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COMPLEX QUATERNIONS, THEIR CONNECTION TO TWISTOR THEORY*)

Vl. Souček

Institute of Mathematics, Charles University, Sokolovská 83, 180 00 Praha 8, Czechoslovakia

The projective space of complex quaternions is defined as a basic example of a new type of complex-quaternionic manifolds. It is shown that this manifold has a quite nonstandard topology and the relevance of it to the twistor correspondence is discussed.

1. INTRODUCTION

The use of complex quaternions in mathematical physics is far from being new, many relativistic notions have been naturally expressed in terms of complex quaternions [1 – 3]. Here we want to describe another field, where the space $\mathbb{C}\mathbb{H}$ of complex quaternions comes quite naturally into mathematical physics.

The twistor theory has been quite successful recently, many nice applications to various problems (for example Yang-Mills fields, instantons, monopoles) were found recently. To show the connection of the space $\mathbb{C}\mathbb{H}$ to the twistor theory we describe first briefly the basic twistor diagram, which expresses the basic twistor geometry lying behind twistor theory.

After a short discussion of the basic properties of the space $\mathbb{C}\mathbb{H}$ (§2.) we shall describe (§3.) the properties of the projective space of complex quaternions $\mathbb{P}^1(\mathbb{C}\mathbb{H})$ and its relevance to twistor theory.

Basic twistor diagram:

Let us consider 4-dimensional complex vector space $\mathbb{T}$. We shall use the flag manifolds of vector subspaces of $\mathbb{T}$ defined by

$$\mathbb{P}^3(\mathbb{C}) = \{L_1 \subset \mathbb{T} \mid \dim L_1 = 1\} ,$$

$$G_{2,4} = \{L_2 \subset \mathbb{T} \mid \dim L_2 = 2\} ,$$

$$\mathbb{F}_{1,2} = \{[L_1, L_2] \mid L_1 \subset L_2 \subset \mathbb{T}, \dim L_1 = 1, \dim L_2 = 2\} .$$

The Grasmanian $G_{2,4}$ can be considered to be the conformal compactification of the complex Minkowski space $\mathbb{C}M$. Set-valued maps φ, ψ defined using natural forgetting

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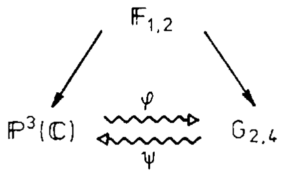
projections in the basic twistor diagram



$$\varphi(L_1) = \{L_2 \mid L_2 \supset L_1\} \sim P^2(\mathbb{C})$$
$$\psi(L_2) = \{L_1 \mid L_1 \subset L_2\} \sim P^1(\mathbb{C})$$

are fundamental maps giving the twistor correspondence. The images $\varphi(L_1)$ in the complex Minkowski space $\mathbb{C}M$ are usually called the α-planes.

2. THE RING $\mathbb{C}\mathbb{H}$

The space $\mathbb{C}\mathbb{H}$ of complex quaternions is the space of all numbers $q = q_0 + i_1 q_1 + i_2 q_2 + i_3 q_3$; $q_0, \dots, q_3 \in \mathbb{C}$; ($i_1, i_2, i_3$ are the quaternion units, the complex unit i in $\mathbb{C}$ is supposed to commute with all i_1, i_2, i_3). There are two conjugations in $\mathbb{C}\mathbb{H}$:

$$q \to q^+ = q_0 - i_1 q_1 - i_2 q_2 - i_3 q_3$$
$$q - q^* = q_0^* + i_1 q_1^* + i_2 q_2^* + i_3 q_3^*$$

and we shall denote $\|q\|^2 = q \cdot q^+ \in \mathbb{C}$.

The set $N = \{q \in \mathbb{C}\mathbb{H} \mid \|q\|^2 = 0\}$ is the set of points in $\mathbb{C}\mathbb{H}$ without an inverse. for $q \notin N$ we have $q^{-1} = q^+ \cdot \|q\|^{-2}$.

The space $\mathbb{C}\mathbb{H}$ is often identified with the complex Minkowski space $\mathbb{C}M$ under identification

$$x = [x_0, \dots, x_3] \in \mathbb{C}M \to q = x_0 + i i_1 x_1 + i i_2 x_2 + i i_3 x_3 .$$

It is easy to see that $x_\mu x^\mu = \|q\|^2$, the set N corresponds to the complex light cone in $\mathbb{C}M$.

The Lorentz group action on $\mathbb{C}\mathbb{H}$ is described by ([1])

complex Lorentz group: $q \to AqB$; $A, B \in \mathbb{C}\mathbb{H}$; $\|A\|^2 = \|B\|^2 = 1$,

real Lorentz group: $q \to AqA^{+*}$; $A \in \mathbb{C}\mathbb{H}$; $\|A\|^2 = 1$.

(Minkowski space M is invariant subspace).

Two interesting differential operators are defined on $\mathbb{C}\mathbb{H}$. They are an analytic extension of the two generalizations of Cauchy-Riemann equations to real quaternions.

(i) The function $f: \mathbb{H} \to \mathbb{H}$ is said to be differentiable, iff $\lim_{h \to 0} [f(q + h) - f(q)] \cdot h^{-1}$, $q \in \mathbb{H}$ exists. It can be shown ([3]) that only a linear function has this property. Such a function can be described as a solution of a differential operator equation $D_1 f = 0$.

(ii) The function f is said to be regular iff $D_2 f = 0$, where the operator $D_2 f = (\partial/\partial q_0 + i_1 \partial/\partial q_1 + i_2 \partial/\partial q_2 + i_3 \partial/\partial q_3) f$ was introduced by Fueter (for more details see [2]).

The differential operators D_1, D_2 can be extended to holomorphic mappings from $\mathbb{C}\mathbb{H}$ to $\mathbb{C}\mathbb{H}$. After restriction to real Minkowski space $M \subset \mathbb{C}M$ a nice physical interpretation can be given to those operators. The operator D_1 is identical with the Penrose twistor operator, the operator D_2 can be identified with the Weyl operator for massless fields. For more details see ([4]).

3. THE PROJECTIVE SPACE $\mathbb{P}^1(\mathbb{C}\mathbb{H})$

If the space $\mathbb{C}\mathbb{H}$ is a model for flat (complex) Minkowski space, it seems to be natural that some sort of manifold, modeled over open subsets of $\mathbb{C}\mathbb{H}$ should describe curved (complex) spacetimes; such types of spacetimes have been discussed very often recently [5, 6]. To begin we can try to study the simplest possible nontrivial example – the projective space. Let us compare the situation with complex and quaternion cases. If $\mathbb{C}$ is the model for Euclidean space $\mathbb{R}_2$, then $\mathbb{P}^1(\mathbb{C})$ is conformal compactification of $\mathbb{R}_2$, i.e. S_2; if $\mathbb{H}$ is the model for $\mathbb{R}_4$, then $\mathbb{P}^1(\mathbb{H})$ is conformal compactification of $\mathbb{R}_4$, i.e. S_4. It seems to be natural that $\mathbb{P}^1(\mathbb{C}\mathbb{H})$ should be a conformal compactification of $\mathbb{C}M$, which was shown ([6]) to be the Grasmanian $G_{2,4}$. But it is not the case, an unexpected suprise is hidden in the complex-quaternion version of the projective space.

Let us define $\mathbb{P}^1(\mathbb{C}\mathbb{H}) = [\mathbb{C}\mathbb{H} \times \mathbb{C}\mathbb{H}] \setminus [0, 0]/\sim$, where

$$[q_1, q_2] \sim [q'_1, q'_2] \Leftrightarrow \exists \lambda \in \mathbb{C}\mathbb{H}, \ \|\lambda\|^2 \neq 0 \,, \quad [q'_1, q'_2] = [q_1\lambda, q_2\lambda] \,.$$

The space $\mathbb{P}^1(\mathbb{C}\mathbb{H})$ is a topological space (with factor-topology). We can divide it into two parts

$$[\mathbb{C}\mathbb{H} \times \mathbb{C}\mathbb{H}] \setminus [0, 0] = B \cup C \,; \quad \mathbb{P}^1(\mathbb{C}\mathbb{H}) = (B/\sim) \cup (C/\sim)$$

and after some effort we find that

$$B/\sim \ \cong G_{2,4} \,, \quad C/\sim \ \cong \mathbb{P}^3(\mathbb{C})$$

so

$$\mathbb{P}^1(\mathbb{C}\mathbb{H}) \cong G_{2,4} \cup P^3(\mathbb{C}) \,.$$

The topology in the whole $\mathbb{P}^1(\mathbb{C}\mathbb{H})$ is a nonstandard one, it is not Hausdorff topology. We can prove the following facts on this topology:

1) For every $\beta \in B/\sim \ \cong G_{2,4}$ it holds that

$$\text{clos}\,(\beta) = \beta \cup \psi(\beta) \,.$$

2) For every $\gamma \in C/\sim\ \cong \mathbb{P}^3(\mathbb{C})$ it holds that

$$\bigcap_{\mathcal{O}\ \text{open},\ \gamma\in\mathcal{O}} \mathcal{O} = \gamma \cup \varphi(\gamma)\,.$$

If we restrict the topology only to $G_{2,4}$ or $\mathbb{P}^3(\mathbb{C})$, we shall recover the usual topology on them. So "strangeness" or the topology describes exactly the twistor correspondence φ, ψ between $G_{2,4}$ and $\mathbb{P}^3(\mathbb{C})$.

The nice example described above shows a possible way to a new, highly nonstandard type of a manifold modeled over open subsets of $\mathbb{P}^1$ $(\mathbb{C}\mathbb{H})$. It is possible that this approach can be useful for further investigation of curved (complex) spacetimes. Some objects living on such manifolds could give new insight into the twistor description of massless fields and other related questions.

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