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I CONFERENZA

HOLOMORPHICITY IN QUATERNIONIC ANALYSIS.

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The basic problem of quaternionic analysis from the very beggining was what is the proper generalization of the notion of holomorphicity. In the complex analysis there is a lot of equivalent conditions for holomorphicity. The question is which one can be suitably generalized to quaternions. We shall discuss first the two most natural definitions - the existence of quaternionic derivative and the power series definition.

(I.0) Notation.

The field of quaternions will be denoted by H . A typical element of H can be written as $q = x_0+i_1x_1+i_2x_2+i_3x_3$. The quaternionic units enjoy the usual properties $i_1^2=i_2^2=i_3^2=-1$; $i_1i_2i_3 = -1$. (Note that all other usual properties of quaternionic units follows from these relations.) The conjugation in H is given by

$$q^{\dagger} = x_0-i_1x_1-i_2x_2-i_3x_3$$

and the norm $|q| = (qq^{\dagger})^{\frac{1}{2}}$ can be used to express the inverse element for every $q \in H$, $q \neq 0$, namely $q^{-1} = q^{\dagger}/|q|^2$.

(I.1) Differentiable functions.

Definition:

A function $f : H \rightarrow H$ is called (left) differentiable at q , if the limit
$$\frac{df}{dq} = \lim_{h\to 0} h^{-1}(f(q+h)-f(q))$$
exists.

Remark.

It is the most natural definition at the first sight, but,as will be shown, it leads to very restricted class of functions (a subclass of linear ones). This fact has been rediscovered many times and abandoned after, just because there seems to be not very much of interesting things to be said about. But it is worth to study it more closely to get an unexpected connection with other parts of mathematics, resp. of mathematical physics.

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(I.2) Theorem:

If $\frac{df}{dq}$ exists, then $f(q) = a + qb$ $(a,b \in H)$.

Proof:

Let $f = f_0+i_1f_1+i_2f_2+i_3f_3$. As in the complex case, $\frac{df}{dq}$ exists **if and only if** the limits along all four axes are equal one to another and equal to $\frac{df}{dq}$. Using the notation $\partial_j=\partial/\partial x_j$, the condition looks like $\partial_0 f=-i_1\partial_1 f=-i_2\partial_2 f=-i_3\partial_3 f=\frac{df}{dq}$. To simplify further considerations we shall use 'complex' variables

$\zeta = x_0+i_3x_3$, $\eta = x_2+i_3x_1$

and we shall define two functions $g = f_0+i_3f_3$, $h = f_2+i_3f_1$ and consider them as the functions of variables $\zeta,\eta,\bar{\zeta},\bar{\eta}$. Then:

$$1)\ \partial_0 f=-i_3\partial_3 f \Longleftrightarrow \begin{matrix}(\partial_0+i_3\partial_3)(f_0+i_3f_3)=0\\(\partial_0+i_3\partial_3)i_2(f_2+i_3f_1)=0\end{matrix} \Longleftrightarrow \frac{\partial g}{\partial\bar{\zeta}} = \frac{\partial h}{\partial\zeta} = 0 \qquad (1)$$

$$2)\ i_1\partial_1 f=i_2\partial_2 f \Longleftrightarrow \begin{matrix}i_2(\partial_2-i_3\partial_1)(f_0+i_3f_3)=0\\(\partial_2+i_3\partial_1)i_2i_2(f_2+i_3f_1)=0\end{matrix} \Longleftrightarrow \frac{\partial g}{\partial\eta} = \frac{\partial h}{\partial\bar{\eta}} \qquad (2)$$

$$3)\ \partial_0 f=-i_2\partial_2 f \Longleftrightarrow \begin{matrix}\partial_0(f_0+i_3f_3)+i_2\partial_2 i_2(f_2+i_3f_1)=0\\\partial_0 i_2(f_2+i_3f_1)+i_2\partial_2(f_0+i_3f_1)=0\end{matrix} \Longleftrightarrow \begin{matrix}\partial_0 g-\partial_2 h=0\\\partial_0 h+\partial_2 g=0\end{matrix}$$

which can be written also as $\frac{\partial g}{\partial\zeta} - \frac{\partial h}{\partial\eta} = 0$, $\frac{\partial h}{\partial\bar{\zeta}} + \frac{\partial g}{\partial\bar{\eta}} = 0$ (3)

But now the relations (1),(2) tell us that $g=g(\zeta,\bar{\eta})$, $h=h(\bar{\zeta},\eta)$ and using a procedure like $\frac{\partial^2 g}{\partial\zeta^2} = \frac{\partial}{\partial\zeta}\left(\frac{\partial h}{\partial\eta}\right) = 0$, etc., we obtain immediately that g and h have to be linear in their variables. It is now easy to check the possibilities for coefficients and to see that the only possibility left is $f(q) = a+qb$ $(a,b \in H)$.

(I.3) Power series.

Quaternions don't commute, hence the reasonable generalization of the term a_nz^n from the complex case is the term

$a_0qa_1q\dots qa_{n+1}$, $a_i \in H$.

But the definition of holomorphicity using sums of such terms leads to a quite general class of functions (the same as the sum of monomials, generated by $x_0,\dots x_3$ with quaternionic coefficients, i.e. to real-analytic mappings). It can be seen immediately from the following formulas:

$$
\begin{aligned}
x_0 &= \tfrac{1}{4}(q - i_1 q\cdot i_1 - i_2 q\cdot i_2 - i_3 q\cdot i_3)\\
x_1 &= -\tfrac{1}{4} i_1(q - i_1 q\cdot i_1 + i_2 q\cdot i_2 + i_3 q\cdot i_3)\\
x_2 &= -\tfrac{1}{4} i_2(q + i_1 q\cdot i_1 - i_2 q\cdot i_2 + i_3 q\cdot i_3)\\
x_3 &= -\tfrac{1}{4} i_3(q + i_1 q\cdot i_1 + i_2 q\cdot i_2 - i_3 q\cdot i_3)
\end{aligned}
\qquad (4)
$$

Hence to investigate such functions would be the same as to study real-analytic mappings from R_4 to R_4. Nevertheless, formulas (4) will be useful later.

(I.4) Fueter equation.

The most interesting notion of quaternionic holomorphicity came from the attempt to generalize Cauchy-Riemann equation. The analogy is clear - the operator $\frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ from complex analysis is replaced here by the operator $\frac{1}{4}(\partial_0 + i_1\partial_1 + i_2\partial_2 + i_3\partial_3)$. The Fueter's school developed Cauchy integral formulas and Laurent series expansion for solutions of the equation. The modern account of basic results together with further new results can be found in [Su] . A lot of further function theory was developed in more general setting (functions on R_n with values in the Clifford algebra) in [BDS],[L].

Definition:

We shall say that the function $f\colon \Omega \to H$ is (left) regular in $\Omega \subset H$ if

$$\partial^{+} f := \tfrac{1}{4}(\partial_0 + i_1\partial_1 + i_2\partial_2 + i_3\partial_3)(f_0 + i_1 f_1 + i_2 f_2 + i_3 f_3) = 0$$

is satisfied in Ω . The function f is antiregular in Ω,if $\partial f=0$.

Remarks:

1. The operators $\partial/\partial z$ and $\partial/\partial\bar{z}$ seems to have a nice quaternionic counterpart in ∂ , ∂^{+}(where $\partial := \frac{1}{4}(\partial_0 - i_1\partial_1 - i_2\partial_2 - i_3\partial_3)$).

2. There is, of course, a possibility to investigate right regular functions, satisfying $f.\partial=0$, resp. $f.\partial^{+}=0$ instead. The properties of such functions would be the mirror image of those of left regular functions.

3. Let us note that $\partial\partial^{+} f = \partial^{+}\partial f = 1/16\, \Delta f$ (where Δ denotes the Laplace operator), hence if f is regular, then f is harmonic (i.e. every component is).

(I.5) *H*-valued forms.

Before studying properties of regular functions more closely, we would like to follow the analogy with the complex case a little bit further. It is the standard procedure in the complex case to identify complex functions with 0-forms and to consider the de Rham operator $d : \mathcal{E}^0_c \to \mathcal{E}^1_c$ (where $\mathcal{E}^k_c$ is the space of (complex-valued) k-forms). The operator d splits then into two pieces $d = \partial + \bar\partial$, where

$$\partial : \mathcal{E}^0_c \longrightarrow \mathcal{E}^{1,0}_c \ , \ \bar\partial = \mathcal{E}^0_c \longrightarrow \mathcal{E}^{0,1}_c \quad \text{and} \quad \mathcal{E}^1_c = \mathcal{E}^{1,0}_c \oplus \mathcal{E}^{0,1}_c .$$

Let us try now to find the analogy of such splitting in the quaternionic case. First thing to do is to consider H-valued forms on H. Such forms are considered currently in quaternionic analysis (see [Su]), even in mathematical physics (see [A]). They can be defined as the tensor product $\mathcal{E}^*_H = \mathcal{E}^* \otimes_R H$. The exterior product can be defined by the usual formula ([Su], p.203) and the exterior derivative acts on them componentwise. The Stokes theorem holds in the usual form for such forms. (Note only that the exterior multiplication - because of noncommutativity of coefficients - need not have usual properties, but all multiplicative properties of them are quite naturally coming from multiplicative properties of quaternionic coefficients and those of real differential forms and need not be discussed in more details.) The most natural H-valued 1-forms are

$$dq = dx_0 + i_1 dx_1 + i_2 dx_2 + i_3 dx_3 \ , \quad dq^\dagger = dx_0 - i_1 dx_1 - i_2 dx_2 - i_3 dx_3 \ .$$

But the straightforward generalization of the operators ∂, $\bar\partial$, i.e. the operators $f \longmapsto (\partial f)\, dq$; $f \longmapsto (\partial^\dagger f)\, dq^\dagger$ doesn't work because there is clearly no hope that these two operators would give the decomposition of the de Rham operator d. The spaces $\{f.dq \in \mathcal{E}^1_H\}$, $\{g.dq \in \mathcal{E}^1_H\}$ have both the dimension 1 (as the right H-vector spaces) and cannot give together the whole space $\mathcal{E}^1_H$ (which has the dimension 4).

Even more, to have the decomposition of $\mathcal{E}^1_H$ into subspaces, we have first look for a suitable basis of $\mathcal{E}^1_H$ as the H-vector space. And there is no natural candidate for it, it seems to be difficult to complete dq, $dq^\dagger$ to a basis for $\mathcal{E}^1_H$.

To solve the problem we can use the properties, described in (4). Let us define 1-forms $dq^1 = -i_1 dq i_1$, $dq^2 = -i_2 dq i_2$, $dq^3 = -i_3 dq i_3$. Then the relations (4) tells us that they form together with dq the basis for $\mathcal{E}^1_H$.

The idea to use this basis and to decompose $\mathcal{E}^1_H$ into four one-dimensional spaces is against the basic spirit of quaternions. Fixed quaternionic units i_1, i_2, i_3 are chosen only to have a suitable coordinate description, but basic notions of quaternionic analysis has to be independent of the choice of units. To keep the splitting independent of the choice of quaternionic units, we have to consider operators

$$\mathfrak{D} f = dq\partial f$$

$$\vec{\mathfrak{D}} f = dq^1\partial^1 f + dq^2\partial^2 f + dq^3\partial^3 f ,$$

where $\partial^1 f = -i_1\partial f i_1$, $\partial^2 f = -i_2\partial f i_2$, $\partial^3 f = -i_3\partial f i_3$ and to split $\mathcal{E}^1_H$ as

$$\mathcal{E}^1_H = \mathcal{E}^{1,(1)}_H \oplus \mathcal{E}^{1,(3)}_H$$

where $\mathcal{E}^{1,(1)}_H = \{\varepsilon dq \mid \varepsilon \in \mathcal{E}^0_H\}$ and $\mathcal{E}^{1,(3)}_H = \{h_1 dq^1 + h_2 dq^2 + h_3 dq^3 \mid h_i \in \mathcal{E}^0_H\}$.

(I.6) <u>Lemma</u>: We have $d = \mathfrak{D} + \vec{\mathfrak{D}}$.

<u>Proof</u>:

$$\mathfrak{D} + \vec{\mathfrak{D}} = dq\partial - i_1 dq\partial i_1 - i_2 dq\partial i_2 - i_3 dq\partial i_3 = 4\mathrm{Re}(dq\partial) = d,$$

where Re means the real part of the corresponding quaternionic expression.

<u>Remark</u>:

The difference between holomorphic ($\bar{\partial}f=0$) and antiholomorphic ($\partial f=0$) functions in the complex case is only a question of the choice of an orientation in R_2. But in quaternionic analysis it seems to be inevitable that the solutions of the corresponding equations $\mathfrak{D}f=0$ and $\vec{\mathfrak{D}}f=0$ are quite different in character. How the solutions look like?

(I.7)

(i) <u>The solutions of $\mathfrak{D}f=0$</u>.

It is clear that $\mathfrak{D}f=0$ if and only if f is the antiregular function (in Fueter's sense).

(ii) <u>The solutions of $\vec{\mathfrak{D}}f=0$</u>.

<u>Theorem</u>:

The function f is differentiable if and only if $\vec{\mathfrak{D}}f=0$.

Moreover, then $\mathfrak{D}f = dq(\frac{df}{dq})$.

Proof:

The function f is differentiable

$$\frac{df}{dq} = \partial_0 f = -i_1\partial_1 f = -i_2\partial_2 f = -i_3\partial_3 f \iff$$

$$\iff \frac{df}{dq} = \partial f \quad \& \quad \partial_0 f-\partial f=-i_1\partial_1 f-\partial f=-i_2\partial_2 f-\partial f=-i_3\partial_3-\partial f=0 \iff$$

$$\iff \frac{df}{dq} = \partial f \quad \& \quad df - dq\partial f = 0 .$$

But $df - dq\partial f = \vec{\mathfrak{D}} f$.

(I.7) Remark.

We have seen that in order to gain the full analogy with the complex case, we have to consider both differentiable and regular functions as the corresponding generalization of holomorphic and antiholomorphic functions of a complex variable. Note that the question of an orientation can enter in a different way into the picture. If we define $dq^{\dagger j} = -i_j dq^{\dagger} i_j$ and $\partial^{\dagger j} = -i_j \partial^{\dagger} i_j$, we have a similar decomposition

$$d = \mathfrak{D}^{\dagger} + \vec{\mathfrak{D}}^{\dagger},$$

where $\mathfrak{D}^{\dagger} f = dq\,\partial$ and $\vec{\mathfrak{D}}^{\dagger} f = dq^{\dagger 1}\partial^{\dagger 1} + dq^{\dagger 2}\partial^{\dagger 2} + dq^{\dagger 3}\partial^{\dagger 3}.$

So we can see that in complex analysis the analogues of $\mathfrak{D}$ and $\vec{\mathfrak{D}}^{\dagger}$ coincide, which is the reason why it was difficoult to find a proper quaternionic analogues for them.

(I.8) Remark.

The splitting of H-valued 1-forms, described in (I.7), can be developed further and a splitting of 2-forms and 3-forms can be described to complete the whole picture (see [So1]).

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