CLIFFORD ANALYSIS FOR HIGHER SPINS

VLADIMÍR SOUČEK
Mathematical Institute
Charles University
Sokolovská 83
186 00 Praha
Czech Republic

Abstract. Higher spin analogues of the (massless) wave and Dirac equation on Minkowski space are well understood in dimension 4. They appear usually under the name massless field equations. In the paper, higher-dimensional analogues of the massless field equations are studied in Riemannian setting and basic results of the function theory are described. Another set of conformally invariant equations - the so called twistor equations for any spin - were studied in the physical case by R. Penrose and others. We are discussing the higher-dimensional analogues of these equations (in Riemannian situation) and their possible role in Clifford analysis. Multiplicative properties of solutions of the twistor equation are discussed.

Key words: Clifford analysis, massless field equations, twistor equations, higher spins

1. Introduction

The Dirac equation and its solutions studied in Clifford analysis are considered either for Clifford-valued or for spinor-valued functions (see (Delanghe *et al.* , 1992)). Higher spin analogues of the Dirac equation are well known in the physical situation (on four-dimensional Minkowski space), they are usually called massless field equations ((Penrose, Rindler , 1984)). They include, besides the vawe and the Dirac equation, Maxwell equations (spin 1 case) and linearized gravity (spin 2 case). Methods of Clifford analysis were applied already to Maxwell equations (see (Jancewicz , 1988)), but the function theory for higher spin and higher-dimensional cases were not studied yet. Higher-dimensional analogues of massless field equations were defined in an abstract representation theory language in (Fegan , 1975) or (Baston, Eastwood , 1991). The purpose of the paper is to pave a few first roads into this wast teritory and to show that all this subject should form an integral part of Clifford analysis. It indicates as well that a study of further conformally invariant equations with values in other spin representations from the point of view of function theory could be of interest.

Another important issue coming back again and again since Fueter's discovery of his equation is the fact that the space of solutions of the Dirac equation is not an algebra - product of two solutions need not be a solution. The basic reason behind it is clearly the fact that Dirac equation is (in principle) an equation for a spinor field. From that point of view it is understandable that only natural multiplication available for spinor fields is (a variant of) tensor multiplication which leads directly to higher spin fields. No version of a product for two spin $\frac{1}{2}$ fields is known which

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would garantee that the product of two such solutions is always a solution of spin 1 massless field equation.

On the other hand, the twistor equation (which is not very interesting in spin $\frac{1}{2}$ case, because solutions are linear) is becoming more interesting for higher spins; the space of its solutions is bigger and bigger (higher order polynomials). Moreover, it is possible to define a product in such a way that the product is commutative and product of two solutions is again a solution (of higher spin). Hence solutions of twistor equations (considered for all spins at the same time) offer an interesting generalization of Cauchy-Riemann equations to higher dimensions with the property that the space of solutions form a commutative algebra. The solutions with specific spins are analogues of homogeneous solutions of Cauchy-Riemann equations. All that together with a lot of relationships among solutions of massless field equations and twistor equations for any spin is well understood in dimension 4 due to the work of R. Penrose ((Penrose, Rindler , 1984), see also (Hitchin, 1980)), but higher dimensional analogues were not studied yet. So at the end of the paper we discuss basic multiplicative properties for solutions of the twistor equation in higher dimensions. THe paper gives a review of cartain results in this area but complete proofs of theorems given will be published in another paper.

2. The Cartan Powers of the Spinor Representation

Let us consider negative definite Euclidean space R^n, the corresponding Clifford algebra $R_{0,n} = C$ and its complexification C^c. As is standard in Clifford analysis, we consider functions defined on domains in R^n but their values will generally be in higher spin representations of the group Spin (n). In this section, we are going to describe these irreducible representations together with their elementary properties and we are setting the notation used in other sections.

It is necessary to distinguish even- and odd-dimensional cases.

The case $n = 2k$.

There is exactly one (up to isomorphism) irreducible representation S of the Clifford algebra C^c which splits, as a representation of the even part $(C^c)^+$, into two irreducible parts denoted by S_A and $S_{A'}$. The corresponding dual (contragredient) representations are denoted by S^A, resp. $S^{A'}$. All finite-dimensional irreducible representations of the group Spin (n) are classified by their highest weights $\mu \in \Lambda_W$, where

$$\Lambda_W = \{(\mu_1, \ldots, \mu_k) \in Z^k \cup (Z + \frac{1}{2})^k | \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{k-1} \geq |\mu_k|\}$$

(a lot of useful details concerning the discussion in this section can be found in (Fulton, Harris , 1991), Chap.19). The basic spinor representations S_A, resp. $S_{A'}$ have the highest weights $\alpha = (\frac{1}{2}, \ldots, \frac{1}{2})$, resp. $\alpha' = (\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$. The spinor representations $S_A, S_{A'}$ are selfdual for k even and they are dual to one another for k odd.

The higher spin representations we are interested in can be realized as invariant subspaces of suitable tensor products of S_A and $S_{A'}$. It is convenient to use the Penrose abstract index notation (for details see (Penrose, Rindler , 1984)) where indices are used as abstract (not coordinates) labels. Whenever same symbols appear up and down inside a tensor product, it will indicate the contraction in the corresponding pieces of the product, a round bracket around a set of indices means the symmetrisation. The tensor product $S_{A\ldots E}$ of p copies of the representation S_A decomposes into a sum of irreducible representations. The symmetric power $S_{(A\ldots E)} := Sym^p(S_A)$ is an invariant subspace but it is an irreducible representation only in dimension 4; it decomposes further in higher dimensions.

The Clifford algebra C^c can be identified with the space $End(S) \cong S \otimes S^*$. For the even and odd parts, we get $(C^c)^+ \cong S_A^A \oplus S_{A'}^{A'}$ and $(C^c)^- \cong S_A^{A'} \oplus S_{A'}^A$. Hence the complexification $C^n = R^n \otimes_C C$ is a subset of $S_A^{A'} \oplus S_{A'}^A$ and for a given vector $v \in C^n$, we denote by $v_A^{A'}$, resp. $v_{A'}^A$ the corresponding components of the sum. The projections of C^n into $S_A^{A'}$, resp. $S_{A'}^A$ will be denoted by $(C^n)_A^{A'}$ resp. $(C^n)_{A'}^A$. These both spaces are isomorphic (as representations) to the fundamental representation C^n.

The case $n = 2k + 1$.

The situation is different in odd dimensions. There are two inequivalent irreducible representation S_A, resp. $S_{A'}$ of C^c but they are isomorphic as representations of $(C^c)^+$, hence as Spin (n)-representations. They are selfdual representations.

The set Λ_W of dominant weights is given by

$$\Lambda_W = \{(\mu_1, \ldots, \mu_k) \in Z^k \cup (Z + \frac{1}{2})^k | \mu_1 \geq \mu_2 \geq \ldots \geq \mu_{k-1} \geq \mu_k \geq 0\}.$$

The basic spinor representation S_A has the highest weight $\alpha = (\frac{1}{2}, \ldots, \frac{1}{2})$.

The Clifford algebra is described as $C^c \cong S_A^A \oplus S_{A'}^{A'}$. The projection of vectors in C^n into S_A^A are hence denoted as v_A^A. It is convenient to note that in the discussion below, a transition from even- to odd-dimensional case can be usually done simply by converting all primed indices to unprimed ones.

We need below a notion of the Cartan product (let us consider now any dimension, even or odd). If V_1 and V_2 are two irreducible representations of Spin (n) and if v_1, resp. v_2 are their highest weight vectors, then $v_1 \otimes v_2$ is a highest weight vector of an irreducible component of the product $V_1 \otimes V_2$ called the Cartan product of V_1 and V_2; let us denote it by $V_1 \boxtimes V_2$. In terms of highest weights, it is characterized by the sum of highest weights of V_1 and V_2. The corresponding orthogonal projection $\pi^{Car} : V_1 \otimes V_2 \mapsto V_1 \boxtimes V_2$ makes it possible to define the Cartan product of any two elements $v_1 \in V_1, v_2 \in V_2$; we define $v_1 \boxtimes v_2 = \pi^{Car}(v_1 \otimes v_2)$. The Cartan product is associative. Indeed, if V_i, $i =, 1, 2, 3$ are three irreducible representations and v_i, $i = 1, 2, 3$ are the corresponding highest weight vectors, then both $(V_1 \boxtimes V_2) \boxtimes V_3$ and $V_1 \boxtimes (V_2 \boxtimes V_3)$ have $v_1 \otimes v_2 \otimes v_3$ as their highest weight vector. It implies associativity of the product of vectors.

If $v \in S_A$ is a highest weight vector, then $v \otimes \ldots \otimes v \in S_A \otimes \ldots \otimes S_E$ is a highest weight vector of an irreducible piece of the product. This piece is characterised by its highest weight $p \cdot \alpha = (\frac{p}{2}, \ldots, \frac{p}{2})$ and will be denoted by $S_{\{A\ldots E\}}$ or shortly $(S_A)^p$; it is usually called the Cartan p-th power of S_A. The orthogonal projection (with respect to the Killing form) from $S_{A\ldots E}$ to $S_{\{A\ldots E\}}$ will be denoted again by π^{Car} and indicated by braces $\{\}$ around the corresponding set of indices. Using the projection, we can define the Cartan product of any two elements $s_{A\ldots E} \in$

$S_{\{A\ldots E\}}, t_{F\ldots H} \in S_{\{F\ldots H\}}$ by

$$s_{A\ldots E} \boxtimes t_{F\ldots H} = s_{\{A\ldots E}t_{F\ldots H\}} = \pi^{Car}(s_{A\ldots E} \otimes s_{F\ldots H}).$$

Due to the fact that the Cartan power is a subspace of the symmetric power, it is clear that the multiplication

$$\boxtimes : (S_A)^p \times (S_A)^q \mapsto (S_A)^{p+q}$$

is commutative. This property is playing an important role in the discussion below. The same notation will can used in the primed version.

3. Conformally Invariant First Order Equations.

An important feature of the (massless) wave and Dirac equations is that they are conformally invariant. We are looking for their higher spin analogues and it is natural to require conformal invariance for these generalizations as well. There is a classification of all conformally invariant first order differential equations (see (Fegan , 1975)). We are going to consider equations for fields with values in the p-th Cartan power $(S_A)^p$ of the basic spinor representation (there is the corresponding primed case in even dimensions with the same theory). In this cases, there are two first order conformally invariant equations available. They are constructed as follows (we are going to treat only even-dimensional case, the odd-dimensional case means just to forget all primes).

Definition 3.1 *Let us consider the product $(C^n)_A^{A'} \otimes (S_A)^p$ of irreducible representations. Its decomposition into two irreducible parts is*

$$(C^n)_A^{A'} \otimes (S_A)^p \cong E_1 \oplus E_2, \tag{1}$$

where E_1, resp. E_2 are characterized by their highest weights $\beta_1 = (\frac{p}{2}+1, \frac{p}{2}, \ldots, \frac{p}{2})$, resp. $\beta_2 = (\frac{p}{2}, \ldots, \frac{p}{2}, \frac{p}{2}-1)$. Let us consider fields f on R^n with values in $(S_A)^p$. The de Rham differential d is applied to such a field componentwise. If π_1, resp. π_2 denote the corresponding projections onto parts with heighest weights β_1 and β_2, then we have the following two equations for f :

$$\pi_1(d\,f) = 0 \tag{2}$$

and

$$\pi_2(d\,f) = 0. \tag{3}$$

The first one is called the twistor equation and the second one the massless field equation for spin $\frac{p}{2}$.

The massless field equations and the twistor equations are defined here on a flat space but they are well defined, using Levi-Civita covariant derivative, on spin manifolds, see e.g. (Souček , 1986).

The coordinate description and basic properties of their solutions will be given in the paper. The equations are conformally invariant under the assumption that a suitable conformal weight for spinor fields, uniquely determined and different in each case, is considered but conformal properties of solutions will not be discussed here. This is a nice subject and it needs a more space than available here; it is one of places where the behaviour of individual cases for different spins is different so that information available from the standard case is not immediately applicable.

4. A Coordinate Description

To write down a coordinate form of these two types of equations is a delicate and a subtle task. An explicite decomposition proved in spin $\frac{1}{2}$ case in (Bureš, Souček , 1986) has to be used in the considered higher spin case. First it is necessary to understand the decomposition of the spinor spaces $S_{A\{B...E\}}$ and $S_{A'\{B...E\}}$ into irreducible components.

Lemma 4.1 *Let $n = 2k$ or $n = 2k+1$ and let us fix $p \in Z, p > 1$. The weights $\mu_j = (\frac{p}{2}, \ldots, \frac{p}{2}, \frac{p-2}{2}, \ldots, \frac{p-2}{2})$ with $k-j$ components equal to $\frac{p}{2}$ are dominant, the irreducible representations with highest weights μ_j are denoted by $F_j = F(\mu_j)$. Then the irreducible components of $S_{A\{B...E\}}$ and $S_{A'\{B...E\}}$ are characterized by:*

i) the case $n = 2k$:

$$S_{A\{B...E\}} \cong F_0 \oplus F_2 \oplus \ldots,$$

(the sum ends with F_k or F_{k-1}) and

$$S_{A'\{B...E\}} \cong F_1 \oplus F_3 \oplus \ldots,$$

(the sum ends with F_k or F_{k-1}).

ii) the case $n = 2k+1$:

$$S_{A\{B...E\}} \cong F_0 \oplus F_1 \oplus \ldots \oplus F_k.$$

Let $e \in R^n$ be a unit vector. In even dimension, the map e from $S_{A\{B...E\}}$ to $S_{A'\{B...E\}}$ given by $s_{A...E} \mapsto e^A_{A'} s_{AB...E}$ is an isomorphism and the inverse map $\tilde{e}$ is given by $s_{A'B...E} \mapsto -e^{A'}_A s_{A'B...E}$. In odd dimensions, the same is true when primed indices are substituted by unprimed ones.

The restriction of e (resp. of $\tilde{e}$) to F_j has values in F_{j+1} and F_{j-1} (where $F_{-1} = F_{k+1} = \{0\}$).

The following theorem is the key theorem showing an explicit form of the decomposition of the tensor product in the Def.3.1. It makes possible to write down the corresponding projections explicitely.

Theorem 4.1 *Let $e_1, \ldots, e_n$ be an orthonormal basis in R^n and let $\epsilon_1, \ldots, \epsilon_n$ be the dual basis. Then the components E_i in the decomposition (1) are given by (in odd dimensions all primes should be forgotten)*

$$E_1 = \left\{ \sum_j \epsilon_j \otimes (e_j)_{\{A}^{A'} s_{A'B...E\}} \, | s_{A'B...E} \in F_1 \right\}$$

and

$$E_2 = \Big\{ \sum_j \epsilon_j \otimes (s_j)_{AB\dots E} \,\big|\, (s_j)_{AB\dots E} \in F_0 \text{ such that there exist } (s'_j)_{AB\dots E} \in F_2 \text{ with } \sum_j (e_j)^A_{A'} [(s_j)_{AB\dots E} - (s'_j)_{AB\dots E}] = 0. \Big\}$$

The corresponding projections are given by

$$P_1\left(\sum_j \epsilon_j \otimes (s_j)_{A\dots E}\right) = \frac{1}{n}\sum_j \epsilon_j \otimes (e_j)_{\{A}^{A'}\left(\sum_k (e_k)^F_{A'}(s_k)_{FB\dots E\}}\right),$$

$$P_2\left(\sum_j \epsilon_j \otimes (s_j)_{A\dots E}\right) = \sum_j \epsilon_j \otimes \left[(s_j)_{A\dots E} + \frac{1}{n}(e_j)_{\{A}^{A'}\left(\sum_k (e_k)^F_{A'}(s_k)_{FB\dots E\}}\right)\right].$$

Using Theorem 4.1., an explicit form of the twistor and massless field equations can be deduced.

Theorem 4.2 *The massless field equation (3) for a spinor field $f_{A\dots E}$ with values in $(S_A)^n$ is given by*

$$(Df)_{A'B\dots E} = \sum_j (e_j)^A_{A'} \frac{\partial f_{AB\dots E}}{\partial x_j} = 0 \tag{4}$$

and the twistor equation (2) is equivalent to the set of equations ($j = 1, \dots, n$)

$$\frac{\partial f_{AB\dots E}}{\partial x_j} = \frac{1}{n}(e_j)_{\{A}^{A'}\left(D(f)_{A'B\dots E\}}\right). \tag{5}$$

5. Massless field equations

It is easy to see that the spin $\frac{1}{2}$ case reduces back to the Dirac equation

$$D^A_{A'} f_A = \sum_{i=1}^n (e^i)^A_{A'} \frac{\partial f_A}{\partial x_i} = 0 \tag{6}$$

studied in Clifford analysis (see e.g. (Bureš, Souček , 1986)). Note that the contraction in the index A substitutes the usual Clifford multiplication.

To relate and to apply the wealth of results available in Clifford analysis to solutions of massless field equation with higher spin, it is important to understand the relation between the massless field equation and the so called twisted Dirac equation. By that we mean the equation

$$D(f)_{A'B\dots E} = \sum_j (e_j)^A_{A'} \frac{\partial f_{AB\dots E}}{\partial x_j} = 0 \tag{7}$$

for fields with values in $S_A \otimes (S_A)^{p-1}$. This system of equations decomposes into many copies of the standard Dirac equation, the term $(S_A)^{p-1}$ in the tensor product plays only auxiliary role and can be treated componentwise. The twisted Dirac equation is an elliptic system of equations, while the massless field equation is (a slightly) overdetermined system of equations. The form of the massless field equation shown in Theorem 4.2 confirms the important fact that solutionsof the massless field equations are special solutions of the twisted Dirac equation. constrained by the additional requirement that values of the field belong to $S^p \subset S_A \otimes S^{p-1}$. Let us state it explicitely.

Theorem 5.1 *The twisted massless field equation (7) restricted to the space of spinor fields with values in S^p coincides with the massless field equation (3).*

It is clear that Theorem 5.1 is a key information making possible to apply many results already known in Clifford analysis to higher spin case. We are going to show how the basic facts such as Cauchy theorem, Cauchy integral formula, Laurent series and Residue Theorem can be formulated for higher spins. We are going to formulate theorems only for even-dimensional case, small changes needed for the odd-dimensional case are easy to make.

In our notation, the basic $(n-1)$-form $d\sigma$ of Clifford analysis looks like

$$(d\sigma)^A_{A'} = \sum(-1)^{j+1}(e_j)^A_{A'}dx_1 \wedge \ldots \wedge d\widehat{x_j} \wedge \ldots \wedge dx_n. \tag{8}$$

The Cauchy theorem is an example where the classical result cannot be applied directly, but the computation goes in the same way as in the classical case.

Theorem 5.2 (Cauchy) *Let $f_{A\ldots E}$ be a function on a domain $\Omega \subset R^n$ with values in $(S_A)^p$. Then the form*

$$\omega = (d\sigma)^A_{A'} f_{A\ldots E} \tag{9}$$

is closed on Ω iff f is a solution of the massless field equation (3).

To formulate the Cauchy integral formula, let A_n denote the area of the unit sphere in R^n.

Theorem 5.3 (Cauchy integral formula) *Let $f_{A\ldots E}$ be a solution of the massless field equation (3) on a domain Ω and let $\Omega' \subset\subset \Omega$ be a relatively compact subdomain with a smooth boundary. Then for each point $x \in \Omega'$,*

$$f_{A\ldots E}(x) = \frac{1}{A_n}\int_{\partial\Omega'} \frac{(y-x)_{\{A}^{A'}}{|y-x|^n}(d\sigma_y)^F_{A'} f_{FB\ldots E\}}(y). \tag{10}$$

Cauchy integral formula is an example of a theorem which can be deduced by applying the theorem known in spin $\frac{1}{2}$ case (we can write the standard formula for fields twisted by S^{p-1} and to apply the projection π^{Car} to the both sides of the equation.

Weierstrass theorem or Mean Value Theorem, for example, hold without a change and they are immediate consequences of the standard version applied componentwise.

As for Taylor and Laurent series, let us recall the standard sets of monogenic functions $V_\alpha(x), |\alpha| = k$ having values in the even part of the Clifford algebra (see

(Delanghe *et al.* , 1992)). In spinor notation, they belong either to S_A^B or to $S_{A'}^{B'}$ They form a basis for the space of inner spherical monogenics of degree k.

Similarly, the functions W_α, $|\alpha| = k$ form a basis for outer spherical monogenics of degree k. They are vector valued, so they will be interpreted as functions with values in $S_A^{A'} \oplus S_{A'}^A$.

Theorem 5.4 (Laurent expansion) *Let a field $f_{A\ldots E}$ with values in $(S_A)^p$ be a solution of the spin $\frac{p}{2}$ massless field equation (3) in an annular domain $\Omega \subset R^n$ Then*

$$f_{A\ldots E}(x) = \sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k}(V_\alpha)_{\{A}^F(x)(\mu_\alpha)_{FB\ldots E\}}\right) + \sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k}(W_\alpha)_{\{A}^{A'}(x)(\mu_\alpha)_{A'B\ldots E\}}\right)$$

the convergence being normal on each closed annular subdomain. The coefficients in the expansion are given by

$$(\mu_\alpha)_{FB\ldots E} = \int_{\partial B}(W_\alpha(x))_F^{A'}(d\sigma_x)_{A'}^G f_{GB\ldots E}(x),$$

$$(\mu_\alpha)_{A'B\ldots E} = \int_{\partial B}(V_\alpha(x))_{A'}^{B'}(d\sigma_x)_{B'}^F f_{FB\ldots E}(x),$$

where the ball B is any concentric ball with its boundary inside Ω.

With Laurent series at our disposal, we can define the residue for a solution with a pointwise singularity and to prove the Residue Theorem.

Definition 5.1 *Let a field $f_{A\ldots E}$ with values in $(S_A)^p$ be a solution of the spin $\frac{p}{2}$ massless field equation (3) in $B(x)\setminus\{x\}$, where $B(x)$ is a ball with the center in x*

Then the first coefficient $(\mu_\alpha)_{A'B\ldots E}$ of the negative part of the Laurent series is called the residue of f at the point x and denoted by $res_x(f)$.

Theorem 5.5 *Let $\Omega' \subset\subset \Omega$ be a relatively compact subdomain with a smooth boundary (oriented by its outer normal) and let $\{x_i\}_{i\in I}$ be a finite set of points in Ω.*

Then for every solution $f_{A\ldots E}$ of (3) in $\Omega \setminus (\cup_{i\in I})$ we have

$$\int_{\partial\Omega'}(d\sigma)_{A'}^A f_{A\ldots E} = \sum_{i\in I} res_{x_i}(f).$$

So we have shown how the basic amount of standard function theory can be extended to higher spin cases. It needs clearly a cartain amount of work to go through Clifford analysis and to try to extend its results to higher spins, it is not always straightforward (e.g. it is not easy to implement the standard proof of the Cauchy-Kowalewski theorem). Note also that ideas used for the proof of generalized integral formulae in (Bureš, Souček , 1985) are applicable in the case of massless field equations.

6. Twistor equations

Let us recall (see Sect.2) that the Cartan product of irreducible representations is associative and that the Cartan multiplication between Cartan powers of the spinor representations $\boxtimes : (S_A)^k \times (S_A)^l \mapsto (S_A)^{k+l}$ is commutative. Let $(S_A)^0$ be defined as the space of constants C. Then the sum $(S_A)^\infty = \oplus_{k=0}^\infty (S_A)^k$ (only finite number of terms nontrivial) is an infinite-dimensional commutative algebra.

We propose now to consider the twistor equation for functions with values in the space $(S_A)^\infty$.

Definition 6.1 *A function $f = \sum_{j=0}^\infty f_j, f_j \in (S_A)^j$ on a domain in R^n with values in $(S_A)^\infty$ is called a solution of the twistor equation (2), if $T(f) = 0$, i.e. if $T(f_j) = 0$ for all $j = 1, 2, \ldots$.*

The space of solutions of the twistor equations introduced above is an analogue of the space of all polynomials in complex function theory. Suitable completions of the space can be, of course, considered. A main and remarkable feature of this generalization of holomorphicity to higher dimensions is the fact that if the product used is the Cartan product, then the space of solutions is a commutative algebra.

Without going into a more detail study of the properties of solutions of the twistor equation, we need not use spinor algebra notation, so that formulas are kept simple and understandable.

Theorem 6.1 *If $f_j \in (S_A)^j$ and $g_k \in (S_A)^k$; $j, k = 0, 1, \ldots$ are two solutions of the twistor equation, then $f_j \boxtimes g_k = g_k \boxtimes f_j$ is again a solution of the twistor equation. Hence the space of all solutions of the twistor equation with values in V^∞ is a commutative algebra.*

To see why it is so, let $e_1, \ldots, e_n$ be an orthonormal basis, $(x_1, \ldots, x_n)$ the corresponding coordinates and $\epsilon_1, \ldots, \epsilon_n$ the dual basis, then

$$\begin{aligned} T(f_j \boxtimes g_k) &= \pi^{Car}(d(f_j \boxtimes g_k) = \pi^{Car}\left(\sum_i \epsilon_i \otimes \frac{\partial (f_j \boxtimes g_k)}{\partial x_i}\right) \\ &= \left[\sum_i (\epsilon_i \boxtimes \frac{\partial f_j}{\partial x_i}) \boxtimes g_k + \sum_i (\epsilon_i \boxtimes (\frac{\partial g}{\partial x_i}) \boxtimes f\right] \\ &= T(f_j) \boxtimes g + T(g_k) \boxtimes f = 0 \end{aligned}$$

Few remarks are in order at this place. History of Clifford analysis started in the 30's with works of Fueter and his coworkers studying analogues of Cauchy -Riemann equations for quaternion functions. At that time and many times independently after, it was found that a natural generalization of holomorphicity in quaternionic case – namely the requirement of differentiability in quaternionic sense – is too strong condition; all differentiable functions in this sense are linear. It is worth to point out that the differentiability condition can be expressed using a set of partial differential equations similar to Cauchy-Riemann equations. This time, however, the set of equations is highly overdetermined and, as a consequence, the space of solutions is too small. In the quaternionic case, it is possible to identify functions with quaternionic values with spin $\frac{1}{2}$ fields (Fueter equation than coincides with the Dirac equation). It was pointed out some time ago ((Souček , 1986)) that (using this identification), the mentioned analogue of Cauchy-Riemann equations, describing differentiability in quaternionic sense, concides with the twistor equation for spin $\frac{1}{2}$ fields. Hence in dimension 4, we see that quaternionic differentiability, if considered for all possible spins, leads to a rich family of solutions and that the space of solutions has nice multiplicative properties. It brings back a question whether the twistor equation, considered for all spins, could not be a suitable generalization of Cauchy-Riemann equations to higher dimensions.

Another result proved in dimension 4 (see (Hitchin, 1980)) brings a new light to the discussion. It is shown in the paper that the space of solutions of the twistor equation of a given spin is the image of the 0-th order cohomology group on the twistor space of a certain homogeneity. The sum of these cohomology groups with different homogeneity forms clearly a commutative ring, so the algebra structure of the space of solutions of the twistor equation for all spins is translated by the Penrose transform to the commutative structure of polynomials in several complex variables.

There are many others interesting interlations among solutions of massless field equations, twistor equations and other conformally invariant equations which are well understood in dimension 4 (see (Penrose, Rindler , 1984)), it would be valuable to understand their generalization to higher dimensions.

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