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HYPERCOMPLEX DIFFERENTIAL FORMS APPLIED TO THE DE RHAM AND THE DOLBEAULT COMPLEX

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Introduction.

The main aim of the paper is to generalize the following result in complex analysis to higher dimensions. Let $\Omega \subset C$ be open, let $\mathcal{O}(\Omega)$ be the ring of holomorphic functions in Ω and let $f \in \mathcal{O}(\Omega)$. Then locally around each point w of Ω, f admits a holomorphic primitive F, i.e. $d/dz(F) = f$, in a neighbourhood of w. Of course this result is not true globally. Hence one may construct the quotient space

$$H = \mathcal{O}(\Omega)/\frac{d}{dz}\mathcal{O}(\Omega),$$

which is naturally isomorphic to the de Rham cohomology space $H^1(\Omega, C)$, so that dim H equals to the **number** of holes in Ω.

In our generalization the role of the Cauchy-Riemann system in the plane is taken over by special systems of differential equations in n dimensions, occuring in hypercomplex analysis (see [1], [2]).

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Furthermore we obtain our results by means of a splitting of the vector-valued de Rham complex on an open set $\Omega \subset \mathbb{R}^n$, which leads to the representation of the de Rham cohomology spaces using Clifford algebra valued differential forms such as the left monogenic forms (see [7]). In this way the isomorphism $H \cong H^1(\Omega,\mathbb{C})$ is generalized to higher dimensions.

We also apply our theory to several complex variables in order to obtain a similar representation for the Dolbeault cohomology spaces $H^{(p,q)}(\Omega,\mathfrak{C}_n)$, $\Omega \subset \mathbb{C}^n$ open, with values in the complex Clifford algebra $\mathfrak{C}_n$. Hence we generalize the isomorphism $H = H^1(\Omega,\mathbb{C})$ both to $\mathbb{R}^n$ and to $\mathbb{C}^n$.

Our method works as follows. In the first section we start from a complex

$$0 \longrightarrow A_{-1} \xrightarrow{i} A_0 \xrightarrow{d_0} \cdots \xrightarrow{d_{n-1}} A_n \longrightarrow 0 \qquad (1)$$

and a subcomplex

$$0 \longrightarrow A_{-1} \xrightarrow{i} B_0 \xrightarrow{\delta_0} \cdots \longrightarrow B_{n-1} \xrightarrow{\delta_{n-1}} 0 \qquad (2)$$

and we prove conditions implying that the homology spaces of (1) are isomorphic to those of (2).

In the second section we consider a splitting of (1) which follows from a splitting $A_j = A_j^1 \oplus A_j^2$ and which leads to a special subcomplex of (1) to which we may apply the results of the first section, to obtain a representation for the homology spaces of (1).

In the third section we apply this theory to the de Rham and the Dolbeault complex. For the Dolbeault complex we make use of so called weak complex monogenic functions (see [8]).

1. Homology of subcomplexes.

In this section we shall discuss the following problem. Consider a complex of vector spaces and linear maps:

$$0 \longrightarrow A_{-1} \xrightarrow{i} A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} A_n \longrightarrow 0 \qquad (1)$$

(i.e. $d_j d_{j-1} = 0$) and consider its corresponding homology spaces

$$H_j^A = \ker d_j / \operatorname{im} d_{j-1} .$$

Furthermore, consider a subcomplex of (1):

$$0 \longrightarrow A_{-1} \xrightarrow{\ i\ } B_0 \xrightarrow{\delta_0} B_1 \xrightarrow{\delta_1} \dots \xrightarrow{\delta_{n-2}} B_{n-1} \xrightarrow{\delta_{n-1}} 0 \qquad (2)$$

(i.e. $B_j \subset A_j$, $j=0,\dots,n-1$ and $\delta_i = d_i|B_i$) and its homology spaces

$$H_j^B = \ker \delta_j / \operatorname{im} \delta_{j-1} .$$

Then we shall establish conditions implying that

$$H_j^A \cong H_j^B , \quad j=0,\dots,n-1 .$$

To that end, notice that the inclusions $\ker \delta_j \subset \ker d_j$ induce the natural maps $\phi_j : H_j^B \to H_j^A$, where for $a \in H_j^B$, $\phi_j(a) = a + \operatorname{im} d_{j-1}$. As $\operatorname{im} \delta_{j-1} \subset \operatorname{im} d_{j-1}$, these maps ϕ_j are well defined on H_j^B and we shall prove

Lemma 1.

Consider the complex of quotient spaces induced by (1) and its subcomplex (2) :

$$0 \longrightarrow A_0/B_0 \xrightarrow{\tilde{d}_0} \dots \xrightarrow{\tilde{d}_{n-2}} A_{n-1}/B_{n-1} \xrightarrow{\tilde{d}_{n-1}} A_n \longrightarrow 0 \qquad (3)$$

If the sequence (3) is exact at the j-th place, $j=0,\dots,n-1$; then

(i) ϕ_j is onto

(ii) ϕ_{j+1} is a monomorphism.

Proof.

First notice that the maps $\tilde{d}_j$, given by $\tilde{d}_j(a) = d_j(a) + B_{j+1}$, are well defined because $d_j(B_j) = \delta_j(B_j) \subset B_{j+1}$.

As to (i), suppose that (3) is exact at the j-th place. Then for every $\beta \in \ker d_j$, there exists $\alpha \in A_{j-1}$ such that $d_{j-1}\alpha - \beta = b \in B_j$. But then

$$d_j b = d_j d_{j-1} \alpha - d_j \beta = 0 ,$$

so that $\delta_j b = d_j b = 0$ or $b \in \ker \delta_j$.

Hence $b + \text{im}\delta_{j-1} \in H_j^B$ and so

$$\phi_j(b+\text{im}\delta_{j-1}) = b + \text{im}\, d_{j-1} = \beta + \text{im}\, d_{j-1} \in H_j^A ,$$

which means that ϕ_j is onto.

As to (ii), let $j=1,\dots,n-1$. Then we have to prove that if $\beta \in \ker\delta_{j+1}$ is such that $\beta \in \text{im}d_j$, then $\beta \in \text{im}\delta_j$ and so $\beta + \text{im}\,\delta_j = 0$.

Take any $\beta \in \ker\delta_{j+1} \cap \text{im}d_j$. Then for some $\alpha \in A_j$, $\beta = d_j(\alpha)$. Hence $\tilde{d}_j(\alpha) = 0$ and so there exists $\gamma \in A_{j-1}$ such that

$\tilde{d}_{j-1}(\gamma) = \alpha$, i.e. $d_{j-1}(\gamma) - \alpha = \varepsilon \in B_j$.

But then $\delta_j(-\varepsilon) = d_j(-\varepsilon) = d_j(\alpha) = \beta$, so that $\beta \in \text{im}\delta_j$. ∎

The previous Lemma immediately leads to

Theorem 1.

If the sequence (3) is exact for all $j=0,\dots,n-1$; then the maps ϕ_j, $j=0,\dots,n-1$ are isomorphisms between the homology spaces H_j^A and H_j^B , $j=0,\dots,n-1$.

2. Subcomplexes associated to the splitting of a complex.

Consider again the complex (1) of vector spaces and linear maps:

$$0 \longrightarrow A_{-1} \xrightarrow{\ i\ } A_0 \xrightarrow{\ d_0\ } A_1 \xrightarrow{\ d_1\ } \dots \xrightarrow{\ d_{n-1}\ } A_n \longrightarrow 0 .$$

Furthermore, assume that A_j may be decomposed as a direct sum of subspaces A_j^1 and A_j^2 :

$$A_j = A_j^1 \oplus A_j^2 , \quad j=0,\dots,n ,$$

where $A_0^1 = 0$ and $A_n^2 = 0$ and denote by π_j^1 and π_j^2 the projection operators from A_j onto the spaces A_j^1 and A_j^2 respectively.

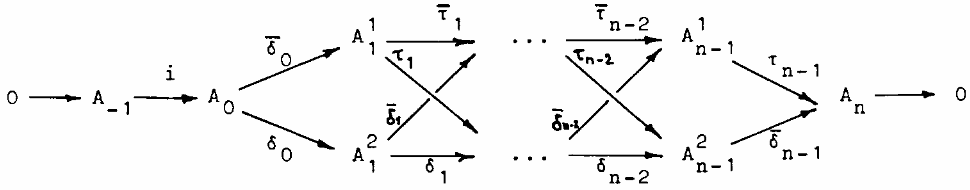
Then we may introduce

Definition 1.

Put for $j=0,\dots,n$ $\quad \bar{\tau}_j = \pi_{j+1}^1 \circ d_j | A_j^1 , \quad \tau_j = \pi_{j+1}^2 \circ d_j | A_j^1$

$$\bar{\delta}_j = \pi_{j+1}^1 \circ d_j | A_j^2 , \quad \delta_j = \pi_{j+1}^2 \circ d_j | A_j^2 .$$

Then the following diagram is called the splitting of the complex (1):



As an immediate consequence of such splitting we have

Lemma 2.

All squares occuring in the diagram (4) anticommute, i.e. for $j=0,\dots,n-2$

$$\delta_{j+1}\delta_j + \tau_{j+1}\bar{\delta}_j = 0\ , \quad \bar{\delta}_{j+1}\delta_j + \bar{\tau}_{j+1}\bar{\delta}_j = 0\ ,$$

$$\bar{\tau}_{j+1}\bar{\tau}_j + \bar{\delta}_{j+1}\tau_j = 0\ , \quad \tau_{j+1}\bar{\tau}_j + \delta_{j+1}\tau_j = 0\ .$$

Proof.

We only prove the first identity. We have that

$$\delta_{j+1}\delta_j + \tau_{j+1}\bar{\delta}_j = \delta_{j+1}\circ\pi^2_{j+1}\circ d_j|A^2_j + \tau_{j+1}\circ\pi^1_{j+1}\circ d_j|A^2_j =$$

$$= \pi^2_{j+1}\circ d_{j+1}\circ(\pi^2_{j+1} + \pi^1_{j+1})\circ d_j|A^2_j = \pi^2_{j+1}\circ d_{j+1}\circ d_j|A^2_j = 0\ . \blacksquare$$

Now, a splitting (4) of the complex (1) gives rise to a subcomplex of (1) in the following way. Put

$$B_0 = \{\ f\in A_0\ |\ \bar{\delta}_0 f = 0\ \}\ ,\quad B_j = \{\ f\in A^2_j\ |\ \bar{\delta}_j f = 0\}\ ,\ i=1,\dots,n-1.$$

Then by Lemma 2 we obtain a complex of the form (2):

$$0 \longrightarrow A_{-1} \xrightarrow{\ i\ } B_0 \xrightarrow{\ \delta_0\ } B_1 \xrightarrow{\ \delta_1\ } \dots \xrightarrow{\ \delta_{n-2}\ } B_{n-1} \xrightarrow{\ \delta_{n-1}\ } 0\ ,$$

which is called the subcomplex of (1) derived from the splitting (4). Indeed, let $f\in B_j$, then $\delta_j f\in B_{j+1}$. Moreover $\delta_{j+1}\delta_j f = -\tau_{j+1}\bar{\delta}_j f = 0$. Notice that when $f\in B_{n-1}$, $d_{n-1}f = \bar{\delta}_{n-1}f = 0$.

Of course we shall use the complex (2) in order to compare its homology spaces with ones of (1). First of all we have

Lemma 3.

Let $d^!_j = \pi^1_{j+1}\circ d_j$ and assume that $d^!_{j-1}: A_{j-1} \longrightarrow A^1_j$, $j=1,\dots,n-1$ is onto.

Then the sequence

$$0 \longrightarrow A_0/B_0 \xrightarrow{\tilde{d}_0} \cdots \xrightarrow{\tilde{d}_{n-2}} A_{n-1}/B_{n-1} \xrightarrow{\tilde{d}_{n-1}} A_n \longrightarrow 0$$

is exact at the j-th place.

<u>Proof</u>.

As clearly $\tilde{d}_{j+1}\tilde{d}_j = 0$, we have to show that for all $\omega \in A_j$ such that $d_j\omega \in B_{j+1}$, there exists $\phi \in A_{j-1}$ such that $d_{j-1}\phi - \omega \in B_j$. Put $\omega^1 = \pi_j^1(\omega)$, $\omega^2 = \pi_j^2(\omega)$. Then we have that

$$d_j\omega = \bar{\tau}_j\omega^1 + \bar{\delta}_j\omega^2 + \tau_j\omega^1 + \delta_j\omega^2 \in B_{j+1}$$

and so $\pi_{j+1}^1 \circ d_j\omega = \bar{\tau}_j\omega^1 + \bar{\delta}_j\omega^2 = 0$. As $d'_{j-1}: A_{j-1} \longrightarrow A_j^1$ is onto, there exists $\phi \in A_{j-1}$ such that $\omega^1 = d'_{j-1}\phi$.

Furthermore, if $\phi^k = \pi_{j-1}^k(\phi)$; $k=1,2$; $\omega^1 = d'_{j-1}\phi^1 + d'_{j-1}\phi^2 = $

$= \bar{\tau}_{j-1}\phi^1 + \bar{\delta}_{j-1}\phi^2$. This leads to

$$d_{j-1}\phi - \omega = (\bar{\tau}_{j-1}\phi^1 + \bar{\delta}_{j-1}\phi^2 + \tau_{j-1}\phi^1 + \delta_{j-1}\phi^2) - (\omega^1 + \omega^2) =$$
$$= \tau_{j-1}\phi^1 + \delta_{j-1}\phi^2 - \omega^2 \in A_j^2,$$

and by Lemma 2

$$\bar{\delta}_j(d_{j-1}\phi - \omega) = \bar{\delta}_j\tau_{j-1}\phi^1 + \bar{\delta}_j\delta_{j-1}\phi^2 - \bar{\delta}_j\omega^2 =$$
$$= -\bar{\tau}_j\bar{\tau}_{j-1}\phi^1 - \bar{\tau}_j\bar{\delta}_{j-1}\phi^2 + \bar{\tau}_j\omega^1 = 0.$$

Hence $d_{j-1}\phi - \omega \in B_j$, which completes the proof for $j=1,\dots,n-1$. For $j=0$, let $\omega \in A_0$ be such that $d_0\omega \in B_1$. Then $\bar{\delta}_0\omega = 0$, so that $\omega \in B_0$. ■

From Theorem 1 and Lemma 3 we immediately obtain

<u>Theorem 2</u>.

Suppose that a splitting (4) of the complex (1) is given such that $d'_{j-1}: A_{j-1} \longrightarrow A_j^1$ is onto for all $j=1,\dots,n-1$.

Then $H_j^A = H_j^B$, $j=0,\dots,n-1$.

Examples.

(1) Let $\Omega \subset R^n$ be open and let E be a real or complex vector space. Then we shall now consider the spaces $C_\infty(\Omega,\Lambda^j R^n)\otimes E$ of smooth j-forms on Ω with coefficients in E.

If we denote this space by $E^j(\Omega,E)$, we can consider the de Rham complex

$$0 \longrightarrow E \xrightarrow{\ i\ } E^0(\Omega,E) \xrightarrow{\ d\ } E^1(\Omega,E) \xrightarrow{\ d\ } \dots \xrightarrow{\ d\ } E^n(\Omega,E) \longrightarrow 0 \quad (5)$$

i.e. $A_{-1} = E$, $A_j = E^j(\Omega,E)$ and $d_j = d = \sum_j dx_j \frac{\partial}{\partial x_j}$.

The maps π_j^1, π_j^2, δ_j, $\bar{\delta}_j$, τ_j, $\bar{\tau}_j$, d_j' occuring in a splitting of this complex will be denoted simply by π^1, π^2, δ, $\bar{\delta}$, τ, $\bar{\tau}$, d'. Furthermore, for the corresponding subcomplex we use the notation $M^k(\Omega,E)$ for B_k.

(2) Let $\Omega \subset C^n$ be open and let E be a complex vector space. Then we denote by $E^{(p,q)}(\Omega,E)$ the space of all smooth (p,q)-forms f in Ω, i.e. f may be written in the form

$$f(z) = \sum_{I,J} f_{I,J}(z)dz_I \wedge d\bar{z}_J ,$$

where

$$dz_I = dz_{i_1}\wedge\dots\wedge dz_{i_p}, \quad d\bar{z}_J = d\bar{z}_{j_1}\wedge\dots\wedge d\bar{z}_{j_q} .$$

Let $\partial = \sum_{j=1}^n dz_j \frac{\partial}{\partial z_j}$, $\bar{\partial} = \sum_{j=1}^n d\bar{z}_j \frac{\partial}{\partial \bar{z}_j}$. Then we consider the Dolbeault complex

$$0 \longrightarrow O^p(\Omega,E) \xrightarrow{\ i\ } E^{(p,0)}(\Omega,E) \xrightarrow{\ \bar{\partial}\ } \dots \xrightarrow{\ \bar{\partial}\ } E^{(p,n)}(\Omega,E) \longrightarrow 0 \quad (6)$$

where $O^p(\Omega,E)$ is the space of holomorphic p-forms in Ω.

Here $A_{-1} = O^p(\Omega,E)$, $A_j = E^{(p,q)}(\Omega,E)$, $d_j = \bar{\partial}$.

We use the notation $\bar{\partial}'$ instead of d_j' and $M^{(p,q)}(\Omega,E)$ instead of B_q.

3. Applications.

3.1. Monogenic differential forms in R^{n+1}.

Let $E = \mathcal{C}_n$ be the real (or complex) Clifford algebra over R^n with negative definite quadratic form. Then a general element $a \in \mathcal{C}_n$ may be written in the form

$$a = \sum_{A \subset N} a_A e_A \ , \quad N = \{1,\dots,n\}, \quad a_A \in R \quad (\text{or } C\ ),$$

where $e_A = e_{\alpha_1} \dots e_{\alpha_h}$, $A = \{\alpha_1,\dots,\alpha_h\}$ with $\alpha_1 < \dots < \alpha_h$ and the product in $\mathcal{C}_n$ is determined by the relations

$e_i e_j + e_j e_i = -2\delta_{ij}$, $i,j=1,\dots,n$, $(e_1,\dots,e_n)$ being an orthogonal basis of R^n. An involution in $\mathcal{C}_n$ is given by

$$a^- = \sum_A a_A e_A^- \ ,$$

where $e_A^- = e_{\alpha_h}^- \dots e_{\alpha_1}^-$, $e_j^- = -e_j$, $j=1,\dots,n$.

Let $\Omega \subset R^{n+1}$ be open and $f \in C_1(\Omega,\mathcal{C}_n)$. Then f is called left monogenic in Ω if $Df = 0$, where

$D = \sum_{j=0}^{n} e_j \frac{\partial}{\partial x_j}$ is a generalized Cauchy-Riemann operator and $e_0 = e_\emptyset = 1$ (see e.g. [1]).

Furthermore, in [7] the notion of monogenic differential forms has been introduced, generalizing the theory of monogenic functions. Here we give a description by means of a splitting of the de Rham complex.

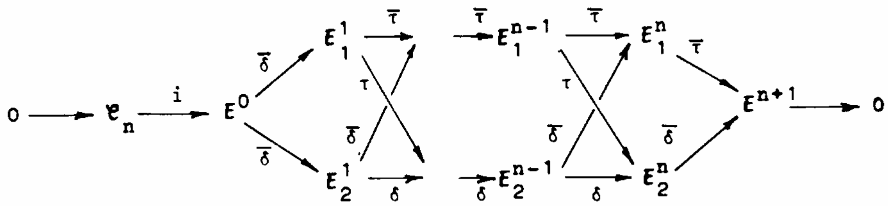
Let $\Omega \subset R^{n+1}$ be a fixed open set. Then we put $E^j = E^j(\Omega,\mathcal{C}_n)$ and we introduce the basic differentials $dZ_j = dx_j - e_j dx_0$, $j=1,\dots,n$, where $(x_0,\dots,x_n)$ are the coordinates in R^{n+1}.

Moreover we denote by $E_2^j = A_j^2$ the space of j-forms in $E^j(\Omega,\mathcal{C}_n)$ which may be written in the form

$$f = \sum_{k_1 < \dots < k_j} dZ_{k_1} \wedge \dots \wedge dZ_{k_j} f_{k_1 \dots k_j} \ ,$$

$f_{k_1 \dots k_j}$ being $\mathcal{C}_n$-valued smooth functions in Ω , whereas we denote $A_j^1 = E_1^j = dx_0 \wedge E_2^{j-1}$, i.e. the space E_1^j is the space of j-forms of the form $dx_0 \wedge \omega$, $\omega \in E_2^{j-1}$.

It is easy to see that $E^j = E_1^j \oplus E_2^j$, whence we have the splitting



Furthermore the operators $\delta, \bar\delta, \tau, \bar\tau$ are given in

<u>Lemma 4</u>.

We have that $\tau = 0$, $\bar\tau = \delta$, where $\delta = \sum_{j=1}^{n} dZ_j \frac{\partial}{\partial x_j}$ and $\bar\delta = dx_0 D$.

<u>*Proof*</u>.

It is easy to see that $d = dx_0 D + \sum_{j=1}^{n} dZ_j \frac{\partial}{\partial x_j}$.

Let $f \in E_2^j$. Then $dx_0 \wedge Df \in E_1^{j+1}$ and $\sum_{j=1}^{n} dZ_j \wedge \frac{\partial}{\partial x_j} \in E_2^{j+1}$.

Hence we have that

$$\bar\delta = dx_0 D \quad \text{and} \quad \delta = \sum_{j=1}^{n} dZ_j \frac{\partial}{\partial x_j} .$$

Furthermore, as every $f \in E_1^j$ is of the form $f = dx_0 \wedge g$, $g \in E_2^{j-1}$,

$$df = \delta f = - dx_0 \wedge \delta g \in E_1^{j+1} ,$$

so that $\tau = 0$ and $\bar\tau = \delta$. ∎

Notice that $d' = \pi^1 \circ d = dx_0 D$ and that for $f \in E_2^j$

$$dx_0 \wedge Df = \sum_{k_1 < \ldots < k_j} dx_0 \wedge dZ_{k_1} \wedge \ldots \wedge dZ_{k_j} \, Df_{k_1 \ldots k_j} .$$

Hence $f \in M^j(\Omega, \mathscr{C}_n)$ if and only if $f \in E_2^j$ and the coefficients of f are left monogenic, i.e. $Df_{k_1 \ldots k_j} = 0$. $M^j(\Omega, \mathscr{C}_n)$ is therefore called the space of left monogenic j-forms in Ω .

We now represent the de Rham cohomology by means of these forms. First we have

<u>Lemma 5</u>.

The operator $d' : E^{j-1} \longrightarrow E_1^j$, $j = 1, \ldots, n$, is surjective.

<u>*Proof*</u>.

We can even prove that $\bar\delta : E_2^{j-1} \longrightarrow E_1^j$ is surjective, which implies the same for d'. Let $dx_0 \wedge g \in E_1^j$, $g \in E_2^{j-1}$. Then we have to solve the equation $dx_0 \wedge Df = dx_0 \wedge g$.

Using the definition of E_2^{j-1}, this leads to the equation $Df = g$, $g \in C_\infty(\Omega)$, which, in view of [1] always admits a solution $f \in C_\infty(\Omega)$.

Hence, applying Theorem 2, the de Rham cohomology space $H^j(\Omega,\mathcal{C}_n)$, $j=0,\dots,n$, coincide with the homology space of the complex

$$0 \longrightarrow \mathcal{C}_n \xrightarrow{\ i\ } M^0(\Omega,\mathcal{C}_n) \xrightarrow{\ \delta\ } \dots \xrightarrow{\ \delta\ } M^n(\Omega,\mathcal{C}_n) \longrightarrow 0 \qquad (8)$$

3.2. Spinor valued differential forms (see [2]).

First notice that in even dimensions there are two basic Spin(n+1)--modules, whereas there is only one in odd dimensions (see e.g. [6]).

Let E be such a basic Spin(n+1)-module. Then we shall consider the spaces

$$\mathcal{E}^j = C_\infty(\Omega,\Lambda^j R^{n+1} \otimes_R E)$$

of E-valued differential forms in $\Omega \subset R^{n+1}$ open.

As $\Lambda^j R^{n+1} \otimes_R E$ is again a Spin(n+1)-module, it decomposes into irreducible submodules and it may be proved by induction that one of these irreducible pieces, which we denote by E_1^j, is isomorphic to a basic Spin(n+1)-module. In the case of 1-forms this was proved in [3]; for the decomposition in general we refer to [10].

Let us denote by E_2^j the invariant subspace, which is complementary to E_1^j in $\Lambda^j R^{n+1} \otimes_R E$. Then we have the splitting

$$\mathcal{E}^j = \mathcal{E}_1^j \oplus \mathcal{E}_2^j ,$$

where

$$\mathcal{E}_1^j = C_\infty(\Omega,E_1^j) , \quad \mathcal{E}_2^j = C_\infty(\Omega,E_2^j) .$$

In order to apply Theorem 2 we first prove

Lemma 6.

The operators $d'_{j-1}: \mathcal{E}^{j-1} \longrightarrow \mathcal{E}_1^j$ are onto.

Proof.

We shall use a coordinate description. Of course, it is sufficient to prove that the maps $\bar{\tau}: E_1^{j-1} \longrightarrow E_1^j$ are onto.

Let $f_0,\dots,f_n$ be an orthonormal basis of R^{n+1} and let $(x_0,\dots,x_n)$ be the corresponding coordinate system in R^{n+1}.

Then $f_0,\dots,f_n$ generates the Clifford algebra $\mathcal{C}_{n+1}$. Furthermore, $e_j = -f_0 f_j$, $j=1,\dots,n$ are generating bivectors for the even subalgebra $\mathcal{C}_{n+1}^+$.

As $Spin(n+1) \subset \mathcal{C}^+_{n+1} \cong \mathcal{C}_n$ (see [5]) and $Spin(n+1)$ spans $\mathcal{C}^+_{n+1}$, the space E is at the same time the representation space for $\mathcal{C}^+_{n+1}$. It was shown in [11] that the operator $\bar{\tau}$ acts on $\phi \in E_1^{j-1}$ in the same way as $(\sum_{j=0}^{n} f_j \frac{\partial}{\partial x_j})$. Furthermore, the equation $(\sum_{j=0}^{n} f_j \frac{\partial}{\partial x_j})\phi = \psi$, $\psi \in E_1^j$ is equivalent with

$$(\sum_{j=0}^{n} e_j \frac{\partial}{\partial x_j})\phi = -f_0\psi \qquad (9)$$

where $e_0 = -f_0 f_0 = 1$.

But as the whole Clifford algebra $\mathcal{C}_n$ may be written as the sum of basic $Spin(n+1)$-modules, the solvability of (9) again follows from [1], p.160.

Let us denote $M^j = \ker \bar{\delta}_j E^j$, $j=0,\dots,n$; then again we have proved that the homology spaces of the complex

$$0 \longrightarrow E \xrightarrow{\ i\ } M^0 \xrightarrow{\ \delta\ } \cdots \xrightarrow{\ \delta\ } M^n \longrightarrow 0 \qquad (10)$$

are isomorphic to the cohomology spaces $H^j(\Omega,E)$ for $j=0,\dots,n$.

Example 1.

Let us now specialize to the case $n=3$ (i.e. R^4) in the previous application. Then still other possibilities are available. In the case $n=3$ we take $E = H$, where H is the field of (real) quaternions. Notice that, as $Spin(4) = Sp(1)xSp(1)$, H can be turned into a $Spin(4)$-module either by multiplication with the first or the second factor of the elements in $Sp(1)xSp(1)$. In this case it is easy to describe the full decomposition of H-valued differential forms in $\Omega \subset R^4$ into irreducible pieces with respect to the action of $Spin(4)$.

Using the highest weights (see e.g. [6]) for the description of the irreducible representations of $Spin(4)$, we obtain the following splitting.

Suppose that the highest weight of H is $(\frac{1}{2},\frac{1}{2})$. The highest weight of the other basic spinor module is the $(\frac{1}{2},-\frac{1}{2})$. Hence

$\Lambda^1 R^4 \otimes_R H$ splits into $V(\frac{3}{2},\frac{1}{2}) \oplus V(\frac{1}{2},-\frac{1}{2})$,

$\Lambda^2 R^4 \otimes_R H$ splits into $V(\frac{3}{2},\frac{3}{2}) \oplus V(\frac{1}{2},\frac{1}{2}) \oplus V(\frac{3}{2},-\frac{1}{2})$,

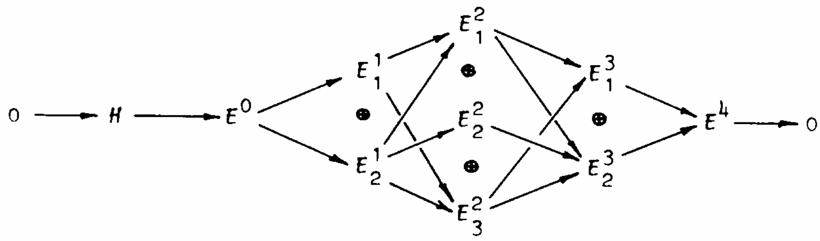
$\Lambda^3 R^4 \otimes_R H$ splits into $V(\frac{3}{2},\frac{1}{2}) \oplus V(\frac{1}{2},-\frac{1}{2})$.

So, denoting $\mathcal{E}_1^1 = C_\infty(\Omega, V(\frac{1}{2},-\frac{1}{2})$, $\mathcal{E}_2^1 = C_\infty(\Omega, V(\frac{3}{2},\frac{1}{2}))$

$\mathcal{E}_1^2 = C_\infty(\Omega, V(\frac{1}{2},\frac{1}{2}))$, $\mathcal{E}_2^2 = C_\infty(\Omega, V(\frac{3}{2},\frac{3}{2}))$,

$\mathcal{E}_3^2 = C_\infty(\Omega, V(\frac{1}{2},-\frac{1}{2}))$, $\mathcal{E}_1^3 = C_\infty(\Omega, V(\frac{1}{2},-\frac{1}{2}))$, $\mathcal{E}_2^3 = C_\infty(\Omega, V(\frac{3}{2},\frac{1}{2}))$,

we obtain the complete diagram



(11)

This splitting may be compared with one given in [9] , where a coordinate description of some operators in (11) was given. Indeed, using this description, it is possible to prove that the maps

$$d_1' + d_2' : \mathcal{E}^1 \longrightarrow \mathcal{E}_1^2 \oplus \mathcal{E}_2^2$$

and

$$d_2'' : \mathcal{E}^2 \longrightarrow \mathcal{E}_2^3$$

are onto.

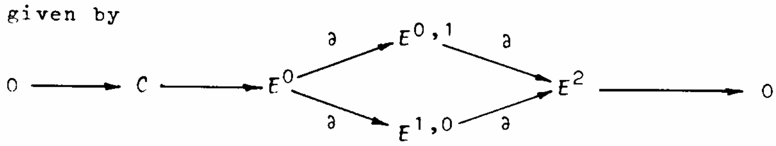
Now, as it was proved that the maps d_0', d_1', d_2' and d_3' are onto, we can apply Theorem 2 by taking $\tilde{\mathcal{E}}_1^1 = \mathcal{E}_1^1$, $\tilde{\mathcal{E}}_1^2 = \mathcal{E}_1^2 \; \mathcal{E}_2^2$ and $\tilde{\mathcal{E}}_1^3 = \mathcal{E}_2^3$.

In this case we again obtain a different description of $H^j(\Omega, E)$.

<u>Example 2</u>.

Let us consider the complex case, where $n=1$ (i.e. in R^2). The scheme described before may be also used here. The results may be compared with the ones proved in e.g. [4].

The standard splitting of the de Rham complex on $\Omega \subset C$ open is given by



(12)

It leads to the complex

$$0 \longrightarrow C \longrightarrow M^0 \xrightarrow{\;\partial\;} M^1 \longrightarrow 0 \qquad (13)$$

where $M^0 = \Gamma(\Omega, \mathcal{O}) \subset \mathcal{E}^0$, $M^1 = \Gamma(\Omega, \mathcal{O}^{1,0}) \subset \mathcal{E}^{1,0}$,

$\mathcal{O}$ and $\mathcal{O}^{1,0}$ being the sheaves of holomorphic functions and of holomorphic 1-forms respectively. The long cohomology sequence associated to the exact sequence of sheaves

$$0 \longrightarrow C \longrightarrow \mathcal{O} \xrightarrow{\partial} \mathcal{O}^{1,0} \longrightarrow 0$$

is given by

$$0 \to \Gamma(\Omega,C) \to \Gamma(\Omega,\mathcal{O}) \to \Gamma(\Omega,\mathcal{O}^{1,0}) \to H^1(\Omega,C) \to H^1(\Omega,\mathcal{O}) \ \ldots$$

But as $\bar{\partial} f = g$ always admits a solution for every $\Omega \subset C$, $H^1(\Omega,\mathcal{O}) = 0$. Hence

$$H^1(\Omega,C) = \Gamma(\Omega,\mathcal{O}^{1,0})/_{\text{im}\ \Gamma(\Omega,\mathcal{O})},$$

which is nothing else but the isomorphism

$$H^1(\Omega,C) = M^1/_{\text{im}\ M^0},$$

obtained from Lemma 6 and Theorem 2.

3.3. Weak complex monogenic forms in C^{n+1}.

Let $E = \mathscr{C}_n$ be the complex Clifford algebra over R^n and consider the $\mathscr{C}_n$-valued differential operator $2D_{\bar{z}} = D_x + iD_y$ in C^{n+1}, where $D_x = \sum_{j=0}^{n} e_j \frac{\partial}{\partial x_j}$ and $D_y = \sum_{j=0}^{n} e_j \frac{\partial}{\partial y_j}$ (see [8]).

Notice that $D_{\bar{z}} = \sum_{j=0}^{n} e_j \frac{\partial}{\partial \bar{z}_j}$ and that an e_0-valued solution of $D_{\bar{z}} f = 0$ is a holomorphic function of several complex variables, i.e. $f \in \mathcal{O}(\Omega, C)$.

Furthermore, when $h \in \mathcal{O}(\Omega,\mathscr{C}_n)$ and when in $\mathcal{E}(\Omega,\mathscr{C}_n) = C_\infty(\Omega,\mathscr{C}_n)$, $(D_x+iD_y)f = g$; then also $(D_x+iD_y)(fh) = gh$. Hence the space $M(\Omega,\mathscr{C}_n)$ of solutions to $(D_x+iD_y)f = 0$ as well as the space $(D_x+iD_y)\mathcal{E}(\Omega,\mathscr{C}_n)$ are right modules over $\mathcal{O}(\Omega,\mathscr{C}_n)$. Especially as $(D_x+iD_y)f = 1$ admits solutions in $\mathcal{E}(C^{n+1},\mathscr{C}_n)$, $\mathcal{O}(\Omega,\mathscr{C}_n) \subset (D_x+iD_y)\mathcal{E}(\Omega,\mathscr{C}_n)$, i.e. for every $h \in \mathcal{O}(\Omega,\mathscr{C}_n)$, $(D_x+iD_y)(fh) = h$.

Definition 2.

An open subset $\Omega \subset C^{n+1}$ is called non isotropic, if Ω is P-convex with respect to the operator $P = \sum_{j=0}^{n} \left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)^2$.

Lemma 7.

Ω is non isotropic if and only if $D_{\bar{z}}: \mathcal{E}(\Omega,\mathcal{C}_n) \longrightarrow \mathcal{E}(\Omega,\mathcal{C}_n)$ is surjective.

Proof.

Consider the operator $D_{\bar{z}} = \frac{\partial}{\partial \bar{z}_0} - \sum_{j=1}^{n} e_j \frac{\partial}{\partial \bar{z}_j}$. Then as $4D_{\bar{z}}D_z = 4D_z D_{\bar{z}} = P$, the equation $D_{\bar{z}} f = g$ admits solutions $f \in \mathcal{E}(\Omega,\mathcal{C}_n)$ for every $g \in \mathcal{E}(\Omega,\mathcal{C}_n)$ if and only if the equation $Pf = g$ does. In view of [12], the stated lemma immediately follows from this.

Notice that, by Holmgren's theorem (see [12]), examples of non isotropic sets Ω are all convex sets in $\mathbb{C}^{n+1}$ and all sets Ω such that $\partial\Omega$ is a C_1-surface without characteristic points with respect to P , i.e. the normal vector field

$$k = \sum_{j=0}^{n} k_j e_j$$

to $\partial\Omega$ does not satisfy $k^- k = \sum_{j=0}^{n} k_j^2 = 0$ (this means that the vector field k is non isotropic at every point of $\partial\Omega$). Of course, intersections of non isotropic domains are still non isotropic.

We shall now apply our theory in a way which is similar to 3.1. Consider the Dolbeault complex

$$0 \longrightarrow \mathcal{O}^p(\Omega,\mathcal{C}_n) \xrightarrow{\ i\ } \mathcal{E}^{(p,0)}(\Omega,\mathcal{C}_n) \xrightarrow{\ \bar{\partial}\ } \dots \xrightarrow{\ \bar{\partial}\ } \mathcal{E}^{(p,n+1)}(\Omega,\mathcal{C}_n) \longrightarrow 0 \qquad (14)$$

Then we put

$$A_{-1} = \mathcal{O}^p(\Omega,\mathcal{C}_n) \ , \quad A_q = \mathcal{E}^{(p,q)}(\Omega,\mathcal{C}_n) \ , \ q=0,\dots,n+1 \ .$$

Furthermore consider the basic differential forms $d\bar{Z}_j = d\bar{z}_j - e_j d\bar{z}_0$; then we call $A_q^2 = \mathcal{E}_2^{(p,q)}$ the right $\mathcal{O}(\Omega,\mathcal{C}_n)$-module of all differential forms in $\mathcal{E}^{(p,q)}(\Omega,\mathcal{C}_n)$ of the form

$$f = \sum_{I,j_1<\dots<j_q} dz_I \wedge d\bar{Z}_{j_1} \wedge \dots \wedge d\bar{Z}_{j_q} f_{I,j_1,\dots,j_q} \ ,$$

whereas we put $A_q^1 = \mathcal{E}_1^{(p,q)} = d\bar{z}_0 \wedge \mathcal{E}_2^{(p,q-1)}$.

We have again $\mathcal{E}^{(p,q)}(\Omega,\mathcal{C}_n) = \mathcal{E}_1^{(p,q)} \oplus \mathcal{E}_2^{(p,q)}$ and we obtain a splitting similar to (7) . Similar to Lemma 4 we have

Lemma 8.

The operators $\tau,\bar{\tau},\delta,\bar{\delta}$ occuring in the splitting of the Dolbeault complex are given by $\tau = 0$, $\delta = \bar{\tau}$ and

$$\delta = \sum_{j=1}^{n} d\bar{Z}_j \frac{\partial}{\partial \bar{z}_j} \;; \qquad \bar{\delta} = d\bar{z}_0 D_{\bar{z}} = \tfrac{1}{2} d\bar{z}_0 (D_x + iD_y) \;.$$

Proof.

Use the identity $\bar{\partial} = \frac{1}{2} d\bar{z}_0 (D_x + iD_y) + \sum_{j=1}^{n} d\bar{Z}_j \frac{\partial}{\partial \bar{z}_j}$. ■

Notice that the operator $\bar{\partial}' = \pi^1 \circ \bar{\partial}$ is now given by $\frac{1}{2} d\bar{z}_0 (D_x + iD_y)$. Hence we have

Lemma 9.

Let Ω be non isotropic. Then the operator

$$\bar{\partial}' : E^{(p,q-1)} \longrightarrow E^{(p,q)}$$

is surjective.

Proof.

It is sufficient to prove that the operator $\bar{\delta} = \frac{1}{2} d\bar{z}_0 (D_x + iD_y)$ is surjective from $E_2^{(p,q-1)}$ onto $E_1^{(p,q)} = d\bar{z}_0 \wedge E_2^{(p,q-1)}$. By construction of $E_2^{(p,q-1)}$, this immediately leads to the equation $(D_x + iD_y) f = g$, $g \in C_\infty(\Omega)$, which, Ω being non isotropic, admits a a solution $f \in C_\infty(\Omega)$.

Hence, applying Theorem 2, the Dolbeault cohomology spaces $H^{(p,q)}(\Omega, \mathcal{C}_n)$, $q=0,\dots,n$ coincide with the homology spaces of the complex

$$0 \longrightarrow \mathcal{O}^{(p)}(\Omega, \mathcal{C}_n) \xrightarrow{\;i\;} M^{(p,0)}(\Omega, \mathcal{C}_n) \xrightarrow{\;\delta\;} \dots \longrightarrow M^{(p,n)}(\Omega, \mathcal{C}_n) \xrightarrow{\;\delta\;} 0 \;.$$

Notice that $f \in M^{(p,q)}(\Omega, \mathcal{C}_n)$ if and only if f is of the form

$$f = \sum_{I, j_1 < \dots < j_q} dz_I \wedge d\bar{Z}_{j_1} \wedge \dots \wedge d\bar{Z}_{j_q} \, f_{I, j_1, \dots, j_q} \,,$$

with $(D_x + iD_y) f_{I, j_1, \dots, j_q} = 0$ in Ω . These forms are called weak complex monogenic (p,q)-forms in Ω .

REFERENCES

[1] F.Brackx, R.Delanghe, F.Sommen: Clifford analysis, Research Notes in Math. 76, Pitman, London, 1982

[2] J. Bureš, V.Souček: Generalized Cauchy-Riemann equations on manifolds, Proc. of the 12th Winter School on abstract analysis, Srní in Suppl.Rend.Circolo Mat. Palermo, ser.II, 6, 1984, 31-42

[3] H.D.Fegan: Conformally invariant first order differential operators, Q. Jour. Math. 27, 1976, 371-378

[4] R.C.Gunning: Lectures on Riemann surfaces, Princeton, 1966

[5] D.Husemoller: Fibre bundles, McGraw Hill, 1966

[6] H.Samelson: Notes on Lie algebras, Van Nostrand, 1969

[7] F.Sommen: Monogenic differential forms and homology theory, to appear in Proc. Royal Irish Acad.

[8] F.Sommen: Clifford analysis in several complex variables, to appear

[9] V.Souček: H-valued differential forms on H, Proc. 11th Winter School on Abstract Analysis, in Suppl.Rend.Circ.Mat.Palermo, ser.II,33, 1984, 293-300

[10] V.Souček: The decomposition of spinor-valued differential forms, to appear

[11] E.M.Stein, G.Weiss: Generalization of the Cauchy-Riemann equations and representations of the rotation group, Amer.J.Math. 90, 1968, 163-196

[12] F. Trèves: Linear Partial Differential Equations with Constant Coefficients, Math. and Appl. 6, Gordon and Breach, 1966

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