PREMIÉRE PARTIE

MATHÉMATIQUES

Présentation de P. Lelong

Les articles qu'on trouvera dans ce livre développent des exposés faits lors du Colloque de Géométrie Complexe tenu à Paris à l'Université Paris 7 à l'occasion du premier Congrès de l'Union Mathématique Européenne en Juillet 1992. On trouvera dans le Sommaire la liste complète des auteurs et le titre de leurs communications, ainsi que la liste et le titre des conférences qui n'ont pas été rédigées pour ce volume. Plusieurs des exposés concernent des problèmes qui intéressent les physiciens en même temps que les mathématiciens et la plupart portent sur des problèmes de Géométrie Complexe.

L'article de *J. Bureš* et *V. Souček* étudie la transformation de Penrose en dimensions paires et ses applications aux équations de Laplace et de Dirac complexes.

L'article de *F. Campana* donne des propriétés des variétés projectives complexes connexes dites de Fano en liaison avec des problèmes de connexité rationnelle.

Dans leur étude des résidus de formes méromorphes, *A. Dickenstein* et *C. Sessa* développent la notion de cohomologie modérée et précisent les propriétés des courants résiduels.

R. Gérard développe des résultats obtenus en collaboration avec T. Tahara et donne des conditions générales qui assurent que le problème de Cauchy pour un système d'équations aux dérivées partielles a une solution analytique lorsqu'existe une solution formelle.

The inverse Penrose transform for the Dirac and complex Laplace equations

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Résumé

Dans un article précédent [8], nous avons décrit la transformation de Penrose en dimension paire pour les champs de masse nulle généralisés. Les outils essentiels sont d'une part, l'isomorphisme de Dolbeault pour la d''-cohomologie, les formules intégrales de l'analyse de Clifford et d'autre part, la théorie des résidus de Leray ; en outre, il contient une formule simple et explicite de la transformation de Penrose inverse. Le but du présent article est de présenter de nombreux exemples de cette transformation et de son inverse en dimensions 4 et 6. Dans ce cadre, on donne des formules explicites de formes sur l'espace des twisteurs et les solutions correspondantes pour les états élémentaires dans l'espace de Minkowski.

1. Introduction.

In this paper we study the Penrose transform in even dimensions for complex Laplace and Dirac equations. This is a generalization of the Penrose transform for massless fields in dimension 4 (see [10],[11],[13],[16],[17]) and it fits into the general scheme of generalized Radon transform introduced by Helgason (see [12]). Such a generalization was studied first by R. Baston and M. Eastwood (see [1],[2]), compare also [14]). Their approach is based on an extended use of sheaf cohomology, resolutions, spectral sequences and

This paper is in final form and no version of it will be submitted for publication elsewhere.

the fundamental facts used are Bott-Borel-Weil theorem and Bernstein-Gelfand-Gelfand resolution. Another point of view was used in [7],[8], where the Penrose transform was presented from analysist's point of view. The Penrose transform is given in terms of differential forms and Dolbeault cohomology, main tools for the description of the inverse Penrose transform are coming from Clifford analysis ([9]). The advantage of the second approach is that it leads to a quite explicit inversion formula. For a given solution of the considered equation, a suitable $\overline{\partial}$-closed form representing the Dolbeault cohomology class which is mapped to the solution by the Penrose transform is given by a simple explicit formula.

The aim of this paper is to discuss the inversion formula in the cases of lower dimensions 4 and 6 with necessary details. We shall consider so called elementary states on $\mathbf{C}^{2n}$ (i.e. polynomial solutions of the equations). In Sections 2 - 5 we summarize basic facts on the Penrose transform, more details and proofs can be found in [7],[8]. In the last Section, we introduce local nonhomogeneous coordinates and we give explicit formulas for the inverse transform in dimensions 4 and 6.

2. Dirac and complex Laplace equation on $\mathbf{C}^{2n}$.

Let $\{P_1, \dots, P_{2n}\}$ be coordinates in $\mathbf{C}^{2n}$, let $\hat{}$ be an antilinear involution

$$P = [P', P''] \in \mathbf{C}^{2n} \mapsto \hat{P} = [\bar{P}'', \bar{P}']; \quad P', P'' \in \mathbf{C}^n$$

and let

$$Q(P, Q) = \sum_{i=1}^{n} (P_i Q_{n+i} + P_{n+i} Q_i)$$

be a nondegenerate symmetric bilinear form on $\mathbf{C}^{2n}$.

Let us denote by

$$\{f_1, \dots, f_n, \hat{f}_1, \dots, \hat{f}_n\}$$

the standard (coordinate) basis of $\mathbf{C}^{2n}$ and

$$W_0 = \text{ span } \{f_1, \dots, f_n\}, \hat{W}_0 = \text{ span } \{\hat{f}_1, \dots, \hat{f}_n\}$$

subspaces of $\mathbf{C}^{2n}$.

Then

$$\mathbf{C}^{2n} = W_0 \oplus \hat{W}_0$$

and the involution $\hat{}$ on $\mathbf{C}^{2n}$ maps W_0 onto $\hat{W}_0$. Let us embed the Euclidean space $\mathbf{R}^{2n}$ into $\mathbf{C}^{2n}$ as the set of fixed points of the complex conjugation $\hat{}$. Let us denote by $\mathcal{C}^C_{2n}$ the complex Clifford algebra of $(\mathbf{C}^{2n}, \mathcal{Q})$. We suppose that $\mathbf{C}^{2n} \subset \mathcal{C}^C_{2n}$ and $\mathrm{Spin}(2n) \subset \mathcal{C}^C_{2n}$.

Let S be the irreducible $\mathcal{C}^C_{2n}$ - module and let $\mathrm{S} = \mathrm{S}^+ \oplus \mathrm{S}^-$ be the decomposition of S into irreducible Spin$(2n)$-modules (called the basic Spin-modules). The $\mathcal{C}^C_{2n}$ - module S can be embedded into Clifford algebra $\mathcal{C}^C_{2n}$ as a maximal left ideal (see [9]) in the following way:

$$\mathrm{S} = \mathcal{C}^C_{2n}.I = \Lambda^*(W_0).I,$$

where $I = f_1\hat{f}_1 ... f_n\hat{f}_n$ is an idempotent, and $\Lambda^*(W_0) \simeq \mathcal{C}(W_0, 0)$(as algebras) is the Clifford algebra of W_0 corresponding to the trivial (zero) bilinear form. If Φ is a holomorphic function on $\mathbf{C}^{2n}$, then the complex Laplace equation is given by

$$\sum_{i=1}^{n} \frac{\partial^2\Phi}{\partial P_i \partial P_{n+i}} = 0. \tag{1}$$

If Φ is a holomorphic spinor field on $\mathbf{C}^{2n}$ with values in S, the Dirac equation is given by

$$D\Phi = \sum_{i=1}^{n} (\hat{f}_i . \frac{\partial \Phi}{\partial P_i} + f_i . \frac{\partial \Phi}{\partial P_{n+i}}). \tag{2}$$

3. The twistor diagram

The Penrose transform is a special case of generalized Radon transforms on homogeneous spaces. Let $\mathcal{Q}$ be a nondegenerate symmetric bilinear form on $\mathbf{C}^{2n+2}$. The isotropic Grassmannian $IG_k \equiv IG_{k;2n+2}$ is the subspace of the Grassmannian G_k consisting of all isotropic subspaces L of dimension k in $\mathbf{C}^{2n+2}$ (i.e. $\mathcal{Q}|_L \equiv 0$). Similarly, the isotropic flag manifold $IG_{1,n+1}$ is a subspace of the ordinary flag manifold of couples of 1-dimensional and $(n+1)$-dimensional isotropic subspaces. The usual double diagram (with natural projections) is then given by two projections

$$\begin{tikzcd} IG_{1,n+1} \arrow[r, "\nu"] \arrow[d, "\mu"] & IG_1 \\ IG_{n+1} & \end{tikzcd} \tag{3}$$

The space IG_1 is a nondegenerate quadric in $\mathbf{P}^{2n+1}(\mathbf{C})$, so $M = \mathbf{C}^{2n}$ can be embedded into it as an open dense subset. This subset is usually considered as a complexification of the $2n$-dimensional Minkowski space by physicists. To describe solutions of complex Laplace and Dirac equation on a domain $\Omega \subset M$, we restrict the diagram given by the maps (3) to

$$\begin{array}{ccc} \Omega' & \xrightarrow{\nu} & \Omega \\ \downarrow{\scriptstyle\mu} & & \\ \Omega'' & & \end{array} \tag{4}$$

where $\Omega' = \nu^{-1}(\Omega) \subset IG_{1,n+1}$ and $\Omega'' = \mu(\Omega') \subset IG_{n+1}$.

The isotropic Grassmanian $IG_n \equiv IG_{n;2n}$ can be defined as a homogeneous space by the following ways:

$$IG_n \equiv SO(2n, \mathbf{C})/H \equiv \mathrm{Spin}_0(2n, \mathbf{C})\ /\tilde{H} \equiv SO(2n, \mathbf{R})/U(n, \mathbf{C}).$$

The groups $SO(2n, \mathbf{C}), SO(2n, \mathbf{R})$ and $U(n, \mathbf{C})$ are defined with respect to our choice of quadratic form $\mathcal{Q}$ in Sect.2, the group $\mathrm{Spin}_0(2n, \mathbf{C})$ is the connected identity component of the group $\mathrm{Spin}(2n, \mathbf{C})$; the subgroups H and $\tilde{H}$ are the isotropic subgroups of the point $W_0 \in IG_n$.

Let ISt_n be the isotropic Stiefel manifold of isotropic n-frames in $\mathbf{C}^{2n}$. Then we have the diagram:

$$\begin{array}{ccc} \mathrm{Spin}_0(2n, \mathbf{C}) & \xrightarrow{\tilde{H}} & IG_n \\ \downarrow{\scriptstyle 2:1} & & \downarrow{\scriptstyle =} \\ \mathrm{SO}(2n, \mathbf{C}) & \xrightarrow{H} & IG_n \\ \downarrow & & \downarrow{\scriptstyle =} \\ ISt_n & \xrightarrow{GL} & IG_n \end{array}$$

We shall use the spaces $\mathrm{Spin}_0(2n, \mathbf{C})$ and ISt_n as spaces of "homogeneous coordinates" on IG_n. Let us denote for any $s \in \mathbf{S}^+ - \{0\}$:

$$\iota(s) = \{Z \in \mathbf{C}^{2n}; Z.s = 0\}.$$

The space

$$\mathbf{S}_{pure} = \{s \in \mathbf{S}^+; \dim \iota(s) = n\}$$

is called the space of (positive) pure spinors. For any $s \in \mathbf{S}^+ - \{0\}$ the vector space $\iota(s)$ is always isotropic. The map

$$\tilde{\iota} : \mathbf{P}(\mathbf{S}_{pure}) \to IG_n$$

given by ι is a biholomorphic map.

Let us define a map $\mathcal{S} : \mathrm{Spin}_0(2n, \mathbf{C}) \to \mathbf{S}^+$ by

$$\mathcal{S}(g) = gI. \tag{5}$$

With respect to the decomposition $\mathbf{C}^{2n} = W_0 \oplus \hat{W}_0$, the group $H \subset SO(2n, \mathbf{C})$ has the following form

$$H = \left\{ h = \begin{pmatrix} A & B \\ 0 & (A^{-1})^t \end{pmatrix} \middle| AB^t + BA^t = 0,\ A \in GL(n, \mathbf{C}) \right\}$$

and the map

$$\det : H \to \mathbf{C} - \{0\};\ \det(h) = \det A,$$

where $\det A$ denotes the determinant of A, is a holomorphic representation of H on $\mathbf{C}$. We can find a holomorphic representation $\sqrt{\det}$ of $\tilde{H}$ on $\mathbf{C}$ such that $(\sqrt{\det})^2 = \det$. The principal role in the considerations is played by the line bundle $\mathbf{L}$ on IG_n defined as a homogeneous bundle given by representation $(\sqrt{det})^{-1}$ of the group $\tilde{H}$.

Denote by κ_n an $\mathrm{SO}(2n, \mathbf{R})$-invariant volume form on IG_n, this is $(\binom{n}{2}, \binom{n}{2})$-form on IG_n. Then there exists a form $\alpha_n \in \mathcal{E}^{(\binom{n}{2},0)}(IG_n, \mathbf{L}^{2n-2})$ which is a holomorphic $\mathrm{SO}(2n, \mathbf{R})$-invariant form on IG_n such that

$$\kappa_n = \alpha_n \wedge \frac{\bar{\alpha}_n}{(\sum_I |M_I|^2)^{n-1}},$$

where M_I are the Plücker coordinates on IG_n, $I \subset \{1, \dots, 2n\}, \#I = n$.

The form $\bar{\alpha}_n$ is an element of $\mathcal{E}^{(0,\binom{n}{2})}(IG_n, \bar{\mathbf{L}}^{2n-2})$, and the expression

$$((\sum_I |M_I|^2)^{n-1})^{-1}$$

is an element of $\mathcal{E}^{(0,0)}(IG_n, (\mathbf{L} \otimes \bar{\mathbf{L}})^{2-2n})$. We shall use below the realization of elements of the cohomology group $H^{(0,\binom{n}{2})}(\Omega, \mathbf{L}^k)$ by the Dolbeault cohomology.

3.1 The Penrose transform for complex Laplace equation.

Let us denote by $\mathcal{HW}(\Omega)$ the space of all holomorphic solutions of the complex Laplace equation on Ω.

We want to construct the map

$$\mathcal{P} : H^{(0,\binom{n}{2})}(\Omega'', \mathbf{L}^{2-2n}) \to \mathcal{HW}(\Omega)$$

called the Penrose transform for complex Laplace equation in Ω.

THEOREM 1.

a) Let Ω be a domain in $\mathbf{C}^{2n}$. If a $\bar{\partial}$-closed form β represents a cohomology class $[\beta] \in H^{(0,\binom{n}{2})}(\Omega'', \mathbf{L}^{2-2n})$, then the function

$$f(P) \equiv \mathcal{P}([\beta])(P) = \int_{\nu^{-1}(P)} \mu^* \beta \wedge \alpha_n$$

depends only on the class $[\beta]$.

b) The Penrose transform $f(P) = \mathcal{P}([\beta])(P)$ is a holomorphic function on the domain Ω and it satisfies the complex Laplace equation (1) there.

c) If Ω satisfies suitable topological conditions (see [2],[5],[7],[9]), the map $\mathcal{P}$ is one-to-one map onto $\mathcal{HW}(\Omega)$.

3.2 The Penrose transform for Dirac equation.

Let us denote the set of all holomorphic solutions of the Dirac equation on Ω by $\mathcal{HS}(\Omega)$. We want to construct a map

$$\mathcal{P} : H^{\binom{n}{2}}(\Omega'', \mathbf{L}^{1-2n}) \to \mathcal{HS}(\Omega)$$

called the Penrose transform for the Dirac equation in Ω.

The double fibration (5) will be lifted for our purposes to the diagram given by the double fibration

$$\begin{array}{ccc} \widetilde{\Omega'} & \xrightarrow{\nu} & \Omega \\ \big\downarrow{\scriptstyle \mu} & & \\ \widetilde{\Omega''} & & \end{array} \tag{6}$$

where

$$\widetilde{\Omega'} = \tilde{\pi}^{-1}(\Omega') \subset \mathbf{C}^{2n} \times \mathrm{Spin}_0(2n, \mathbf{C})$$

is a principal fibre bundle with the group $\widetilde{H}$ over Ω' and the domain

$$\widetilde{\Omega''} = \tilde{\pi}^{-1}(\Omega'') \subset \mathbf{C}^{n} \times \mathrm{Spin}_0(2n, \mathbf{C})$$

is a principal fibre bundle with the same group over Ω''.

The Penrose transform is defined for cohomology classes in the cohomology group $H^{(0,\binom{n}{2})}(\Omega'', \mathbf{L}^{1-2n})$. To compute with them, we need to lift them to the domain $\widetilde{\Omega''}$. The

$\bar{\partial}$-closed form on Ω' with values in powers of $\mathbf{L}$ will be represented by an ordinary $\bar{\partial}$-closed form on $\widetilde{\Omega}'$ satisfying suitable homogeneity conditions.

To define the Penrose transform, we shall need the map $\mathcal{S}(g)$ introduced in Sect.2. Let $\mathbf{S}^+$ denote the trivial bundle with fiber $\mathbf{S}^+$ on Ω'.

THEOREM 2.

a) Let Ω be a domain in $\mathbf{C}^{2n}$. If a $\overline{\partial}$-closed form β represents a cohomology class $[\beta] \in H^{\binom{n}{2}}(\Omega'', \mathbf{L}^{1-2n})$, then the form

$$\mathcal{S}\tilde{\mu}^*\beta \wedge \alpha_n$$

represents a well defined element in $\mathcal{E}^{n(n-1)}(\Omega', \mathbf{S}^+)$ and the Penrose transform $\mathcal{P}([\beta])$ defined by

$$f(P) \equiv \mathcal{P}([\beta])(P) = \int_{\nu^{-1}(P)} \mathcal{S}\tilde{\mu}^*\beta \wedge \alpha_n.$$

is a spinor field on Ω which depends only on cohomology class $[\beta]$.

b) The Penrose transform $f(P) = \mathcal{P}([\beta])(P)$ is a holomorphic spinor field on the domain Ω and it satisfies the Dirac equation (2) there.

c) If Ω satisfies suitable topological conditions (see e.g. [2],[8],[9]), then the map $\mathcal{P}$ is one-to-one map onto $\mathcal{HS}(\Omega)$.

4. The Penrose transforms on $\mathbf{C}^{2n}$.

In the rest of the article we want study the special but interesting case $\Omega = \mathbf{C}^{2n}$ in more details. The corresponding double diagram is given by the maps

$$\begin{array}{ccc} \mathbf{C}^{2n} \times IG_n & \xrightarrow{\nu} & \mathbf{C}^{2n} \\ \Big\downarrow{\scriptstyle\mu} & & \\ \mu(\mathbf{C}^{2n} \times IG_n) & & \end{array} \tag{7}$$

For simplicity, the scalar product given by $\mathcal{Q}$ is denoted by $(.,.)$.

(i) Local coordinates for $\mathbf{F} := \nu^{-1}(\mathbf{C}^{2n}) \simeq \mathbf{C}^{2n} \times IG_n$.

We shall need often "homogeneous coordinates" on IG_n given by the isotropic Stiefel manifold ISt_n defined as the space

$$\{Z = [Z^1, \ldots, Z^n] | Z^i \in \mathbf{C}^{2n}, \operatorname{rank} Z = n, (Z^i, Z^j) = 0;\, i, j = 1, \ldots, n\}.$$

The manifold ISt_n is a principal fiber bundle $\pi : ISt_n \mapsto IG_n$ with a group $\mathrm{GL}(n, \mathbf{C})$ acting from the right.

The projection π maps a matrix $\mathcal{Z}$ into an isotropic subspace generated by columns of $\mathcal{Z}$. The identification $ISt_n/\mathrm{GL}(n,\mathbf{C}) \simeq IG_n$ is fixed by a choice of a point in IG_n and a choice of its isotropic basis. Our choice is the first half $w_1, \ldots, w_n$ of the canonical basis in $\mathbf{C}^{2n}$ and the isotropic space $W_0 \subset \mathbf{C}^{2n}$ generated by it.

The space IG_n is also a homogenous space of the group $SO(2n, \mathbf{C})$. The homogeneous coordinates on IG_n given by the Stiefel manifold or by $SO(2n, \mathbf{C})$ are sufficient for complex Laplace but not for the Dirac operator. We shall need a bigger set of homogeneous coordinates given by the group $\mathrm{Spin}_0(2n, \mathbf{C})$ acting on IG_n. The identifications

$$\mathrm{Spin}_0(2n, \mathbf{C})\ /\tilde{H} \simeq SO(2n, \mathbf{C})/H \simeq IG_n$$

is given by the choice of W_0 for a preferred element in IG_n.

The projection $p : \mathrm{Spin}_0(2n, \mathbf{C}) \ \mapsto ISt_n$ and its restriction $p : \tilde{H} \mapsto GL(n, \mathbf{C})$ will be denoted by the same symbol. Components $Z^i_j = Z^i_j(g)$ of the map p are holomorphic functions on the group $\mathrm{Spin}_0(2n, \mathbf{C})$, they will be often used below.

Homogeneous coordinates on the space $\mathbf{F} = \mathbf{C}^{2n} \times IG_n$ are given by $(P, \mathcal{Z}) \in \tilde{\mathbf{F}}^o := \mathbf{C}^{2n} \times ISt_n$, resp. by $(P, g) \in \tilde{\mathbf{F}} := \mathbf{C}^{2n} \times \mathrm{Spin}_0(2n, \mathbf{C})$, with the action $(P, \mathcal{Z}) \mapsto (P, \mathcal{Z} \cdot \gamma); \gamma \in GL(n, \mathbf{C})$, resp. $(P, g) \mapsto (P, g \cdot p(\tilde{h})); \tilde{h} \in \tilde{H}$.

(ii) Local coordinates for $\mathbf{T} = \mu(\mathbf{F})$.

Homogeneous coordinates on the twistor space are given by $(\eta, \mathcal{Z})$, where $\mathcal{Z} = (Z^1, \ldots, Z^n), Z^i \in \mathbf{C}^{2n}$ are Stiefel coordinates for the vector space associated with the affine space $W \in \mathbf{T}$ and $\eta \in \mathbf{C}^n$ is given by $\eta^i = (P, Z^i), i = 1, \ldots, n,\ P \in W$. The mapping μ is given in these coordinates by $\mu(P, \mathcal{Z}) = (\eta, \mathcal{Z}),\ \eta^i = (P, Z^i)$. The space $\mathbf{T}_0 := \mathbf{C}^n \times ISt_n$ is a principal fibre bundle over the twistor space $\mathbf{T}$ with the group $GL(n, \mathbf{C})$ acting by $(\eta, \mathcal{Z}) \mapsto (\eta \cdot \gamma, \mathcal{Z} \cdot \gamma), \gamma \in GL(n, \mathbf{C})$. The corresponding projection π is holomorphic. We shall also use a bigger coordinate space $\tilde{\mathbf{T}} := \mathbf{C}^n \times \mathrm{Spin}_0(2n, \mathbf{C})$ for $\mathbf{T}$. It is again a principal fiber bundle with the group $\tilde{H}$ acting by $(\eta, g) \mapsto (\eta \cdot p(\tilde{h}), g\tilde{h})$.

All coordinates are then summarized in the double diagram given by the maps

$$\begin{array}{ccc}
\tilde{\mathbf{F}} \sim (P, g) & \xrightarrow{\tilde{\nu}} & \mathbf{C}^{2n} \sim (P) \\
\Big\downarrow{\scriptstyle \tilde{\mu}: \eta^i = (P, Z^i(g))} & & \\
\tilde{\mathbf{T}} \sim (\eta, g) & &
\end{array}$$

The forms with values in $\mathbf{L}^k$ will be represented by equivariant forms on $\tilde{\mathbf{F}}, \tilde{\mathbf{T}}$, resp. $\tilde{\mathbf{F}}_0, \tilde{\mathbf{T}}_0$. Every element $g \in \mathrm{Spin}_0(2n, \mathbf{C})$ is projected to an element $\{Z^1(g), \ldots, Z^n(g)\} \in ISt_n$; the vectors

$$\{Z^1(g), \ldots, Z^n(g), \hat{Z}^1(g), \ldots, \hat{Z}^n(g)\}$$

form a basis of $\mathbf{C}^{2n}$. Let us denote the dual basis (with respect to $(.,.)$) by

$$\{Y^1, \ldots, Y^n, \hat{Y}^1, \ldots, \hat{Y}^n\}.$$

We have

$$P = \sum_{j=1}^{n}((P, Z^j)Y^j + (P, \hat{Z}^j)\hat{Y}^j).$$

Let us note that span $\{\hat{Y^j}\} =$ span $\{Z^j\} = W$.

5. The inverse Penrose transform

Let f be a solution of the complex Laplace (Dirac) equation on $\mathbf{C}^{2n}$. We are going to describe the cohomology class $[\beta]$ on $\mathbf{T}$ such that $\mathcal{P}([\beta]) = f$.

Even more, we also shall write a simple explicit formula for the inverse Penrose transform giving a (special) representative of the cohomology class. Let us define a constant K(n) by

$$K(n)^{-1} = \int_{\mathbf{C}^N} \frac{\overline{\alpha_n}(\mathcal{Z})}{(\sum_J |M_J(\mathcal{Z})|^2)^{n-1}},\ N = \binom{n}{2}.$$

THEOREM 3. *Let $f(P)$ be a holomorphic solution of the complex Laplace equation (1) in $\mathbf{C}^{2n}$. Let us define a form β on the twistor space $\widetilde{\mathbf{T}}$ by*

$$\beta = K(n).(\sum_{j=0}^{n-1} a_j \nabla^j f).\frac{\overline{\alpha_n}(\mathcal{Z})}{(\sum_J |M_J(\mathcal{Z})|^2)^{n-1}},$$

where $\nabla^j f = \frac{\partial^j}{\partial \tau^j}|_0 f((1+\tau)Q)$, $Q = \sum_1^n \eta_j Y^j$, and

$$a_0 = \frac{1}{2}\frac{(2n-2)!}{(n-1)!}, a_{n-1} = 1.$$

$$a_j = \frac{(2n-2)!}{(n+j-1)!}\left[\binom{n-2}{j-1} + \frac{1}{2}\binom{n-2}{j}\right]; j = 1, ..., n-2.$$

The form β is an equivariant form representing a $\overline{\partial}$-closed element of the space $\mathcal{E}^{(0,\binom{n}{2})}(\mathbf{T},\mathbf{L}^{2-2n})$ and we have

$$\mathcal{P}([\beta]) = f,\ P \in \Omega.$$

THEOREM 4. *Let $f(P)$ be a holomorphic solution of the Dirac equation (2) in $\mathbf{C}^{2n}$. Let us define a form β on the twistor space $\widetilde{\mathbf{T}}$ by*

$$\beta = K(n)\frac{(I\overline{g})}{|g|^2}(\sum_{j=0}^{n-1} a_j \nabla^j f).\frac{\overline{\alpha_n}(g)}{\sum_J |M_J(g)|^2)^{n-1}},$$

where $\nabla^j f = \frac{\partial^j}{\partial \tau^j}|_0 f((1+\tau)Q)$, $Q = \sum_1^n \eta_j Y^j$, and

$$a_0 = \frac{(2n-2)!}{(n-1)!}, a_j = \frac{(2n-2)!}{(n+j-1)!}\binom{n-1}{j}; j = 1,\ldots,n-2,\ a_{n-1} = 1.$$

The form β is an equivariant form representing a $\overline{\partial}$-closed element of the space $\mathcal{E}^{(0,\binom{n}{2})}(\mathbf{T},\mathbf{L}^{1-2n})$ and we have

$$\mathcal{P}([\beta]) = f,\ P \in \Omega.$$

6. Elementary states.

A solution of the Dirac equation (resp. complex Laplace equation) on the whole space $\mathbf{C}^{2n}$ which is polynomial and homogeneous of order k in the coordinates $\{P^i\}$ on $\mathbf{C}^{2n}$, is called an elementary state of order k. The set of elementary states of all orders form a dense set in the space of all holomorphic solutions on $\mathbf{C}^{2n}$. We want now to describe a basis of the space of elementary states of order k in lower dimensions (4 and 6) and to show how the inverse Penrose transform works quite explicitely. All maps involved in Theorems 3 and 4 can be written down explicitely.

EXAMPLE. (Elementary states on $\mathbf{C}^4$.)

We use the following coordinates:

a) On $\mathbf{C}^4$.

$$P = (x^1, x^2, y^1, y^2).$$

b) On $\mathbf{F} = \mathbf{C}^4 \times IG_{2,4}$. We take nonhomogeneous coordinate ξ which corresponds to one of standard coordinate neighborhoods on $IG_{2,4}$. The basis of the corresponding isotropic subspace is given by:

$$\begin{aligned} Z_1 &= (1, 0, 0, -\xi) \\ Z_2 &= (0, 1, \xi, 0) \end{aligned} \tag{8}$$

c) On the twistor space $\mathbf{T} = \mu(\mathbf{F})$, we take the coordinates:

$$(\eta, \xi) = (\eta^1, \eta^2, \xi).$$

The projection μ on $\mathbf{T}$ has the form:

$$\mu(x, y, \xi) = (\eta, \xi)$$

with

$$\begin{aligned} \eta^1 &= y^1 - \xi x^2 \\ \eta^2 &= y^2 + \xi x^1 \end{aligned} \tag{9}$$

The corresponding basis is:

$$\begin{aligned} Z_1 &= (1, 0, 0, -\xi) \\ Z_2 &= (0, 1, \xi, 0) \\ \hat{Z}_1 &= (0, -\bar{\xi}, 1, 0) \\ \hat{Z}_2 &= (\bar{\xi}, 0, 0, 1) \end{aligned} \tag{10}$$

and the dual basis

$$\begin{aligned} Y_1 &= \|\xi\|^{-1}(0, -\bar{\xi}, 1, 0) \\ Y_2 &= \|\xi\|^{-1}(\bar{\xi}, 0, 0, 1) \\ \hat{Y}_1 &= \|\xi\|^{-1}(1, 0, 0, \xi) \\ \hat{Y}_2 &= \|\xi\|^{-1}(0, 1, \xi, 0), \end{aligned}$$

where

$$\|\xi\| = (1 + |\xi|^2).$$

For the inverse transform we need the following coordinate form of $Q(\eta,\xi)$ (Q was defined in Theorem 3 by $Q = \sum_1^n \eta_j Y^j$):

$$\begin{aligned} x^1 &= \|\xi\|^{-1}(\eta^2\bar{\xi}) \\ x^2 &= \|\xi\|^{-1}(-\eta^1\bar{\xi}) \\ y^1 &= \|\xi\|^{-1}\eta^1 \\ y^2 &= \|\xi\|^{-1}(\eta^2) \end{aligned} \tag{11}$$

Note that the all 4 functions are holomorphic in the variable η, as a consequence the forms written below are manifestly $\bar{\partial}$-closed.

A direct computation gives:

$$\sum_I |M_I|^2 = \|\xi\|^2.$$

The spinor space $\mathbf{S}^+$ can be identified with

$$\mathbf{S}^+ = \{(a.1 + b.\hat{f}_1\hat{f}_2)I\,; a, b \in \mathbf{C}\},$$

and the corresponding contraction $I\bar{g}/|g|^2$ is given (up to a multiplicative factor) by $I(1+\bar{\xi}f_1f_2)/\|\xi\|$, where ξ is a nonhomogeneous coordinate of the element in ISt_n given by the projection of g. Note that the multiplicative factors in the formula for the form β express the fact that β has values in a suitable line bundle.

A. The complex Laplace equation.

From the general theory (esp. Theorem 3), we get:

THEOREM 5. *Let $\Phi(x,y)$ be a homogeneous polynomial of degree k on $\mathbf{C}^4$ which is a solution of the complex Laplace equation. Let $\widetilde{\Phi}(\eta,\phi) = \Phi(x(\eta,\xi), y(\eta,\xi))$ be the function on $\mathbf{T}$ obtained by substitution (11).*

Then the form

$$\beta_\Phi = \frac{-i}{2\pi}(1+k)\widetilde{\Phi}\frac{d\bar{\xi}}{(1+|\xi|^2)^2}$$

is manifestly $\bar{\partial}$-closed form on $\mathbf{T}$, so it represents an element

$$[\beta_\Phi] \in H^{(0,1)}(\mathbf{T}, \mathbf{L}^{-2}).$$

Moreover, we have

$$\mathcal{P}([\beta_\Phi]) = \Phi.$$

The fact that the class $[\beta_\Phi]$ is the inverse image of Φ can be in this simple situation verified directly. By a direct computations we get the following formulae:

$$\int_{\mathbf{C}} \frac{|\xi|^{2l} d\bar{\xi} \wedge d\xi}{(1+|\xi|^2)^{k+2}} = 2\pi i \sum_{j=0}^{l} (-1)^j \binom{l}{j} \frac{1}{k+1-l+j}, \; l = 0, \ldots, k. \tag{12}$$

and

$$\int_{\mathbf{C}} \frac{|\xi|^{2k} d\bar{\xi} \wedge d\xi}{(1+|\xi|^2)^{k+2}} = \frac{2i\pi}{k+1}. \tag{13}$$

Let us describe the space in the cohomology group which corresponds to elementary states of order k = 0 and 1. Let

$$\kappa = \frac{-i}{2\pi} \frac{d\bar{\xi}}{(1+|\xi|^2)^2},$$

then we have the following representants of cohomology classes and their Penrose transforms:

Let k = 0.

$$\beta_0 = \kappa \rightarrow \mathcal{P}([\beta_0]) = 1, \tag{14}$$

because it immediately follows from (13) that

$$\mathcal{P}([\beta_0]) = \frac{-i}{2\pi} \int_{\mathbf{C}} \frac{d\bar{\xi} \wedge d\xi}{(1+|\xi|^2)^2} = 1.$$

From the formulas (12), (13), we get also the following results for k = 1.

$$\begin{aligned}
\beta_1 &= \frac{2\eta_2 \bar{\xi}}{1+|\xi|^2} \kappa \rightarrow \mathcal{P}([\beta_1]) = x^1 \\
\beta_2 &= \frac{-2\eta_1 \bar{\xi}}{1+|\xi|^2} \kappa \rightarrow \mathcal{P}([\beta_2]) = x^2 \\
\beta_3 &= \frac{2\eta_1}{1+|\xi|^2} \kappa \rightarrow \mathcal{P}([\beta_3]) = y^1 \\
\beta_4 &= \frac{2\eta_2}{1+|\xi|^2} \kappa \rightarrow \mathcal{P}([\beta_4]) = y^2
\end{aligned} \tag{15}$$

Let us show how the formulas work e.g. for $\Phi(x,y)=x^1$. We get

$$\beta=\frac{-i}{2\pi}2.\frac{\eta_2\bar{\xi}d\bar{\xi}}{(1+|\xi|^2)^3}$$

and

$$\mathcal{P}([\beta])=\frac{-i}{2\pi}2\int_{\mathbf{C}}\frac{x^1|\xi|^2d\bar{\xi}\wedge d\xi}{(1+|\xi|^2)^3}=x^1$$

For general k we shall show as an example the procedure for $\Phi(x,y)=(y^1)^k$. The corresponding form is

$$\beta_k=\frac{-i}{2\pi}(1+k)\frac{(\eta^1)^k d\bar{\xi}}{(1+|\xi|^2)^{2+k}}$$

and its Penrose transform is

$$\mathcal{P}([\beta_k])=\frac{-i}{2\pi}(k+1)\int_{\mathbf{C}}\frac{(y^1)^k d\bar{\xi}\wedge d\xi}{(1+|\xi|^2)^3}=(y^1)^k.$$

We can now state a general result:

Let $\mathcal{H}_k(\eta_1,\eta_2)$ denote the set of all homogeneous polynomials of order k in (η_1,η_2) and let $\mathcal{R}_k$ denote the set of all cohomology classes

$$\mathcal{R}_k=\{[\beta(k,P_0,\dots,P_k)]\,;P_j\in\mathcal{H}_k(\eta_1,\eta_2)\}\subset H^{(0,1)}(\mathbf{T},\mathbf{L}^{-2}),$$

where

$$\beta(k,P_0,\dots,P_k)=\frac{-i}{2\pi}(1+k)\frac{\sum_{j=0}^k P_j(\eta_1,\eta_2)\bar{\xi}^j}{(1+|\xi|^2)^k}\frac{d\bar{\xi}}{(1+|\xi|^2)^2},$$

then $\mathcal{P}$ maps the space $\mathcal{R}_k$ on the space of elementary states of degree k on $\mathbf{C}^4$ in one-to-one way.

It is easy to see that the set of classes $\{[\beta(k,P_0,\dots,P_k)]\}$, where $\{P_j\}$ are homogeneous monomials of order k, is a basis of $\mathcal{R}_k$ and that $\dim\mathcal{P}(\mathcal{R}_k)=(k+1)^2$.

B. The Dirac equation.

THEOREM 6. *Let $\Phi(x,y)=(\Phi^0(x,y)+\Phi^{12}(x,y)\hat{f}_1\hat{f}_2)I$ be a holomorphic spinor-valued function on $\mathbf{C}^4$ such that Φ^0 and Φ^{12} are homogeneous polynomials of degree k on $\mathbf{C}^4$ and Φ is a solution of the Dirac equation on $\mathbf{C}^4$. Let $\tilde{\Phi}(\eta,\phi)=\Phi(x(\eta,\xi),y(\eta,\xi))$ be the spinor-valued function on $\mathbf{T}$ obtained by substitution (11).*

Then the form

$$\beta_\Phi = \frac{-i}{2\pi}(2+k)I(1+\bar{\xi}f_2f_1)\widetilde{\Phi}\frac{d\bar{\xi}}{(1+|\xi|^2)^3}$$

is manifestly $\bar{\partial}$-closed form on $\mathbf{T}$ *which represents an element* $[\beta_\Phi] \in H^{(0,1)}(\mathbf{T}, \mathbf{L}^{-3})$ *satisfying* $\mathcal{P}([\beta_\Phi]) = \Phi$.

As an illustration let us describe the correspondence in the case $k = 1$. The basis for elementary states of homogeneity 1 and the corresponding Penrose transforms are given by the following expressions:

$$\begin{aligned}
\beta &= \frac{\eta_1\bar{\xi}^2}{1+|\xi|^2}\kappa \to \mathcal{P}([\beta]) = y^1.I \\
\beta &= \frac{\eta_2\bar{\xi}^2}{1+|\xi|^2}\kappa \to \mathcal{P}([\beta]) = y^2.I \\
\beta &= \frac{\eta_1\bar{\xi}}{1+|\xi|^2}\kappa \to \mathcal{P}([\beta]) = (-x^2 + y^1\hat{f}_1\hat{f}_2)I \\
\beta &= \frac{\eta_2\bar{\xi}}{1+|\xi|^2}\kappa \to \mathcal{P}([\beta]) = (x^1 + y^2\hat{f}_1\hat{f}_2)I \\
\beta &= \frac{\eta_2}{1+|\xi|^2}\kappa \to \mathcal{P}([\beta]) = (x^1\hat{f}_1\hat{f}_2)I \\
\beta &= \frac{\eta_1}{1+|\xi|^2}\kappa \to \mathcal{P}([\beta]) = (x^2\hat{f}_1\hat{f}_2)I
\end{aligned} \tag{16}$$

For general k, let us show how the procedure works e.g. for $\Phi = (y^1)^k I$. We have

$$\beta^k = \frac{-i}{2\pi}(k+2)\frac{(\eta^1)^k d\bar{\xi}}{(1+|\xi|^2)^{3+k}}$$

and

$$\mathcal{P}([\beta_k]) = \frac{-i}{2\pi}(k+2)\int_{\mathbf{C}}\frac{(y^1-\xi x^2)^k(1+\xi\hat{f}_1\hat{f}_2)Id\bar{\xi}\wedge d\xi}{(1+|\xi|^2)^{3+k}} = (y^1)^k I.$$

We can state the following general result:

Denote by $\mathcal{H}_k(\eta_1,\eta_2)$ the set of all homogeneous polynomials of order k in (η_1,η_2), $\mathcal{D}_k$ the set of all cohomology classes

$$\{[\beta(k, P_0, \ldots, P_{k+1})]\ P_j \in \mathcal{H}_k(\eta_1,\eta_2)\} \subset H^{(0,1)}(\mathbf{T}, \mathbf{L}^{-3})$$

where

$$\beta(k, P_0, \ldots, P_k) = \frac{-i}{2\pi}(2+k)\frac{\sum_{j=0}^{k+1} P_j(\eta_1, \eta_2)\bar{\xi}^j}{(1+|\xi|^2)^{k+1}} \frac{d\bar{\xi}}{(1+|\xi|^2)^2}$$

then $\mathcal{P}(\mathcal{D}_k)$ is the space of elementary states of degree k on $\mathbf{C}^4$.

We shall see immediately, that the set of classes $\{[\beta(k, P_0, \ldots, P_{k+1})]\}$ where $\{P_j\}$ are homogeneous monomials of order k is a basis of $\mathcal{P}(\mathcal{D}_k)$ and $\dim\mathcal{P}(\mathcal{D}_k) = (k+1)(k+2)$.

EXAMPLE. (Elementary states on $\mathbf{C}^6$).

Following general theory, we use the following coordinates:
a) On $\mathbf{C}^6$:

$$P = (x^1, x^2, x^3, y^1, y^2, y^3)$$

b) On a part of $\mathbf{F} = \mathbf{C}^6 \times IG_{3,6}$, we take nonhomogeneous coordinates (ξ_1, ξ_2, ξ_3) corresponding to one basic coordinate neigborhood on $IG_{3,6}$. The corresponding isotropic basis is

$$\begin{aligned} Z_1 &= (1, 0, 0, 0, -\xi_1, -\xi_2) \\ Z_2 &= (0, 1, 0, \xi_1, 0, -\xi_3) \\ Z_3 &= (0, 0, 1, \xi_2, \xi_3, 0) \end{aligned} \tag{17}$$

c) On the twistor space $\mathbf{T}$ we take the coordinates:

$$(\eta, \xi) = (\eta^1, \eta^2, \eta^3, \xi_1, \xi_2, \xi_3).$$

The projection μ on Ω' has the form:

$$\mu(x, y, \xi) = (\eta, \xi)$$

with

$$\begin{aligned} \eta^1 &= y^1 - \xi_1 x^2 - \xi_2 x^3 \\ \eta^2 &= y^2 + \xi_1 x^1 - \xi_3 x^3 \\ \eta^3 &= y^3 + \xi_2 x^1 + \xi_3 x^2 \end{aligned}$$

The corresponding basis are:

$$\begin{aligned}
Z_1 &= (1,0,0,0,-\xi_1,-\xi_2)\\
Z_2 &= (0,1,0,\xi_1,0,-\xi_3)\\
Z_3 &= (0,0,1,\xi_2,\xi_3,0)\\
\hat{Z}_1 &= (0,-\bar{\xi}_1,-\bar{\xi}_2,1,0,0)\\
\hat{Z}_2 &= (\bar{\xi}_1,0,-\bar{\xi}_3,0,1,0)\\
\hat{Z}_3 &= (\bar{\xi}_2,\bar{\xi}_3,0,0,0,1)
\end{aligned}$$

and the dual basis

$$\begin{aligned}
Y_1 &= \|\xi\|^{-1}(0,-\bar{\xi}_1,-\bar{\xi}_2,1+|\xi_3|^2,-\xi_3\bar{\xi}_2,\xi_3\bar{\xi}_1)\\
Y_2 &= \|\xi\|^{-1}(\bar{\xi}_1,0,-\bar{\xi}_3,-\xi_2\bar{\xi}_3,1+|\xi_2|^2,-\xi_2\bar{\xi}_1)\\
Y_3 &= \|\xi\|^{-1}(\bar{\xi}_2,\bar{\xi}_3,0,\xi_1\bar{\xi}_3,-\xi_1\bar{\xi}_2,1+|\xi_1|^2)\\
\hat{Y}_1 &= \|\xi\|^{-1}(1+|\xi_3|^2,-\xi_2\bar{\xi}_3,\xi_1\bar{\xi}_3,0,-\xi_1,-\xi_2)\,'\\
\hat{Y}_2 &= \|\xi\|^{-1}(-\xi_3\bar{\xi}_2,1+|\xi_2|^2,-\xi_1\bar{\xi}_2,\xi_1,0,-\xi_3)\\
\hat{Y}_3 &= \|\xi\|^{-1}(\xi_3\bar{\xi}_1,-\xi_2\bar{\xi}_1,1+|\xi_1|^2,\xi_2,\xi_3,0)
\end{aligned}$$

where

$$\|\xi\| = 1+|\xi_1|^2+|\xi_2|^2+|\xi_3|^2.$$

For the inverse transform we have the following coordinate form of $Q(\eta,\xi)$:

$$\begin{aligned}
x^1 &= \|\xi\|^{-1}(\eta^2\bar{\xi}_1+\eta^3\bar{\xi}_2)\\
x^2 &= \|\xi\|^{-1}((-\eta^1\bar{\xi}_1+\eta^3\bar{\xi}_3)\\
x^3 &= \|\xi\|^{-1}((-\eta^1\bar{\xi}_2-\eta^2\bar{\xi}_3)\\
y^1 &= \|\xi\|^{-1}(\eta^1(1+|\xi_3|^2)+\eta^2(-\xi_2\bar{\xi}_3)+\eta^3(\xi_1\bar{\xi}_3))\\
y^2 &= \|\xi\|^{-1}(\eta^1(-\xi_3\bar{\xi}_2)+\eta^2(1+|\xi_2|^2)+\eta^3(-\xi_1\bar{\xi}_2))\\
y^3 &= \|\xi\|^{-1}(\eta^1(\xi_3\bar{\xi}_1)+\eta^2(-\xi_2\bar{\xi}_1)+\eta^3(1+|\xi_1|^2)
\end{aligned} \tag{18}$$

Note again that all coordiantes are holomorphic functions of η^i. A direct computation gives :

$$\sum_I |M_I|^2 = \|\xi\|^2.$$

A. The complex Laplace equation.

THEOREM 7. *Let $\Phi(x,y)$ be a homogeneous polynomial of degree k which is a solution of the complex Laplace equation on $\mathbf{C}^6$. Let $\Psi(\eta,\xi) = \Phi(x(\eta,\xi),y(\eta,\xi))$, and $\bar{\alpha}_3 = d\bar{\xi}_1 \wedge d\bar{\xi}_2 \wedge d\bar{\xi}_3$.*

Then

$$\beta_\Phi = \frac{-i}{16\pi^2}(6+5k+k^2)\Psi\frac{\bar{\alpha}_3}{\|\xi\|^{4+k}}$$

is $\bar{\partial}$-closed form on $\mathbf{T}$ which represents a class $[\beta_\Phi]$ in $H^{(0,3)}(\mathbf{T},\mathbf{L}^{-5})$ and

$$\mathcal{P}([\beta_\Phi]) = \Phi.$$

The theorem can again be proved by direct computation, all what is needed is the following formula:

$$\int_{\mathbf{C}^3}\frac{(1+|\xi_1|^2)^k\bar{\alpha}_3\wedge\alpha_3}{\|\xi\|^{k+4}} = -(2i)^3\frac{2\pi^2}{6+5k+k^2}. \tag{19}$$

To illustrate it, let us write down the case of the function $\Phi(x,y) = (y^3)^k$ explicitely. It has the inverse Penrose transform

$$\beta_\Phi = \frac{-i}{16\pi^2}(6+5k+k^2)\frac{(\eta_1\xi_3\bar{\xi}_1 - \eta_2\xi_2\bar{\xi}_1 + \eta_3(1+|\xi_1|)^2)^k\bar{\alpha}_3\wedge\alpha_3}{\|\xi\|^{4+k}}$$

and it is straightforward to check that it gives back the function $(y^3)^k$.

B. The Dirac equation.

Let S^+ be the spinor space of positive spinors. We have the identification of spinor space with the subspace of $\mathcal{C}_6^C$ as follows:

$$\mathrm{S}^+ = \{a + b\hat{f}_1\hat{f}_2 + c\hat{f}_1\hat{f}_3 + d\hat{f}_2\hat{f}_3)I \mid a,b,c,d, \in \mathbf{C}\}$$

where $I = f_1\hat{f}_1f_2\hat{f}_2f_3\hat{f}_3$.

The image $\mathcal{S}(g), g \in \mathrm{Spin}_0(2n,\mathbf{C})$ is defined (up to a multiplicative factor) by

$$(1+\xi_1\hat{f}_1\hat{f}_2 + \xi_2\hat{f}_1\hat{f}_3 + \xi_3\hat{f}_2\hat{f}_3)I.$$

Similarly, the element $\frac{I\bar{g}}{|g|^2}$ is given (up to a factor) by

$$\|\xi\|^{-1}I(\bar{\xi}_3\hat{f}_3\hat{f}_2 + \bar{\xi}_2\hat{f}_3\hat{f}_1 + \bar{\xi}_1\hat{f}_2\hat{f}_1 + 1)$$

Let $\Phi : \mathbf{C}^6 \to \mathbf{S}^+$ be a solution of the Dirac equation. Let

$$\Phi(x,y) = (a(x,y)1 + b(x,y)\hat{f}_1\hat{f}_2 + c(x,y)\hat{f}_1\hat{f}_3 + d(x,y)\hat{f}_2\hat{f}_3)I \tag{20}$$

be the coordinate form of Φ.

THEOREM 8. *Let $\Phi(x,y)$ of a form (20) be a solution of the Dirac equation on $\mathbf{C}^6$, such that a,b,c,d are homogeneous polynoms in (x,y) of degree k. Denote $\Psi(\eta,\xi) = \Phi(x(\eta,\xi), y(\eta,\xi))$, then*

$$\beta_\Phi = \frac{-i}{16\pi^2}(12 + 7k + k^2)I(\bar{\xi}_3\hat{f}_3\hat{f}_2 + \bar{\xi}_2\hat{f}_3\hat{f}_1 + \bar{\xi}_1\hat{f}_2\hat{f}_1 + 1)\Psi(\eta,\xi)\frac{\bar{\alpha}_3}{\|\xi\|^{5+k}}$$

is $\bar{\partial}$-closed form on $\mathbf{T}$ which represents a class $[\beta_\Phi]$ in $H^{(0,3)}(\mathbf{T}, \mathbf{L}^{-5})$ and

$$\mathcal{P}([\beta_\Phi]) = \Phi.$$

We can present a simple example here. Take the following solution of the Dirac equation $\Phi(x,y) = (y^1)^k I$. The procedure announced in the theorem above gives us the form

$$\beta = K(3)\frac{\eta_1(1 + |\xi_3|^2)^k\bar{\alpha}_3}{\|\xi\|^{5+k}}$$

and using (19), the corresponding Penrose transform applied on β can be computed explicitely, the result is

$$\begin{aligned}
\mathcal{P}([\beta]) &= (y^1)^k IK(3)(k^2 + 7k + 12)\int_{\mathbf{C}^3}\frac{(1 + |\xi_1|^2)^k\bar{\alpha}_3 \wedge \alpha_3}{\|\xi\|^{k+5}} \\
&= (y^1)^k IK(3)(k^2 + 7k + 12)(K(3)(k^2 + 7k + 12))^{-1} \\
&= (y^1)^k I.
\end{aligned}$$

Bibliography

[1] R. Baston, The algebraic construction of invariant differential operators, D. Phil. Thesis, *Oxford University*, 1985.

[2] R. J. Baston-M. G. Eastwood, The Penrose Transform, its interaction with representation theory, *Oxford University Press*, 1989.

[3] J. Bureš, Integral formulae in complex Clifford analysis, Clifford algebras and their applications (eds. R.Chisholm, A.Common), *NATO ASI series* C183, 1985, 219-226.

[4] J. Bureš-V. Souček, Generalized hypercomplex analysis and its integral formulas, *Complex Variables: Theory and Application*, 1985, 5, 53-70.

[5] J. Bureš-V. Souček, The Penrose transform and Clifford analysis, Proc. of the Winter School Geometry and Physics, Srní 1990, *Suppl. Rend. Circ. Mat. Palermo*, Ser.II, 26, 1991, 97-104.

[6] J. Bureš-V. Souček, The Penrose transform for Dirac equation, *Suppl. Rend. Circ. Mat. Palermo*, to appear.

[7] J. Bureš-V. Souček, The Penrose transform for Laplace equation in higher dimensions, to appear.

[8] J. Bureš-V. Souček, The Penrose transform on isotropic Grassmannians, *Proc. of the Conference "75 years of the Radon transform"*, Vienna, 1992, to appear.

[9] R. Delanghe-F.Sommen-V. Souček, Clifford algebra and spinor-valued functions: a function theory for the Dirac operator, Mathematics and Its Applications 53, *Kluwer Academic Publishers*, Dordrecht, 1992.

[10] M. G. Eastwood-R. Penrose-R. O. Wells Jr., Cohomology and massless fields, *Comm. Math. Physics*, 78, 1981, 305-351.

[11] S. Gindikin-M. Henkin, The Penrose transform and complex integral geometry, *Contemporary problems in Math.*, Vol.17, VINITI 1981, 57-111 (in Russian).

[12] S. Helgason, *Groups and Geometric Analysis*, Academic Press, London 1984.

[13] L. P. Hughston-T. R. Hurd, A $\mathbb{C}P^5$ calculus for space-time fields, *Phys. Reports*, 100, 1983, 273-326.

[14] M. K. Murray, A Penrose transform for the twistor space of an even dimensional conformally flat Riemannian manifold, *Ann. Global Anal. Geom.*, Vol. 4, No.1, 1986, 71-88.

[15] V. Souček, Integral formulae for solutions of massless field equations in higher dimensions, to appear.

[16] R. S. Ward-R. O. Wells Jr., *Twistor Geometry and Field Theory*, Cambridge Univ. Press, 1989.

[17] N. M. J. Woodhouse, Real methods in twistor theory, *Class. Quantum Grav.* 2, 1985, 257-291.