]\phi_m(DQ)f(Q) + \cdots \\ & \qquad \pm |Q|^{n+1}\phi_1(g)\phi_1(DQ)[\mathscr{D}_2 \cdots \mathscr{D}_m f(Q)] \\ & = \frac{1}{\rho^{n+1}} \int_{K_n} [[\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_{m-1}]|Q|^{n+1}]\mathscr{D}_m f(Q) + \cdots \\ & \qquad \pm \phi_1(g)|Q|^{n+1}\mathscr{D}_1[\mathscr{D}_2 \cdots \mathscr{D}_m f(Q)]\end{aligned}$$

where K_n is the ball of the radius ρ around 0. The assumption (3) tells us that the only term surviving the limit $\rho \to 0^+$ will be the term

$$\frac{1}{\rho^{n+1}} \int_{S_n} \{[\phi_1(g)\mathscr{D}_1 \cdots \mathscr{D}_{m-1}]|Q|^{n+1}\}\mathscr{D}_m]f(Q).$$

But the limit of this term will be (because of (2) and (3)) equal to $f(0)$.

6. EXAMPLES

Example 1 Let V_1, V_2 be the basic spinor representation spaces and let us suppose again that they are imbedded in $\mathscr{C}_n^{\mathbb{C}}$. Define ϕ by the left multiplication. Then the Theorem 3 reduces to

$$f(P)\cdot \mathrm{Ind}_\Gamma P = \frac{1}{\kappa_n} \int_\Gamma \frac{(Q-P)^+}{|Q-P|^{n+1}} DQf(Q). \tag{2}$$

The form of the formula is the same as in complex Cl   rd analysis ([17]). The integral formula, described in ([17]), can be obtained by taking $V = \mathscr{C}_n^{\mathbb{C}}$ and defining ϕ by the same method. The only change is that the values of f are now in $\mathscr{C}_n^{\mathbb{C}}$. The formula in complex Clifford analysis can be decomposed into pieces, each one being just the formula (2).

Example 2 Take $n = 3$, $V_1 = S^m V_+$, $V_2 = V_- \otimes S^{m-1} V_+$ and consider the mapping ϕ given by

$$\phi(e_i)(v_1 \odot \cdots \odot v_m) = e_i v_1 \otimes v_2 \odot \cdots \odot v_m, \qquad i = 1, \ldots, n.$$

Let us define

$$\tilde{\phi}(e_i)(w \otimes v_2 \odot \cdots \odot v_n) = e_i w \odot v_2 \odot \cdots \odot v_n.$$

Then $\tilde{\phi}$ can be defined as in Section 3 and it is easy to verify that the conditions (i)–(iv) of Section 3 are satisfied on V_1 (note that they are not satisfied on V_2). Hence Theorem 3 gives us the integral formula for solution of $\mathscr{D}_\phi f$. This formula is just the formula described (in the spinor language) in [19].

The special case $m = 1$ coincides with Example 1.

Example 3 Taking $V_1 = V_2 = \mathbb{C}\mathbb{H}$ and the map ϕ as in Example 5, Section 2, then the Cauchy integral formula gives us the integral formula for solution of (complexified) Fueter equation (see [20]).

The complex quaternions were used by Imaeda (under the name biquaternions) for the description of classical electrodynamics ([10]). The residue theorem described there can be obtained by taking the multiplication from the right in the definition of $V = \mathbb{C}\mathbb{H}$.

Example 4 Take $V = \mathscr{C}_n^{\mathbb{C}}$, $\phi_1(e_i)(v) = e_i v$, $\phi_2(e_i)(v) = e_i^+ v$, the operator $\mathscr{D}_{\phi_1, \phi_2}$ is the (complex) Laplace operator. If we take the mapping g as in Example 2 of Section 5, then Theorem 4 gives us the integral formula from ([4]).

Example 5 Let us take again $V = \mathscr{C}_n^{\mathbb{C}}$ and define $\phi_1 = \cdots = \phi_n$ by the left multiplication. Then Theorem 4 gives us (the complexification of) the formula for left k-monogenic functions (see [5]). The function g can be taken as in Example 1 of Section 5.

Example 6 After the restriction to the Euclidean slice $\mathbb{R}_{n+1} \subset \mathbb{C}_{n+b}$

we shall obtain various integral formulas from Clifford analysis as special slices. In such real versions of Theorem 3 or 4 we need not suppose that Ω is null-convex (i.e. every $\Omega \subset \mathbb{R}_{n+1}$ is null-convex), from the homological point of view the situation is much more simple because of the fact that $H_n(\mathbb{R}_n - \{P\}, \mathbb{Z}) \simeq \mathbb{Z}$ is almost trivial. But even so, the homological formulation of Cauchy integral formulas improve the standard versions and make the theorem more versatile.

(a) Example 3, restricted to $\mathbb{R}_{n+1}$, gives us the integral formula described in [2] for regular functions in Clifford analysis.

(b) For $n = 3$, $V = \mathbb{H} \subset \mathbb{C}\mathbb{H} \subset \mathscr{C}_n^{\mathbb{C}}$ given by the left multiplication, the real form of Theorem 3 gives us the Cauchy integral formula for regular function on $\mathbb{R}_4 \simeq \mathbb{H}$ (in the sense of Fueter) described in [6].

(c) The real form of Example 5 is just the integral formula from [5].

Another type of integral formula is available after the restriction to another, hyperbolic real subspaces of $\mathbb{C}_{n+1}$. But the situation is much more complicated because to do a simple restriction is not possible. The character of singularities is quite different, the character of the corresponding integral formula differs a lot from "elliptic" ones and to do a restriction means, in fact, that a suitably deformed contour of integration (avoiding still singularities in $\mathbb{C}_{n+1}$) looks like a Cartesian product of a $(n-1)$-dimensional cycle with S_1 and after the integration over S_1 being done, the corresponding "hyperbolic" integral formula will be obtained. The examples of such a procedure can be found in [19], where this procedure is used to derive the integral formulas for massless fields in [15] from that of Example 2 and in [4], where the integral formulas of Riesz for solutions of wave equation [16] are derived from that of Example 4. Moreover, a new integral formula for spinor fields in higher dimensions is derived in [4], using this procedure.

References

[1] L. P. Hughston and R. Ward, Advances in twistor theory, *Research Notes in Math.* **37**, Pitman, 1979.

[2] F. Brackx, R. Delanghe, and F. Sommen, Clifford analysis, *Research Notes in Math.* **76**, Pitman, 1982.

[3] V. Bartik, A. V. Ferreira, M. Markl, and V. Souček, Index and Cauchy integral formula in complex quaternionic analysis (to appear in Simon Stevin).

[4] J. Bureš, Some integral formulas in complex Clifford analysis, *Proc. of 12th Winter School 1983*, Suppl. di Rend. Circolo Math. di Palermo.

[5] R. Delanghe and F. Brackx, Hypercomplex function theory Hilbert modules with reproducing kernels, *Proc. London Math. Soc.* **37** (1978), 545–576.

[6] R. Fueter, Integralsätze für reguläre Funktionen einer Quaternionenvariablen, *Comm. Math. Helv.* **10** (1937), 306–315.

[7] L. Gross, Norm invariance of mass-zero equations under the conformal group, *J. Math. Phys.* **5** (1964), 687–698.

[8] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Reidel, Dordrecht, 1984.

[9] D. Husemoller, *Fibre Bundles*, McGraw-Hill, New York, 1966.

[10] K. Imaeda, A new formulation of electromagnetism, *Nuovo Cimento 32B* (1976), 138–162.

[11] J. Měska, Regular functions of complex quaternionic variable, *Czech. Math. J.* **34** (1984), 130–145.

[12] W. Massey, Singular homology theory, *Grad. Texts on Math.* **70**, Springer, 1980.

[13] J. Milnor, *Singular Points of Complex Hypersurfaces*, Princeton, 1968.

[14] R. Penrose, Structure of space-time, in *Battelle Rencontres*, 1967, Ed. C. M. de Witt, J. A. Wheeler.

[15] R. Penrose, Null hypersurface initial data for classical fields of arbitrary spin and for general relativity, *Gen. Rel. Grav.* **12** (1980), 225–264.

[16] M. Riesz, A geometric solution of the wave equation in space-time of even dimension, *Comm. Pure Appl. Math.* Vol. XIII (1960), 329–351.

[17] J. Ryan, Complexified Clifford analysis, *Complex Variables* (1982), 119–149.

[18] F. Sommen, Monogenic differential forms and homology theory (to appear in *Proc. Royal Irish Academy*).

[19] V. Souček, Boundary value type and initial value type integral formulas for massless fields, *Twistor Newsletters* **14**, 1982.

[20] V. Souček, Complex-quaternionic analysis applied to spin-massless fields, *Complex Variables* **1** (1983), 327–346.

[21] V. Souček, Holomorphicity in quaternionic analysis, *Seminario di Variable Complesse*, Bologna, 1982.