A Quick Introduction to Functional Analysis, NMAF006

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Contents

Notation

$\overline{z}$, $\overline{u}$, $\overline{\Omega}$	complex conjugate to z, a solution close to u, closure of the set Ω
ℓ_2	the space of such sequences $\{x_n\} \subset \mathbb{C}$ that the series $\sum |x_n|^2$ converges.
$\mathcal{C}(\Omega)$, $\mathcal{C}^k(\Omega)$	the space of continuous functions (k times continuously differentiable functions) on the set Ω
$L^p(\Omega)$, $L^p_\rho(\Omega)$	(weighted) Lebesgue space on the set Ω (with weight ρ)
$W^{k,p}(\Omega)$, $W^{k,p}_\rho(\Omega)$	(weighted) Sobolev spaces on the set Ω (with weight ρ)
$\mathscr{L}(X,Y)$, $\mathscr{L}(X)$	the space of continuous linear operators between the spaces X, Y (on the space X)
$\mathscr{C}(X,Y)$, $\mathscr{C}(X)$	the space of compact linear operators between the spaces X, Y (on the space X)
$\mathcal{U}(x)$	neighborhood of a point x
$\mathcal{D}(\boldsymbol{T})$	domain of operator $\boldsymbol{T}$
$\mathcal{R}(\boldsymbol{T})$	the range of the operator $\boldsymbol{T}$
$\mathcal{M}^{m\times n}$	the set of all $m \times n$ dimensional matrices over $\mathbb{R}$ ($\mathbb{C}$)
$\boldsymbol{T} : X \to Y$	operator from X to Y (instead of $\boldsymbol{T}$ we will write $\boldsymbol{L}$ for the unbounded and $\boldsymbol{K}$ for the compact operator)
$\sigma(\boldsymbol{T})$, σ_{C}, σ_{P}, σ_{R}	spectrum of the operator $\boldsymbol{T}$; continuous, point and residual spectrum
$\big(f, g\big)$, $\langle f, \boldsymbol{T}\rangle$	scalar product, duality
$\mathrm{card}(B)$	the cardinality of the set B

1 Operator trivia

What we will consider to be known:

- *Vector space* X over $\underbrace{\mathbb{R} \text{ or } \mathbb{C}}_{\text{scalars}}$ (but there are also scalars from e.g. $\mathbb{Q}$). Where it does not matter whether it is $\mathbb{R}$ or $\mathbb{C}$, we will sometimes use the notation $\mathbb{K}$ (thus meaning "either $\mathbb{R}$ or $\mathbb{C}$"). In addition to the term "vector space" (VP), the term "linear space" (LP) or "linear vector space" (LVP) is also used.

- *Linearly independent* (LN) *set* in VP: $M \subseteq X$ is LN if $a_1x_1 + \cdots + a_nx_n = 0 \implies a_1 = a_2 = \cdots = a_n = 0$ for all possible tuples $(x_1, \ldots x_n) \subseteq M$ and all scalars $a_j \in \mathbb{K}$.

  Remark: Even if M is infinite, we only consider finite sums (ie, all possible "arbitrarily long but finite" sums). It is necessary to realize that the concept of convergence is not defined in general VP, and thus the very concept of infinite sum has no meaning in general VP.

  "Finite sums belong to algebra, infinite sums to analysis."

- *Base* X ($X \neq \emptyset$, $X \neq \{0\}$):

  1. If there exists a *finite* LN set B in X such that its linear envelope

     $$\mathrm{Lin}\,(B) := \Big\{ \sum_{j=1}^{n} a_j x_j,\ x_j \in B, a_j \in \mathbb{K}, n \in \mathbb{N} \Big\}$$

     is equal to X (we say that B *generates* X), then we will call such a set the *[vector] basis of* X. Its cardinalit is determined in a unique way (as it can be shown), this number is then called the *dimension of* X: $\dim X := \mathrm{card}(B) \in \mathbb{N}$.

  2. If $\forall n \in \mathbb{N}$ exists in X a LN set with n elements, we say that $\dim X = \infty$. In this case, the notion of basis is more strict: the basis of X over $\mathbb{K}$ is in this case such an infinite set B that satisfies:

     (a) B is LN (in the sense of all finite lin. combinations - see above).

     (b) $\forall x \in X\ \exists n(x) \in \mathbb{N}$ and the corresponding finite number of basis elements $x_1, \ldots, x_{n(x)}$ and coefficients $a_j \in \mathbb{K}$ such that

     $$x = \sum_{j=1}^{n(x)} a_j x_j \,.$$

     Remark: Here too, in principle, it is about finite sums of elements selected from an infinite set (for different x it can be about different sets of elements of the base). This infinite basis is called the *Hammel basis of* X *over* $\mathbb{K}$. The question is whether every VP X (which is not of finite dimension) has a Hammel basis. The answer *yes* is a consequence of the *axiom of choice* (for those who do not accept it, the answer would be *no*).

Exercise $\underbrace{\mathbb{R}}_{\text{VP}}$ over $\underbrace{\mathbb{R}}_{\text{scalars}}$ has dimension 1: $\forall x \in \mathbb{R} \exists a = x \in \mathbb{R}$ that $x = a \cdot 1$. So the base is $\{1\}$.

$\underbrace{\mathbb{R}^n}_{\text{VP}}$ over $\underbrace{\mathbb{R}}_{\text{scalars}}$ has dimension n .

$\underbrace{\mathbb{R}}_{\text{VP}}$ over $\underbrace{\mathbb{Q}}_{\text{scalars}}$ has dimension ∞: namely, every finite base of real numbers generates using a countable set coefficients (from $\mathbb{Q}$) only countably many elements, and $\mathbb{R}$ is uncountable.

This solves the problem of the dimension of $\mathbb{R}$ over $\mathbb{Q}$. Note that the dimension ∞ can be determined even without knowing the answer to the question of the existence of a basis, i.e. without the need for the axiom of choice. However, if we accept the axiom of choice, then there exists a Hammel basis $\mathbb{R}$ over $\mathbb{Q}$, i.e., $\exists B \subset \mathbb{R}$ such that B is an LN in the sense of the above definition of a $\forall x \in \mathbb{R}\ \exists n(x) \in \mathbb{N}\ \exists b_1, \ldots, b_{n(x)} \in B\ \exists q_1, \ldots, q_{n(x)} \in \mathbb{Q}$, that $x = \sum_{j=1}^{n(x)} q_j b_j$. Consider that B is necessarily uncountable (otherwise we will only generate countably many elements).

- *Norm on LP*: $\|\cdot\| : X \to [0, \infty)$, such that $\|x + y\| \leq \|x\| + \|y\|$,

$$\|ax\| = |a| \cdot \|x\|,$$
$$\|x\| = 0 \iff x = 0.$$

$(X, \|\cdot\|)$ is then a normed linear space (NLP). We can define a convergence in such a space:

$$x_n \xrightarrow{\|\cdot\|} x \iff \forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ \ \|x - x_n\| < \varepsilon$$

and therefore infinite sums aleso make sense. It is possible to define Cauchy sequences as well

$$\{x_n\} \text{ cauchy } \iff \forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall m, n \geq n_0\ \ \|x_m - x_n\| < \varepsilon$$

and *completeness of* X *in a norm*:

$$(X, \|\cdot\|) \text{ is complete in the norm } \|\cdot\| \iff (\{x_n\} \text{ cauchy } \implies \exists x \in X\ x_n \to x)\ .$$

If $(X, \|\cdot\|)$ is complete in the norm $\|\cdot\|$, we call it a *Banach* space (B-space).

- We also consider to be known that under the condition of $\dim X < \infty$, all norms on X are equivalent. Recall that norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, if $\exists c_1, c_2 > 0$, s.t. $c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1\ \forall x \in X$.

- Equivalent norms preserve the notion of convergence ($x_n \xrightarrow{\|\cdot\|_1} x \iff x_n \xrightarrow{\|\cdot\|_2} x$) and also the property of being cauchy (for a sequence), therefore also the notion of completeness of X is preserved in different norms. Especially, for all finite dimensional spaces X it holds that if it is complete in a norm $\|\cdot\|$, it is also complete in all possible norms on X.

  This does not hold in the case of an infinite dimension. For example $\mathcal{C}([-1,1])$ is complete in the supremum norm $\|f\|_\infty := \max_{[-1,1]} |f(x)|$, but it is not complete in the integral norm $\|f\|_1 := \int_{-1}^{1} |f|$. (consider $\operatorname{arctg} nx \to \frac{\pi}{2} \operatorname{sgn} x$ in $\|\cdot\|_1$).

- *Scalar product on LP*: $(\cdot,\cdot) : X \times X \to \mathbb{K}$ (therefore, if X is over $\mathbb{C}$, scalar product has complex values), is a mapping such that $\forall x, y, z \in X$ we have:

$$(x, y) = \overline{(y, x)},$$
$$(x + y, z) = (x, z) + (y, z),$$
$$(x, x) \geq 0, \text{ and } (x, x) = 0 \iff x = 0,$$
$$(\alpha x, y) = \alpha\,(x, y)\ \ \forall \alpha \in \mathbb{K}.$$

- A space $\big(X, (\cdot,\cdot)\big)$ endowed with a scalar product is called a space with a scalar product, sometimes also a *unitary space*.
- It is easy to show that $\|x\| = \sqrt{(x,x)}$ has all the properties of a norm, and therefore:
  X is unitary $\implies$ X je NLP (in the so-called "norm generated by a scalar product")
- If X is *complete in the norm generated by scalar product*, we call it a *Hilbert space* (H-space), and therefore
  X Hilbert s. $\implies$ X Banach s.
- For any unitary space X we have the *Cauchy-Schwarz inequality*:

$$\forall x, y \in X: \qquad |\big(x,y\big)| \leq \|x\| \cdot \|y\|, \qquad \text{kde } \|x\| = \sqrt{(x,x)}.$$

- If X is unitary, we say that $x, y \in X \setminus \{0\}$ are *perpendicular* in X, if $\big(x,y\big) = 0$. We denote this by $x \perp y$.

Exercise L^2, L^2_ρ, ℓ_2, $W^{1,2}$, $W^{k,2}$, $W^{k,2}_\rho$ are Hilbert spaces.
$\mathcal{C}([a,b])$, L^p for $p \neq 2$ are Banach and not Hilbert spaces.

The existence of a norm (of the scalar product, or of the metric) defines the so-called geometric properties (distance, convergence, and perpendicularity) on the LP.

We now recall various concepts and properties related to mappings on vector spaces.

1. Let X, Y be LP (ie, we don't need geometry). We say, that the mapping is
   - *operator*: $\boldsymbol{T} : X \to Y$
   - *functional*: $\boldsymbol{T} : X \to \mathbb{K}$

   Every functional is also an operator. So we will WLOG define properties for operators only.
2. Let X, Y be an LP. The operator $\boldsymbol{T} : X \to Y$ is
   - *linear*: $\boldsymbol{T}(ax + by) = a\boldsymbol{T}(x) + b\boldsymbol{T}(y) \qquad \forall x, y \in X \ \forall a, b \in \mathbb{K}$
   - *nonlinear*: not linear

   Remark: from the linearity of $\boldsymbol{T}$ it follows that $\boldsymbol{T}(0) = 0$ (choose $a = b = 0$).
3. X, Y NLP (here, in addition to the linearity of spaces, we also need to have a norm), $\boldsymbol{T} : X \to Y$ is linear
   - *bounded*: $\forall K > 0 \ \exists C > 0 \ \ \|x\|_X \leq K \implies \|\boldsymbol{T}x\|_Y \leq C$ (showing "bounded sets to bounded sets"), equivalent to $\exists C > 0 \ \forall x \in X \ \ \|\boldsymbol{T}x\|_Y \leq C \|x\|_X$.
   - *unbounded*: not bounded, i.e. $\exists K > 0 \ \forall C > 0 \ \exists x_C \in X \ \ \|x_C\| \leq K \wedge \|\boldsymbol{T}x\| > C$
4. Be X, Y NLP, $T : X \to Y$ is
   - *continuous*: $x_n \to x \implies \boldsymbol{T}x_n \to \boldsymbol{T}x$ (the so-called "Heine definition")
   - *discontinuous*: not continuous

Next, we will consider only Banach (or Hilbert) spaces, i.e. we will always have completeness.

Definition 1.1 (Operator Norm) Let us have a linear operator $\boldsymbol{T} : X \to Y$. We define a number

$$\|\boldsymbol{T}\|_{\mathscr{L}(X,Y)} := \sup_{\|x\|_X \leq 1} \|\boldsymbol{T}x\|_Y .$$

This number can also be infinity (e.g. for some unbounded operator). For lin. however, we see the operator:

$$x \neq 0 \implies \left\| \boldsymbol{T}\Big(\frac{x}{\|x\|_X}\Big) \right\|_Y \leq \|\boldsymbol{T}\|_{\mathscr{L}(X,Y)} \qquad (\|\boldsymbol{T}\| \text{ is the supremum of such})$$

$$\|\boldsymbol{T}x\|_Y \leq \|\boldsymbol{T}\|_{\mathscr{L}(X,Y)} \cdot \|x\|_X$$

The last inequality also applies to $\|\boldsymbol{T}\| = \infty$, $\forall x \neq 0$. If $\|\boldsymbol{T}\| < \infty$, then both sides are finite and the inequality holds $\forall x \in X$ (ie, including $x = 0$).

Remark We let $\|\boldsymbol{T}\| = \infty$ to have this equivalence:

Lemma 1.2 (On the characterization of boundedness) For lin. operators we have:

$$\boldsymbol{T} \text{ bounded} \iff \|\boldsymbol{T}\| < \infty$$

and in that case $\|\boldsymbol{T}\|$ has properties of a norm (verify).

Proof. "$\Rightarrow$" implication: $\boldsymbol{T}$ bounded: (choose $K = 1: \ \exists C \ \forall \|x\| \leq 1 \ \|\boldsymbol{T}x\| \leq C) \implies \|\boldsymbol{T}\| \leq C$.

"$\Leftarrow$" implication: $\|\boldsymbol{T}\| < \infty$: $\|\boldsymbol{T}x\| \leq \|\boldsymbol{T}\| \cdot \|x\| \ \forall x \in X$, i.e., $\|T\| \leq K \implies \|\boldsymbol{T}x\| \leq K\|x\|$. ☐

Lemma 1.3 (On the equivalence of operator continuity and boundedness) Let X, Y be Banach. If $\boldsymbol{T} : X \to Y$ is a *linear* operator, then

$$\overset{(1)}{\boldsymbol{T} \text{ bounded}} \iff \overset{(2)}{\boldsymbol{T} \text{ continuous}} \iff \overset{(3)}{\|\boldsymbol{T}\| < \infty}.$$

Proof. We have already proved the equivalence of (1) and (3). We will show:

(1) $\Rightarrow$ (2): If $\boldsymbol{T}$ is bounded, then $\exists C > 0$, $\|\boldsymbol{T}(x_n - x)\| \leq C\|x_n - x\|$. From the linearity of $\boldsymbol{T}$ then follows $\|\boldsymbol{T}x_n - \boldsymbol{T}x\| \leq C\|x_n - x\|$. Now if $x_n \to x$, then from here we have $\boldsymbol{T}x_n \to \boldsymbol{T}x$ and thus $\boldsymbol{T}$ is continuous.

(2) $\Rightarrow$ (1): The continuity implies, among other things, that if $x_n \to 0$, then $\boldsymbol{T}x_n \to 0$. Therefore $\forall \varepsilon$ (e.g. for a fixed $\varepsilon = 1$) $\exists \delta > 0$ that $\|x_n\| < \delta \Rightarrow \|\boldsymbol{T}x_n\| < \varepsilon = 1$. Let $\|x\| < K$ now, then $\left\|\frac{x}{K}\delta\right\| < \delta \Rightarrow \left\|\frac{\boldsymbol{T}x}{K}\delta\right\| < 1 \implies \|\boldsymbol{T}x\| < \frac{K}{\delta} =: C$. ☐

Definition 1.4 (Space of bounded linear operators)

$$\mathscr{L}(X,Y) \coloneqq \{\boldsymbol{T} : X \to Y, \ X, Y \text{ NLP}, \ \boldsymbol{T} \text{ linear constrained}\}.$$

It can be shown that $\mathscr{L}(X,Y)$ itself is an NLP with the norm $\|\cdot\|_{\mathscr{L}(X,Y)}$ defined by Definition 1.1. Moreover, if Y is Banach, then also $\mathscr{L}(X,Y)$ is complete in the norm $\|\cdot\|_{\mathscr{L}(X,Y)}$ and thus Banach. In particular, the operator spaces $\mathscr{L}(X,\mathbb{R})$, $\mathscr{L}(X,\mathbb{C})$ are always Banach.
Other notations used:

$$\mathscr{L}'(X) \coloneqq \mathscr{L}(X,\mathbb{K}),$$
$$\mathscr{L}(X) \coloneqq \mathscr{L}(X,X).$$

Theorem 1.5 (Effect of finite and infinite dimensions) *Let $\boldsymbol{T} : X \to Y$, X, Y Banach, T linear,* $\dim X = n \in \mathbb{N}$. *Then $\boldsymbol{T}$ is bounded and therefore continuous.*

Proof. Let us choose a fixed basis $\{x_j\}_{j=1}^n$ in the space X. Then

$$x \in X \quad \implies \quad x = \sum_{j=1}^n \alpha_j x_j, \quad \implies \quad \boldsymbol{T}x = \sum_{j=1}^n \alpha_j T x_j$$

$$\|\boldsymbol{T}x\| \le \sum_{j=1}^{n} |\alpha_j| \, \|\boldsymbol{T}x_j\| \le \underbrace{\max_{j=1,\dots n} \|\boldsymbol{T}x_j\|}_{C} \cdot \sum_{j=1}^{n} |\alpha_j| = c \, \|x\|_1 \le \tilde{c} \, \|x\|$$

where $\|a\|_1 = \sum |a_j|$ is the so-called *Manhattan norm*, one of the possible norms of X, and the last inequality follows from the equivalence of all norms for $\dim X < \infty$. □

Remark In the finite dimension, all linear operators are already continuous. A natural question arises: is this also true for $\dim X = \infty$? The answer is No: X, Y NLP, $\dim X = \infty$, then $\exists \boldsymbol{T} : X \to Y$ linear and unbounded.

Exercise $X = \mathcal{C}^1([a,b])$ with norm $\|f\| = \sup\limits_{[a,b]} |f(x)|$. (In this norm, X is not complete, why?)
$Y = \mathcal{C}([a,b])$ with the same norm. (Y is Banach, why?)

Let now $f_n(x) = \sin nx, \;\; f_n \in X, \;\; \|f_n\| = 1. \qquad f_n'(x) = n \cos nx, \;\; f_n' \in Y, \;\; \|f_n'\| = n.$
A bounded set is mapped to an unbounded set $\Longrightarrow$ the differentiation operator is unbounded.

Exercise Let $\dim X = \infty$, Y Banach. Take $\{x_1, x_2, \dots\}$ LN a countably infinite set of nonzero elements in X. WLOG $\|x_j\| = 1$ (Otherwise we take $\frac{x_j}{\|x_j\|}$ instead.)
According to the so-called *Zorn's lemma* (it is equivalent to the axiom of choice), every LN set can be enlarged to became a basis. So let's supplement it with elements $\{z_\alpha\}_{\alpha \in A}$, A is the so-called *index set*.
Then according to the properties of the base $B \coloneqq \{x_j\}_{j=1}^{\infty} \cup \{z_\alpha\}_{\alpha \in A}$ it holds $\forall x \in X \; \exists n(x), m(x) \in \mathbb{N} \; \exists a_j, b_j$ scalars, that

$$x = \sum_{j=1}^{n(x)} a_j x_j + \sum_{k=1}^{m(x)} b_k z_k.$$

We define $\boldsymbol{T}x \coloneqq \sum\limits_{j=1}^{n(x)} a_j \boldsymbol{T} x_j + \sum\limits_{k=1}^{m(x)} b_k \boldsymbol{T} z_k$ for such that $x \in X$.

This defines $\boldsymbol{T}$ on all of X if we define Tx_j and Tz_α. We define them as follows:

$$Tx_j = j \;\; \forall j \in \mathbb{N}$$
$$Tz_\alpha = 0 \;\; \forall z_\alpha, \; \alpha \in A$$

Then $\boldsymbol{T}$ is linear on X (verify) with $\|x_n\| = 1$ but $\|\boldsymbol{T}x_n\| = n \;\; \forall x_n$.

2 Fundamentals of spectral analysis

2.1 Motivation: solving one ODR

Exercise Consider an ordinary differential equation with an initial condition:

$$\begin{aligned} y'' + y &= f(x) \quad \text{ on } (0,a) \text{ for } a>0, \\ y(0) &= 1, \\ y'(0) &= 0, \end{aligned} \tag{2.1}$$

where $f \in C([0,a])$.

The solution of this problem for $f \equiv 0$ is $y_H = \cos x$, as we can easily find, for example, by the characteristic polynomial method. To find one (particular) solution of the equation with the right-hand side $f(x)$, we can use, for example, the method of variation of constants. From the ansatz of

$$y_P = c_1(x)\cos x + c_2(x)\sin x \tag{2.2}$$

we get equations for $c_1(x)$, $c_2(x)$

$$\begin{aligned} c_1'\cos x + c_2'\sin x &= 0 \\ -c_1'\sin x + c_2'\cos x &= f(x), \end{aligned} \tag{2.3}$$

and it follows from

$$\begin{aligned} c_1' &= -f\sin x, \\ c_2' &= f\cos x. \end{aligned} \tag{2.4}$$

Thus, the functions $c_1(x) = -\int_0^x f(t)\sin t \,\mathrm{d}t$, $c_2(x) = \int_0^x f(t)\cos t\,\mathrm{d}t$ solve the equation (2.3).[1]

We are getting

$$y_P = -\cos x\int_0^x f(t)\sin t\,\mathrm{d}t + \sin x\int_0^x f(t)\cos t\,\mathrm{d}t = \int_0^x f(t)\left(\sin x\cos t - \cos x\sin t\right)\,\mathrm{d}t = \int_0^x f(t)\sin(x-t)\,\mathrm{d}t, \tag{2.5}$$

so finally

$$y(x) = y_H + y_P = \cos x + \int_0^x f(t)\sin(x-t)\,\mathrm{d}t. \tag{2.6}$$

By substituting, one can make sure that the function $y(x)$ given by the prescription (2.6) is a solution to the problem (2.1).

Remark When substituting (2.6) into (2.1), the following lemma may come in handy, which allows the integral to be derived both according to the parameter and according to the limits.

Lemma 2.1 (On the differentiation of the integral according to the parameter and limits) Let $a, b \in \mathcal{C}^1((\alpha,\beta))$, $a((\alpha,\beta)) \subset (A,B)$, $b((\alpha,\beta)) \subset (A,B)$, $g \in \mathcal{C}^1((\alpha,\beta)\times(A,B))$. Further, let the functions $a, b, g, \frac{\partial g}{\partial x}$ be bounded on their domains. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} g(x,t)\,\mathrm{d}t = \int_{a(x)}^{b(x)} \frac{\partial g}{\partial x}(x,t)\,\mathrm{d}t + g(b(x))\,b'(x) - g(a(x))\,a'(x) \quad \text{for each } x\in(\alpha,\beta). \tag{2.7}$$

[1] Note: We could of course choose for c_1 or c_2 other primitive functions to $-f(x)\sin x$ or $f(x)\cos x$ (differing only by a constant). However, this choice guarantees that y_p satisfies the initial conditions.

Proof. Since g is continuous in the second variable, there exists $G \in \mathcal{C}^1((\alpha, \beta) \times (A, B))$ such that

$$\frac{\partial G}{\partial t}\,(x,t) = g(x,t), \quad (x,t) \in (\alpha,\beta) \times (A,B). \tag{2.8}$$

According to the Newton-Leibniz formula, one can write

$$\int\limits_{a(x)}^{b(x)} g(x,t)\ \mathrm{d}t = G(x,b(x)) - G(x,a(x)) \quad \Big/ \frac{\mathrm{d}}{\mathrm{d}x}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \int\limits_{a(x)}^{b(x)} g(x,t)\ \mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\,(G(x,b(x)) - G(x,a(x))) = \\ &= \underbrace{\frac{\partial G}{\partial x}\,(x,b(x)) - \frac{\partial G}{\partial x}\,(x,a(x))}_{\text{derivation only according to the 1st variable}} + \underbrace{\frac{\partial G}{\partial t}\,(x,b(x))}_{\text{according to (2.8)}}\,b'(x) - \underbrace{\frac{\partial G}{\partial t}\,(x,a(x))}_{\text{according to (2.8)}}\,a'(x), \end{aligned} \tag{2.9}$$

while in the last equality the first two terms are derivatives only according to the 1st variable, the third term is $g(x,b(x))$ and the fourth $g(x,a(x))$ according to (2.8).

The proof is completed by verifying the equality

$$\frac{\partial G}{\partial x}\,(x,c) - \frac{\partial G}{\partial x}\,(x,d) = \int\limits_{d}^{c} \frac{\partial g}{\partial x}\,(x,t)\ \mathrm{d}t. \tag{2.10}$$

Indeed, if $G(x,t)$ is primitive to $g(x,t)$ in the variable t, then $\frac{\partial G}{\partial x}\,(x,t)$ is primitive to $\frac{\partial g}{\partial x}\,(x,t)$ in the variable t under the given assumptions. We leave the details to the reader as a simple exercise. $\square$

Let us now consider the modification of the task (2.1):

$$\begin{aligned} y''(x) + y(x) &= f(x)y(x) \quad \text{on } (0,a) \text{ for } a > 0, \\ y(0) &= 1, \\ y'(0) &= 0. \end{aligned} \tag{2.11}$$

So we have a kind of "feedback" in the term on the right side of the equation. Only on the basis of analogy with the first task, we dare to make the following statement:

If there exists a function $y \in \mathcal{C}\,([0,a])$ that satisfies the relation

$$y(x) = \cos x + \int\limits_0^x f(t)\sin(x-t)y(t)\ \mathrm{d}t, \tag{2.12}$$

then this function is already an element of $\mathcal{C}^1\,([0,a])$ and solves the (2.11).

We verify this statement using lemma (2.1). First of all, if $y \in \mathcal{C}\,([0,a])$, the integrand in the equation (2.12) is continuous, i.e. is $y \in \mathcal{C}^1([0,a])$ and we have

$$y'(x) = -\sin x + \int\limits_0^x f(t)\cos(x-t)y(t)\ \mathrm{d}t + 0, \tag{2.13}$$

from which by the same reasoning we have $y'(x) \in \mathcal{C}^1([0,a])$, i.e. $y \in \mathcal{C}^2([0,a])$, and

$$y''(x) = -\cos x - \int\limits_0^x f(x)\sin(x-t)y(t)\ \mathrm{d}t + f(x)y(x). \tag{2.14}$$

From the last three relations, we get $y'' + y = f(x)y(x)$, as well as $y(0) = 1$, $y'(0) = 0$.

So we verified that

$$\boxed{\text{If there exists } y \in \mathcal{C}([0,a]) \text{ such that (2.12) holds, this function is a solution to the problem (2.11).}}$$

We have not yet solved the problem (2.11), we have only reformulated it. However, we will show that with a suitable interpretation of this reformulation, we will be able to answer the question of the existence (and uniqueness) of the solution.

Let's break it down one more time

$$y(x) = \underbrace{\cos x}_{=:u(x)} + \int_0^x \sin(x-t) f(t) y(t) \, \mathrm{d}t \tag{2.15}$$

and denote $K(x,t) = \sin(x-t)f(t)$ as the so-called integration factor, i.e.

$$y(x) = u(x) + \int_0^x K(x,t) y(t) \, \mathrm{d}t \,, \tag{2.16}$$

which is a reformulation of the problem (2.11) into a more general integral equation (known in mathematical literature as Volterra's equation of the second kind).

However, we now turn to an even more general formulation. We will denote

$$\boldsymbol{T}y(x) := \int_0^x K(x,t)y(t) \, \mathrm{d}t = \int_0^x \sin(x-t) f(t) y(t) \, \mathrm{d}t \,. \tag{2.17}$$

The mapping $\boldsymbol{T} : \mathcal{C}\left([0,a]\right) \to \mathcal{C}\left([0,a]\right)$ is (obviously) a linear operator. The problem of finding a solution (2.11), or (2.16) can then be understood as a search for a solution to an operator equation

$$y = u + \boldsymbol{T}y \tag{2.18}$$

on the Banach space $\mathcal{C}([0,a])$. This equation can also be written in the form

$$(\boldsymbol{Id} - \boldsymbol{T})y = u, \tag{2.19}$$

where $\boldsymbol{Id}$ is the identity operator on $\mathcal{C}([0,a])$. If we showed the existence of the "inverse operator to $\boldsymbol{Id} - \boldsymbol{T}$", we could write

$$y = (\boldsymbol{Id} - \boldsymbol{T})^{-1} u \tag{2.20}$$

and thereby solved the assigned task.

Formulating the problem in the form of the equation (2.20) leads us to the following questions:

- What are the properties of the operator $\boldsymbol{T}$ from (2.20)?
- Under what conditions does the operator inverse to $\boldsymbol{Id} - \boldsymbol{T}$ exist and what are its properties?
- Is y introduced using the equation (2.20) really a solution to our problem?

Let's answer the first question first. We show that the operator $\boldsymbol{T}$ is linear and bounded, so it is continuous on the space $\mathcal{C}([0,a])$, i.e. $\boldsymbol{T} \in \mathscr{L}(\mathcal{C}\left([0,a]\right))$.

Recall the standard norm for $\mathcal{C}([0,a])$: $\|y\|_{\mathcal{C}([0,a])} = \sup\limits_{[0,a]}|y(x)| \quad \big(=: \|y\|_\infty \big)$.

Proof. Linearity is obvious, for boundedness we determine it first

$$||\boldsymbol{T}y||_\infty = \sup_{x\in[0,a]} \Big| \int_0^x \sin(x-t) f(t) y(t) \, \mathrm{d}t \Big| \le \sup_{x\in[0,a]} \int_0^x |f(t)||y(t)| \, \mathrm{d}t \le a \, \|f\|_\infty \, \|y\|_\infty \,, \tag{2.21}$$

where $\|f\|_\infty = \max\limits_{[0,a]}|f(x)|$ and $\|y\|_\infty = \max\limits_{[0,a]}|y(x)|$. We see

$$\|\boldsymbol{T}\|_{\mathscr{L}(\mathcal{C}([0,a]))} = \sup_{\|y\|_\infty \le 1} ||\boldsymbol{T}y||_\infty \le \sup_{\|y\|_\infty \le 1} a\,\|f\|_\infty\,\|y\|_\infty \le a\,\|f\|_\infty < \infty, \tag{2.22}$$

thus $\boldsymbol{T}$ is a bounded operator (if the interval $[0,a]$ is bounded) . $\square$

We have the following general theorem prepared to answer other questions. Note that it is not too general. Basically, it is a description of our task in the operator version.

Theorem 2.2 (On the von Neumann operator series) *Let X be a Banach space, $\boldsymbol{T} \in \mathscr{L}(X)$. Let us define $\boldsymbol{T}^0 \equiv \boldsymbol{Id}$, $\boldsymbol{T}^{j+1}y = \boldsymbol{T}(\boldsymbol{T}^j y)$ the so-called iterated operator. Furthermore, let at least one of the following three conditions be satisfied:*

a) $||\boldsymbol{T}||_{\mathscr{L}(X)} < 1$,

b) $\sum\limits_{j=0}^{\infty} ||\boldsymbol{T}^j||_{\mathscr{L}(X)} < \infty$,

c) $\sum\limits_{j=0}^{\infty} ||\boldsymbol{T}^j y||_X < \infty \ \ \forall y \in X$.

Then

1. *$\forall u \in X$ there exists a unique $y \in X$ such that $(\boldsymbol{Id} - \boldsymbol{T})y = u$.*
2. *If we define the mapping "$u \mapsto y$" from the previous point and denote it $(\boldsymbol{Id} - \boldsymbol{T})^{-1}$, the following applies:*

$$(\boldsymbol{Id} - \boldsymbol{T})^{-1} \circ (\boldsymbol{Id} - \boldsymbol{T}) = (\boldsymbol{Id} - \boldsymbol{T}) \circ (\boldsymbol{Id} - \boldsymbol{T})^{-1} = \boldsymbol{Id}, \tag{2.23}$$

and

$$(\boldsymbol{Id} - \boldsymbol{T})^{-1} = \sum_{j=0}^{\infty} \boldsymbol{T}^j, \tag{2.24}$$

where we understand the sum sum $\sum\limits_{j=0}^{\infty}$ as $\lim\limits_{n\to\infty} \sum\limits_{j=0}^{n}$ in the sense of convergence in $\mathscr{L}(X)$.

Remark

1. The series (2.24) is called the von Neumann series of the operator $\boldsymbol{T}$
2. In the following we show that a)⇒b)⇒c).

   We have $||\boldsymbol{T}^2 y||_X = ||\boldsymbol{T}(\boldsymbol{T}y)||_X \le ||\boldsymbol{T}||_{\mathscr{L}(X)} ||\boldsymbol{T}y||_X \le ||\boldsymbol{T}||^2_{\mathscr{L}(X)}\,\|y\|_X$, whence $||\boldsymbol{T}^2|| = \sup\limits_{\|y\|_X \le 1} ||\boldsymbol{T}^2 y||_X \le ||\boldsymbol{T}||^2$ and easily by induction

$$||\boldsymbol{T}^j||_{\mathscr{L}(X)} \le ||\boldsymbol{T}||^j_{\mathscr{L}(X)}. \tag{2.25}$$

Thus, if a) holds, $\sum_{j=0}^n ||\boldsymbol{T}^j|| \le \sum_{j=0}^n ||\boldsymbol{T}||^j \le \sum_{j=0}^\infty ||\boldsymbol{T}||^j < \infty$ and the limit transition $n \to \infty$ on the left gives b). If b) holds, then $\sum_{j=0}^n ||\boldsymbol{T}^j y|| \le \|y\| \sum_{j=0}^n ||\boldsymbol{T}^j|| \le \|y\| \sum_{j=0}^\infty ||\boldsymbol{T}^j|| < \infty$, whence c).

Indeed, then a)⇒b)⇒c) and it will suffice to show that condition c) implies the assertion of this Theorem.[2]

Before we prove the theorem, let's make sure that the operator $\boldsymbol{T}$ defined in (2.17) satisfies its assumptions: $\mathcal{C}([0,a])$ is a Banach space and $\boldsymbol{T} \in \mathscr{L}(\mathcal{C}([0,a]))$. In addition, we showed in (2.22) that $||\boldsymbol{T}||_{\mathscr{L}} \le a\,\|f\|_\infty$.

From here we immediately get that for each $\|f\|_\infty < \infty$ there exists such $a > 0$ such that $a\|f\|_\infty < 1$ and therefore $||\boldsymbol{T}|| < 1$. From the assertion of the theorem, we then get the "existence and uniqueness" of the solution to the problem (2.12), and therefore a solution of (2.11) on the corresponding *truncated* interval $[0,a]$ so that

[2]However, we are grateful to have three different conditions: different operators can satisfy a), b), or c), see below.

$a\,\|f\|_\infty < 1$. This is a typical representative of the so-called "local" existence theorems of a solution of a differential equation. The disadvantage of this statement is that this solution existence interval depends on the size of the right-hand side of f.

At the same time, this observation will also serve as a lesson for us. We now show that $\boldsymbol{T}$ satisfies condition (b) without any requirements on the size of a. We just should estimate more carefully.

$$|\boldsymbol{T}y(x)| \leq \int_0^x |f(t)||y(t)|\,\mathrm{d}t \leq x\,\|f\|_\infty\,\|y\|_\infty\,,$$

where $\|f\|_\infty = \max_{[0,x]}|f(t)|$ and $\|y\|_\infty = \max_{[0,x]}|y(t)|$. Note that we are not looking for $\sup_{x\in[0,a]}$ yet. Further

$$|\boldsymbol{T}^2y(x)| \leq \int_0^x |f(t)||\boldsymbol{T}y(t)|\,\mathrm{d}t \leq \|f\|_\infty^2\,\|y\|_\infty \int_0^x t\,\mathrm{d}t = \frac{x^2}{2}\,\|f\|_\infty^2\,\|y\|_\infty\,,$$

whence we get by induction

$$|\boldsymbol{T}^jy(x)| \leq \frac{x^j}{j!}\,\|f\|_\infty^j\,\|y\|_\infty\,.$$

Now we find the supremum over x and get

$$\left\|\boldsymbol{T}^jy\right\| \leq \frac{a^j}{j!}\,\|f\|_\infty^j\,\|y\|_\infty\,,$$

and thus

$$\left\|\boldsymbol{T}^j\right\| = \sup_{\|y\|_\infty\leq 1}\left\|\boldsymbol{T}^jy\right\| \leq \frac{a^j}{j!}\,\|f\|_\infty^j\,.$$

From here

$$\sum_{j=0}^{\infty}\left\|\boldsymbol{T}^j\right\| \leq \exp(a\,\|f\|_\infty) < \infty.$$

Therefore, condition (b) is fulfilled and we have come to the conclusion that if we prove Theorem 2.2, we have shown at the same time the existence and uniqueness of the (classical) solution of the problem (2.11) for any (but bounded) interval $[0,a]$, and for any $f \in \mathcal{C}([0,a])$.

Proof of Theorem 2.2. According to point 2 of the previous remark, it suffices to show that the statement of the theorem follows from assumption c).

Let's define the following sequence of elements $y_n \in X$ (so-called "iterative process")

$$y_0 \in X \text{ be any element of } X$$
$$y_{n+1} := u + \boldsymbol{T}y_n.$$

We have

$$y_1 = u + \boldsymbol{T}y_0$$
$$y_2 = u + \boldsymbol{T}y_1 = u + \boldsymbol{T}u + \boldsymbol{T}^2y_0,$$

induction shows easily

$$y_n = \sum_{j=0}^{n-1}\boldsymbol{T}^ju + \boldsymbol{T}^ny_0. \tag{2.26}$$

We show that the sequence y_n has a limit in X. Since X is Banach and therefore complete, it suffices for the convergence of y_n to show that $\{y_n\}$ is a Cauchy sequence. So let's choose $\epsilon > 0$, consider $n > m$ and do the math

$$y_n - y_m = \sum_{j=m}^{n-1}\boldsymbol{T}^ju + \boldsymbol{T}^ny_0 - \boldsymbol{T}^my_0,$$

so

$$||y_n - y_m|| \le \sum_{j=m}^{n-1} ||\boldsymbol{T}^j u|| + ||\boldsymbol{T}^n y_0|| + ||\boldsymbol{T}^m y_0||.$$

Since condition c) holds, the first term is smaller than ϵ for sufficiently large $n > m$. Likewise the members of $||\boldsymbol{T}^n y_n||$, $||\boldsymbol{T}^m y_0||$ are (as the nth or mth term of a convergent series of the form c)) smaller than ϵ for sufficiently large n, m.

Thus, the sequence $\{y_n\}$ is Cauchy in the Banach space X, therefore it is convergent in X, so there exists $y \in X$ such that $y_n \xrightarrow{X} y$. Since $\boldsymbol{T}$ is continuous, $\boldsymbol{T}y_n \xrightarrow{X} \boldsymbol{T}y$ is also

$$\begin{array}{ccl} y_{n+1} & = u + & \boldsymbol{T}y_n \\ \downarrow & & \quad\downarrow \\ y & = u + & \boldsymbol{T}y \end{array}$$

and y is a solution to the equation $y = u + \boldsymbol{T}y$ (for any $u \in X$). We will show the uniqueness of this solution with an argument. Let there be two solutions, y and z, so let hold

$$y = u + \boldsymbol{T}y$$
$$z = u + \boldsymbol{T}z.$$

By subtracting these equations and labeling $w = y - z$, we obtain the relation $w = \boldsymbol{T}w$.

From there, however, $w = \boldsymbol{T}w = \boldsymbol{T}^2 w = \cdots = \boldsymbol{T}^j w \;\; \forall j \in \mathbb{N}$. So $||w|| = ||\boldsymbol{T}^j w|| \;\; \forall j \in \mathbb{N}$. The series $\sum_{j=0}^{\infty} ||\boldsymbol{T}^j w||$ is, however, a convergent series of type c), i.e

$$||w|| = \lim_{j\to\infty} ||\boldsymbol{T}^j w|| = 0,$$

whence $w = 0$ and therefore $y = z$.

Thus, the problem $y = u + \boldsymbol{T}y$ has $\forall u \in X$ exactly one solution $y \in X$. In other words: We know

$$\left.\begin{array}{l} \boldsymbol{Id} - \boldsymbol{T} \text{ is linear and continuous} \\ \forall u \in X \;\; \exists! y \in X, (\boldsymbol{Id} - \boldsymbol{T})y = u \end{array}\right\} = \boldsymbol{Id} - \boldsymbol{T} \text{ is both } \textit{injective} \text{ and } \textit{surjective}$$

Thus, the mapping $u \mapsto y$ is a well-defined mapping from X to X. Let us denote it by $(\boldsymbol{Id} - \boldsymbol{T})^{-1}$, i.e. $y = (\boldsymbol{Id} - \boldsymbol{T})^{-1}u$, $\forall u \in X$. It is linear and injective. From 2.26 we get

$$\begin{array}{ccc} y_n & = \displaystyle\sum_{j=0}^{n-1} \boldsymbol{T}^j u + & \boldsymbol{T}^n y_0 \\ \downarrow & & \downarrow \\ y & = \displaystyle\sum_{j=0}^{\infty} \boldsymbol{T}^j u + & 0, \end{array}$$

thus we have for all $u \in X : (\boldsymbol{Id} - \boldsymbol{T})^{-1}u = \sum_{j=0}^{\infty} \boldsymbol{T}^j u$ or $(\boldsymbol{Id} - \boldsymbol{T})^{-1} = \sum_{j=0}^{\infty} \boldsymbol{T}^j$ in the sense of equality of operators.

Finally, let's denote

$$S_N := \sum_{j=0}^{N} \boldsymbol{T}^j.$$

Then

$$S_N \circ (\boldsymbol{Id} - \boldsymbol{T}) = \sum_{j=0}^{N} \boldsymbol{T}^j - \sum_{j=1}^{N+1} \boldsymbol{T}^j = \boldsymbol{T}^0 - \boldsymbol{T}^{N+1} = \boldsymbol{Id} - \underbrace{\boldsymbol{T}^{N+1}}_{\downarrow \atop 0}$$

and similarly for $(\boldsymbol{Id} - \boldsymbol{T}) \circ S_N$ $\square$

Remark Later we will see that it holds: if an operator $\boldsymbol{T} : X \mapsto X$ is linear, bounded, injective and on, then its *inverse* $\boldsymbol{T}^{-1}$ (which exists) is also *linear* and *bounded*, i.e. continuous. This introduces a so-called *element of stability* into our task. If the inverse operator (in our case $(\boldsymbol{Id} - \boldsymbol{T})^{-1}$) is continuous, then this means that for

$$u_n \xrightarrow{X} u \;\Rightarrow\; \underbrace{(\boldsymbol{Id} - \boldsymbol{T})^{-1} u_n}_{y_n} \xrightarrow{X} \underbrace{(\boldsymbol{Id} - \boldsymbol{T})^{-1} u}_{y}$$

in other words, "near right-hand sides of the equation u_n correspond to close solutions", or "small changes on the right-hand side of the equation cause small changes in the solution". This is what is called *solution stability*.

Exercise Let's consider

$$y'' + y = x^2 y$$
$$y(0) = 1$$
$$y'(0) = 0.$$

According to the previous theory, the problem has only one solution on any $[0, a]$. We can verify that the function $y(x) = e^{-x^2/2}$ is this solution.
However, the proof of the previous theorem also shows that this solution can be obtained in the form of iterations (ie, it can be approached arbitrarily). Consider $y_0 \equiv 0$ and write the first few iterations. Does it seem to converge to $e^{-x^2/2}$? There are certainly some interesting lessons to be learned from this. ☺
At $y_0 = 0$ we get for y_5

$$\begin{aligned} y_5(x) = {} & \cos x + \tfrac{164925}{2048} x \sin x - \tfrac{164925}{2048} x^2 \cos x - \tfrac{54975}{1024} x^3 \; sin x + \tfrac{165437}{6144} x^4 \cos x + \tfrac{32383}{3072} x^5 \sin x \\ & - \tfrac{154871}{46080} x^6 \cos x - \tfrac{143131}{161280} x^7 \sin x + \tfrac{126481}{645120} x^8 \cos x + \tfrac{12983}{362880} x^9 \sin x - \tfrac{18889}{3628800} x^{10} \cos x \\ & - \tfrac{7}{12960} x^{11} \sin x + \tfrac{1}{31104} x^{12} \cos x \end{aligned}$$

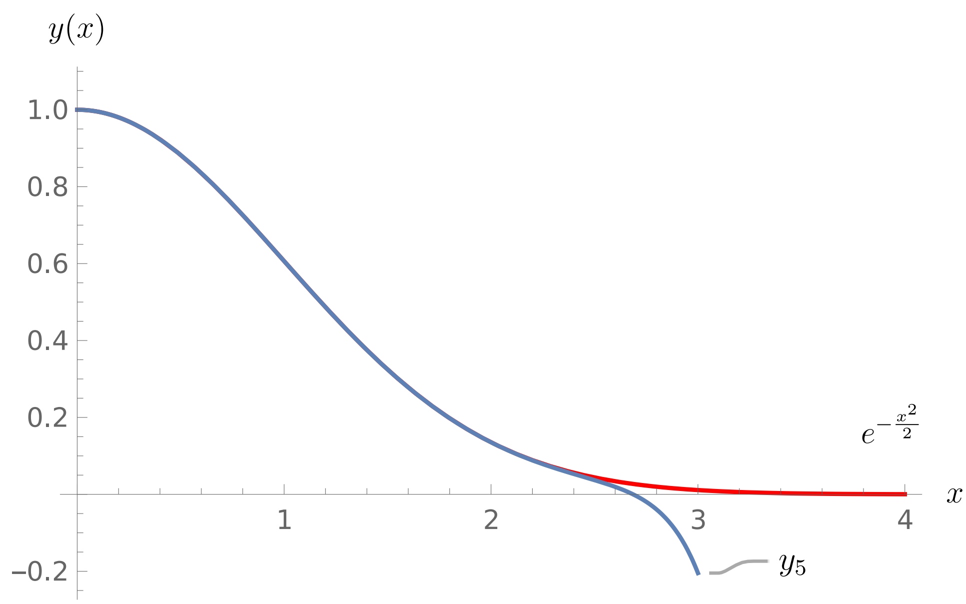


Figure 1: Comparison of the exact solution and the fifth iteration of y_5.

2.2 Basic concepts of spectral analysis

We will examine the operator equation for the unknown $x \in X$

$$(\boldsymbol{T} - \lambda \boldsymbol{Id})x = u, \quad \lambda \in \mathbb{C},\ \boldsymbol{T} \in \mathscr{L}(x),\ u \in X \text{ Banach space} \tag{2.27}$$

The motivation for this is the previous paragraph. Let us denote $\boldsymbol{T}_\lambda := \boldsymbol{T} - \lambda \boldsymbol{Id}$, then $\boldsymbol{T}_\lambda \in \mathscr{L}(X) \Leftrightarrow \boldsymbol{T} \in \mathscr{L}(X)$.

Let us denote the range of the operator $\boldsymbol{T}_\lambda$

$$\mathcal{R}(\boldsymbol{T}_\lambda) := \{y \in X,\ \exists x \in X,\ \boldsymbol{T}_\lambda x = y\} \quad (= \boldsymbol{T}_\lambda(X)).$$

Questions about the solvability of the equation (2.27) can be reformulated in the language of the operator $\boldsymbol{T}_\lambda$ as follows.

In the language of equations	In the language of the operator
$\exists$ a solution for any right-hand side $u \in X$?	Is $\boldsymbol{T}_\lambda$ *surjective*, ie is $\mathcal{R}(\boldsymbol{T}_\lambda) = X$?
If a solution for a given $u \in X$ exists, is it uniquely determined?	Is $\boldsymbol{T}_\lambda$ *injective* on X?
If $\forall u \in \mathcal{R}(\boldsymbol{T}_\lambda)\ \exists! x \in X;\ \boldsymbol{T}_\lambda x = u$, is this solution *stable*? (see remark below)	If $\boldsymbol{T}_\lambda$ is injective, then is $\boldsymbol{T}_\lambda^{-1}$ continuous on $\mathcal{R}(\boldsymbol{T}_\lambda)$?

Remark By the term *stable solution* we mean (to simplify a little) the situation when the equation $\boldsymbol{T}_\lambda x = u$ has uniquely determined solutions for $\forall u \in \mathcal{U}(u_0)$, ie "small changes in $u \in \mathcal{U}(u_0)$" result in "small changes in the solution". This exactly corresponds to the situation where the inverse mapping $\boldsymbol{T}_\lambda^{-1}$ is continuous on $\mathcal{U}(u_0)$. This property is very important in finding an approximate solution. With it, we often approximate the right-hand side of u by some "its close right-hand side" $\overline{u}$ and hope that the solution $\overline{x}$ that corresponds to the right-hand side of $\overline{u}$, will be a close solution to x, corresponding to the right-hand side of u. However, this may not be true for *unstable* operators.

Difference between finite and infinite dimensions.

In finite dimension: $\boldsymbol{T} \in \mathscr{L}(X) \Leftrightarrow \exists$ a matrix $M \in \mathcal{M}^{n\times n}$ such that $\boldsymbol{T}(x) = Mx \ \ \forall x \in X$ (in X we choose one fixed basis).

Then it applies

$$\boldsymbol{T} \text{ is injective } \Leftrightarrow \boldsymbol{T} \text{ is surjective } \quad \Leftrightarrow \quad \underbrace{M \text{ representing } \boldsymbol{T} \text{ is regular.}} \tag{2.28}$$

$$\Updownarrow$$

$$\boldsymbol{T}^{-1} \text{ is injective } \Leftrightarrow \boldsymbol{T}^{-1} \text{ is surjective} \Leftrightarrow \overbrace{M^{-1} \text{ is regular and represents } \boldsymbol{T}^{-1}} \ \text{(ie } \boldsymbol{T}^{-1} \text{ is lin.)} \tag{2.29}$$

Since in finite dimension every linear operator is continuous, so is $\boldsymbol{T}^{-1} \in \mathscr{L}(X)$.

Thus, "all or nothing" holds in finite dimension; it is the so-called finite-dimensional Fredholm alternative for $\boldsymbol{T} \in \mathscr{L}(X)$; $\dim X = n$, which says that exactly one of the following situations holds:

1. $\boldsymbol{T}$ is injective and surjective and has a continuous inversion

2. $\boldsymbol{T}$ is not injective, is not surjective, and has no continuous inversion

In infinite dimension, there is generally no relationship between injectivity and surjectivity.

Exercise Let us define the space of ℓ_2 sequences

$$\ell_2 := \left\{ \{x_n\}_{n=1}^{\infty},\ x_n \in \mathbb{C};\ \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$$

It can be shown that ℓ_2 with norm $||\{x_n\}||_{\ell_2}^2 := \sum_{n=1}^{\infty} |x_n|^2$ is a Banach space (it is even Hilbert's, more later). On ℓ_2, let's define two so-called *shift operators* ("shift operators")

$$A_1 : (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots)$$
$$A_2 : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots).$$

Apparently

$$||A_1 x||_{\ell_2} = ||x||_{\ell_2} \Rightarrow ||A_1|| = \sup_{||x|| \le 1} ||A_1 x|| = 1$$
$$||A_2 x||_{\ell_2} \le ||x||_{\ell_2} \Rightarrow ||A_2|| \le 1,$$

thus both are bounded, hence continuous, i.e. $A_1, A_2 \in \mathscr{L}(\ell_2)$.
Bute

- A_1 is injective, i.e. it assigns different elements to different elements, but it is not surjective (nothing is mapped, e.g. on $(1, 0, 0, \dots)$
- A_2 is surjective but not injective.

However, as far as stability is concerned, even in an infinite dimension the following profound theorem holds.

Theorem 2.3 (On the continuity of the inversion of a bijective mapping) *Let X be a Banach space, $\boldsymbol{A} \in \mathscr{L}(X)$ is surjective and injective. Then $\boldsymbol{A}^{-1} \in \mathscr{L}(X)$, so $\boldsymbol{A}^{-1}$ is a linear and continuous operator.*

Proof. It follows from the so-called open mapping theorem. See Lukeš 4.13-4.16. □

This seems to solve the problem of the stability of the solution: the injectivity and surjectivity is enough. Yes, for linear bounded (i.e. continuous) operators it is so. But for example, for linear and discontinuous, or for non-linear operators, the situation is not so simplef.

Possible operator states

Let $\boldsymbol{T} \in \mathscr{L}(X)$, X is Banach, $\lambda \in \mathbb{C}$, $\boldsymbol{T}_\lambda := \boldsymbol{T} - \lambda \boldsymbol{Id} \in \mathscr{L}(X)$. Then, depending on $\lambda \in \mathbb{C}$, the operator $\boldsymbol{T}_\lambda$ can have different properties in terms of its injectivity, continuity of inversion, and size $\mathcal{R}(\boldsymbol{T}_\lambda)$. The following table summarizes all the possibilities, two of which cannot occur: the one ruled out by the Theorem 2.3 (denoted in the Table by "V1") and the one ruled out by Lemma 2.6, which we formulate and we can prove below (denoted "L1").

The table must be understood as defining different categories to which the $\lambda \in \mathbb{C}$ parameter can belong. So, for example, the upper left corner of the table should be read as follows: "$\lambda \in \mathbb{C}$ is a regular point of $\boldsymbol{T}$, if $\boldsymbol{T}_\lambda$ is prime, $\boldsymbol{T}_\lambda^{-1}$ continuous and $\mathcal{R}(\boldsymbol{T}_\lambda) = X$".

	$\overbrace{\mathcal{R}(\boldsymbol{T}_\lambda) = X}^{\boldsymbol{T}_\lambda \text{ is surjective}}$	$\overbrace{\mathcal{R}(\boldsymbol{T}_\lambda) \neq X,\ \overline{\mathcal{R}(\boldsymbol{T}_\lambda)} = X}^{\boldsymbol{T}_\lambda \text{ is not surjective}}$	$\overbrace{\overline{\mathcal{R}(\boldsymbol{T}_\lambda)} \neq X}^{\boldsymbol{T}_\lambda \text{ is not surjective}}$
$\boldsymbol{T}_\lambda$ injective and $\boldsymbol{T}_\lambda^{-1}$ (which $\exists$) is continuous	λ is a regular point $\boldsymbol{T}$	~~L1~~	$\lambda \in \sigma_{\mathrm{R}}(\boldsymbol{T})$
$\boldsymbol{T}_\lambda$ injective and $\boldsymbol{T}_\lambda^{-1}$ (which $\exists$) is not continuous	~~V1~~	$\lambda \in \sigma_{\mathrm{C}}(\boldsymbol{T})$	$\lambda \in \sigma_{\mathrm{R}}(\boldsymbol{T})$
$\boldsymbol{T}_\lambda$ is not injective, i.e. $\left(\nexists \boldsymbol{T}_\lambda^{-1}\right)$	$\lambda \in \sigma_{\mathrm{P}}(\boldsymbol{T})$	$\lambda \in \sigma_{\mathrm{P}}(\boldsymbol{T})$	$\lambda \in \sigma_{\mathrm{P}}(\boldsymbol{T})$

Table 1: Spectral table for linear bounded operators in infinite dimension.

Remark • $\sigma_{\mathrm{C}}(\boldsymbol{T})$ ... the so-called *continuous spectrum* operator $\boldsymbol{T}$. If $\lambda \in \sigma_{\mathrm{C}}(\boldsymbol{T})$, then the equation $\boldsymbol{T}_\lambda y = u$ does not have a solution for every right $u \in X$, however, for every $\epsilon > 0$ there exists $u_\epsilon \in X$, $\|u_\epsilon - u\|_X < \epsilon$ and at the same time there is a solution to the equation $\boldsymbol{T}_\lambda y = u_\epsilon \in X$ (this is a consequence of $\overline{\mathcal{R}(\boldsymbol{T}_\lambda)} = X$). They are sometimes called "almost solutions". However, at the same time, $\boldsymbol{T}_\lambda$ is unstable ($\boldsymbol{T}_\lambda^{-1}$ is discontinuous), so it does not make good sense to talk about what happens to the solutions when we slightly change the right-hand sides u_ϵ.

- $\sigma_{\mathrm{R}}(\boldsymbol{T})$ ... the so-called *residual spectrum* $\boldsymbol{T}$. Since $\overline{\mathcal{R}(\boldsymbol{T}_\lambda)} \neq X$, there are no solutions for a large part of $u \in X$

- σ_{P} ... so-called *point spectrum* $\boldsymbol{T}$. $\boldsymbol{T}_\lambda$ is not prime, i.e.

$$\exists x_1 \neq x_2,\ \boldsymbol{T}_\lambda x_1 = \boldsymbol{T}_\lambda x_2$$

def. $x \coloneqq x_1 - x_2 \neq 0$, i.e. $\exists x \neq 0$ such that

$$\begin{aligned} \boldsymbol{T}_\lambda x &= 0 \\ (\boldsymbol{T} - \lambda \boldsymbol{Id})x &= 0 \\ \boldsymbol{T}x &= \lambda x. \end{aligned}$$

So $\lambda \in \sigma_P(\boldsymbol{T}) \Leftrightarrow \exists x \neq 0 :\ \boldsymbol{T}x = \lambda x \overset{\text{def.}}{\Longleftrightarrow} \underline{\lambda \text{ is an eigenvalue of } T}$ and $x \neq 0$ is the corresponding eigenvector.

Definition 2.4 (Operator Spectrum) Let X be a Banach space and $\boldsymbol{T} \in \mathscr{L}(X)$. We call the set

$$\sigma(\boldsymbol{T}) \coloneqq \sigma_C(\boldsymbol{T}) \cup \sigma_R(\boldsymbol{T}) \cup \sigma_P(\boldsymbol{T})$$

the spectrum of the operator $\boldsymbol{T}$.

Observation:

- $\lambda \in \sigma(\boldsymbol{T}) \Leftrightarrow \boldsymbol{T}_\lambda$ is not injective or not sujective
- λ regular $\Leftrightarrow \boldsymbol{T}_\lambda$ injective and surjective (and then $\boldsymbol{T}_\lambda^{-1}$ continuous)
- Not every element of the spectrum T is an eigenvalue.

Definition 2.5 (Spectral radius) Let X be a Banach space and $\boldsymbol{T} \in \mathscr{L}(X)$. We define the spectral radius of the operator $\boldsymbol{T}$ by the formula

$$\rho(\boldsymbol{T}) \coloneqq \sup\{|\lambda|;\ \lambda \in \sigma(\boldsymbol{T})\}\ .$$

Observation: If $\rho(\boldsymbol{T}) < \infty$, then $|\lambda| > \rho(\boldsymbol{T}) \Rightarrow \lambda$ is regular.
Finally, we arrive at the promised lemma from the spectral table.

Lemma 2.6 (On the discontinuity of the inversion of a densely defined injective operator) Let X be a Banach space, $\boldsymbol{A} \in \mathscr{L}(X)$. Let

1. $\mathcal{R}(\boldsymbol{A}) \neq X$, $\overline{\mathcal{R}(\boldsymbol{A})} = X$,
2. there exists $\boldsymbol{A}^{-1} : \mathcal{R}(\boldsymbol{A}) \to X$ (ie, $\boldsymbol{A}$ is injective on X).

Then $\boldsymbol{A}^{-1}$ is not continuous.

Proof. For the sake of contradiction, let us assume that the operator $\boldsymbol{A}^{-1}$ is continuous on $\mathcal{R}(\boldsymbol{A})$. Thanks to the property $\mathcal{R}(\boldsymbol{A}) \neq X$ we can choose $y \in X \backslash \mathcal{R}(\boldsymbol{A})$ and thanks to the property $\overline{\mathcal{R}(\boldsymbol{A})} = X$ there exists a sequence $\{y_n\}_{n=1}^\infty \subset \mathcal{R}(\boldsymbol{A})$ such that $y_n \to y$. Moreover we can also find a sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $\boldsymbol{A}x_n = y_n$, which can be written as $x_n = \boldsymbol{A}^{-1}y_n$ due to the existence of the inversion. Due to the continuity of $\boldsymbol{A}^{-1}$ and the

Cauchy property of $\{y_n\}_{n=1}^{\infty}$, the sequence $\{x_n\}_{n=1}^{\infty}$ is also Cauchy. Due to the completeness of the space X, this sequence converges, denote the limit element x. Then, thanks to the linearity and continuity of $\boldsymbol{A}$, we can write

$$\boldsymbol{A}x = \boldsymbol{A}(\lim_{n\to\infty} x_n) \overset{\boldsymbol{A}\text{ cont.}}{=} \lim_{n\to\infty} \boldsymbol{A}x_n = \lim_{n\to\infty} y_n = y$$

Since $\exists x \in X$ such that $\boldsymbol{A}x = y$, $y \in \mathcal{R}(\boldsymbol{A})$. This is a contradiction with $y \notin \mathcal{R}(\boldsymbol{A})$ above. $\square$

Remark (Table 1 in finite dimension) We know that $\boldsymbol{T} \in \mathscr{L}(X)$; $\dim X = n \in \mathbb{N}$ is represented by the matrix $M \in \mathcal{M}^{n\times n}$.

Then:

$$\begin{array}{rcl} \boldsymbol{T} \text{ is injective } \Leftrightarrow \boldsymbol{T} \text{ is surjective} & \Leftrightarrow & \underbrace{M \text{ is regular}} \text{ and represents } \boldsymbol{T} \\ & & \Updownarrow \\ \boldsymbol{T}^{-1} \text{ is injective } \Leftrightarrow \boldsymbol{T}^{-1} \text{ is surjective} & \Leftrightarrow & \overbrace{M^{-1} \text{ is regular}} \text{ and represents } \boldsymbol{T}^{-1} \end{array}$$

In the situation described above, there is also always $\boldsymbol{T}^{-1} \in \mathscr{L}(X)$.

Table 1 can therefore be schematically rewritten as

	1	2, 3	4
	2	3, 4	4
	4	3	1

Table 2: Table 1 rewritten in finite dimension

1. In finite dimension only this situations occur, and so here we have
   1. $\lambda \in \mathbb{C} \to \lambda$ is either regular or already an eigenvalue
   2. $\sigma(\boldsymbol{T}) = \{\lambda \in \mathbb{C}, \lambda \text{ is an eigenvalue } \boldsymbol{T}\} = \{\lambda \in \mathbb{C};\ \lambda \text{ is eigenvalue of } M\}$.

2. Cannot occur in general

3. This column describes the situation $\mathcal{R}(\boldsymbol{T}_\lambda) \neq X$, $\overline{\mathcal{R}(\boldsymbol{T}_\lambda)} = X$. However, this cannot occur in a finite dimension, because $\mathcal{R}(\boldsymbol{T}_\lambda) = \overline{\mathcal{R}(\boldsymbol{T}_\lambda)}$there.

4. It cannot occur because for $\dim X = n$, $\boldsymbol{T}$ is injective $\Leftrightarrow \boldsymbol{T}_\lambda$ is surjective.

The following theorem shows that $\rho(\boldsymbol{T})$ is always finite for $\boldsymbol{T} \in \mathscr{L}(X)$.

Theorem 2.7 *X Banach, $\boldsymbol{T} \in \mathscr{L}(X)$ (ie, $||\boldsymbol{T}|| < \infty$). Then for $|\lambda| > ||\boldsymbol{T}||$ holds*

$$\begin{aligned} a:&\quad \lambda \notin \sigma(\boldsymbol{T}),\ \textit{i.e. } \lambda \textit{ is regular} \\ b:&\quad (T - \lambda \boldsymbol{Id})^{-1} = T_\lambda^{-1} = -\sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}} \in \mathscr{L}(X) \end{aligned}$$

Remark

- From a) it immediately follows $\rho(\boldsymbol{t}T) \leq ||T||$.
- The series in b) is called the von Neumann series of the operator $(\boldsymbol{T} - \lambda \boldsymbol{Id})$. Proposition b) says, among other things, that if $\|\boldsymbol{T}\| < |\lambda|$, the operator $\boldsymbol{T}_\lambda$ is injective, continuous, surjective, and has a continuous inversion.

Proof. If $|\lambda| > ||\boldsymbol{T}||$, then certainly $\lambda \neq 0$. Let us put $\boldsymbol{A} := \frac{1}{\lambda}\boldsymbol{T}$. Then $||\boldsymbol{A}|| = \frac{1}{|\lambda|}||\boldsymbol{T}|| < 1$ and we can use Theorem 2.2 on $\boldsymbol{A}$. This gives us that:

a) $\boldsymbol{Id} - \boldsymbol{A}$ is injective and surjective $\Rightarrow$ $\boldsymbol{T} - \lambda\boldsymbol{Id} = (-\lambda)(\boldsymbol{Id} - \boldsymbol{A})$ is injective and surjective $\xRightarrow{\text{Theorem 2.2}}$ $(\boldsymbol{T} - \lambda\boldsymbol{Id})^{-1}$ is continuous. Hence, λ is regular.

b) Theorem 2.2 gives also

$$
\begin{aligned}
(\boldsymbol{Id} - \boldsymbol{A})^{-1} &= \sum_{k=0}^{\infty} \boldsymbol{A}^k \\
(\boldsymbol{Id} - \frac{1}{\lambda}\boldsymbol{T})^{-1} &= \sum_{k=0}^{\infty} \frac{\boldsymbol{T}^k}{\lambda^k} \Big/ \cdot (-1) \\
\left(\frac{1}{\lambda}\boldsymbol{T} - \boldsymbol{Id}\right)^{-1} &= -\sum_{k=0}^{\infty} \frac{\boldsymbol{T}^k}{\lambda^k} \Big/ \cdot \frac{1}{\lambda} \quad \text{be carefull!}^{3} \\
\underbrace{\lambda^{-1}\left(\frac{1}{\lambda}\boldsymbol{T} - \boldsymbol{Id}\right)^{-1}}_{(\boldsymbol{T}-\lambda\boldsymbol{Id})^{-1}} &= -\sum_{k=0}^{\infty} \frac{\boldsymbol{T}^k}{\lambda^{k+1}}
\end{aligned}
$$

□

[3] The inverse mapping to y=3x is y=x/3, so the inverse mapping has the value of the coefficient reversed. On the left-hand side of the equation, one can (alternatively) proceed:

$$\left(\frac{1}{\lambda}\boldsymbol{T} - \boldsymbol{Id}\right)^{-1} = \lambda(T - \lambda\boldsymbol{Id})^{-1}\,,$$

and then it is clear why we have to divide the equation by λ.

Exercise Consider $\ell_2 := \left\{ \{x_n\}_{n=1}^{\infty},\ x_n \in \mathbb{C},\ \sum\limits_{n=1}^{\infty} |x_n|^2 < \infty \right\}$ the space of all complex sequences that are so-called "summable with the square". It holds that ℓ_2 with the scalar product $\big(\{x_n\}, \{y_n\}\big)_{\ell_2} = \sum\limits_{n=1}^{\infty} x_n \overline{y_n}$ which induces the norm $|\{x_n\}\|_{\ell_2} = \sqrt{\sum_{n=1}^{\infty} |x_n|^2}$, is complete, and therefore a Hilbert (i.e. also a Banach) space.

Consider the operator

$$\begin{aligned} &\boldsymbol{T} : \ell_2 \to \ell_2 \\ &\boldsymbol{T} : (x_1, x_2 \dots, x_k, x_{k+1}, \dots) \mapsto (0, x_1, x_2, \dots, x_{k-1}, x_k, \dots). \end{aligned}$$

Since $||\boldsymbol{T}x||_{\ell_2} = ||x||_{\ell_2}$, $||\boldsymbol{T}|| = \sup\limits_{||x|| \leq 1} ||\boldsymbol{T}x|| = \sup\limits_{||x|| \leq 1} ||x|| = 1.$

Thus $\rho(\boldsymbol{T}) \leq ||\boldsymbol{T}|| = 1$ and therefore $|\lambda| > 1 \Rightarrow \lambda$ is regular. Thus, the entire spectrum $\boldsymbol{T}$ lies in the unit circle in $\mathbb{C}$.

- $\lambda = 0$: We already know that $\boldsymbol{T}$ is not surjective, but it is injective. At the same time, it can be seen that no element from ℓ_2 can be mapped on $(a, 0, 0, \dots)$, $a \in \mathbb{C},\ a \neq 0$. Therefore, no sequence from $\mathcal{R}(\boldsymbol{T})$ can converge to, for example, the element $(1, 0, 0, \dots)$. Therefore, $\overline{\mathcal{R}(\boldsymbol{T})} \neq \ell_2$, from where $0 \in \sigma_R(\boldsymbol{T})$ it follows (from the table 1).

- $|\lambda| \leq 1,\ \lambda \neq 0$

1. We first show that none of these λ is an eigenvalue of $\boldsymbol{T}$. If that were the case, then $\exists x \neq 0$, s.t.

$$\begin{aligned} \boldsymbol{T}x &= \lambda x \\ (0, x_1, x_2, \dots) &= (\lambda x_1, \lambda x_2, \dots), \end{aligned}$$

   i.e.

   i) $\lambda x_1 = 0$

   ii) $\lambda x_k = x_{k-1} \quad \forall k = 2, 3, 4, \dots$

   $x_1 = 0$ follows from i) (because $\lambda \neq 0$), and $x_2 = x_3 = \cdots = 0$ follows from ii) by induction. Thus, $x = 0$, which, however, conflicts with the fact that it should be an eigenvector of $\boldsymbol{T}$.

2. We show that $\boldsymbol{T}_\lambda$ is not surjective, specifically that no $x \in \ell_2$ is mapped on $(1, 0, 0, \dots)$. Let such $x \in \ell_2$ exist. So then

$$\begin{aligned} &\boldsymbol{T}_\lambda x = (1, 0, 0, \dots), \\ &\quad \| \\ &(-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, \dots) \end{aligned}$$

so $1 = -\lambda x_1 \Rightarrow x_1 = -\frac{1}{\lambda}$

$$k = 1, 2, 3, \dots \quad x_k - \lambda x_{k+1} = 0 \ \Rightarrow\ x_{k+1} = \frac{x_k}{\lambda} \ \Rightarrow\ x = \left(-\frac{1}{\lambda}, -\frac{1}{\lambda^2}, -\frac{1}{\lambda^3}, \dots \right)$$

So it seems we found such x, but $||x||_{\ell_2}^2 = \sum\limits_{k=1}^{\infty} \frac{1}{\lambda^{2k}} = \infty$, because it is a geometric series with the quotient $1/\lambda^2$, for which $|\lambda| \leq 1 \Rightarrow |1/\lambda^2| \geq 1$. Since $\boldsymbol{T}_\lambda$ is linear and $(1, 0, 0, \dots) \notin \mathcal{R}(\boldsymbol{T}_\lambda)$, so $(a, 0, 0, \dots) \notin \mathcal{R}(\boldsymbol{T}_\lambda)\ \forall a \in \mathbb{C},\ a \neq 0 \ \Rightarrow\ \overline{\mathcal{R}(\boldsymbol{T}_\lambda)} \neq \ell_2$.

So we are in the last column of the table 1, $\forall \lambda \neq 0,\ |\lambda| \leq 1$. However, since at the same time no such λ is an eigenvalue, $\lambda \in \sigma_R(\boldsymbol{T})$ for all such λ. (This could also be shown independently by showing the injectivity of the $\boldsymbol{T}_\lambda$.)

Conclusion: For this $\boldsymbol{T}$, $\sigma(\boldsymbol{T}) = \sigma_R(\boldsymbol{T}) = \{\lambda \in \mathbb{C}, |\lambda| \leq 1\}$. The spectrum is therefore the entire unit circle, so there are countless elements of the spectrum (while none of them is an eigenvalue). Such an operator is therefore "relatively ugly", but at the same time it does not generate any eigenvectors.

Exercise Perform a spectral analysis of the following operators:

a) Let us have the operator $\boldsymbol{T}:\ell_2 \to \ell_2$

$(x_1, x_2, x_3, \dots) \mapsto \left(x_2, \frac{x_3}{2}, \frac{x_4}{3}, \frac{x_5}{4}, \dots\right)$ Additionally, try to think about what $||\boldsymbol{T}||$ is.

Solution: $\boldsymbol{T} \in \mathscr{L}(\ell_2),\ \sigma(\boldsymbol{T}) = \sigma_P(\boldsymbol{T}) = \{0\},\ \sigma_R(\boldsymbol{T}) = \emptyset,\ \sigma_C(\boldsymbol{T}) = \emptyset$

b) Let us have the operator $\boldsymbol{T}:\ell_2 \to \ell_2$

$(x_1, x_2, x_3, \dots) \mapsto \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots\right)$ Additionally, try to think about what $||\boldsymbol{T}||$ is and whether $0 \in \sigma_C(\boldsymbol{T})$ or $0 \in \sigma_R(\boldsymbol{T})$.

Solution: $\boldsymbol{T} \in \mathscr{L}(\ell_2),\ \sigma(\boldsymbol{T}) = \{0\},\ \sigma_P(\boldsymbol{T}) = \emptyset,\ \sigma_C(\boldsymbol{T}) = \emptyset$

3 Compact operators

We know from the previous chapters that for *linear* operator between Banach spaces $\boldsymbol{T} : X \to Y$ the following applies:

$$\boldsymbol{T} \text{ is continuous } \Leftrightarrow \boldsymbol{T} \text{ is bounded},$$

we write $\boldsymbol{T} \in \mathscr{L}(X,Y)$. Next we will use the notation $\mathscr{L}(X) \coloneqq \mathscr{L}(X,X)$.

Recall that boundedness of the operator means that $\boldsymbol{T}$(bounded set) = bounded set. This statement therefore characterizes the connection for linear operators, in other words $\forall A \subset X$ bounded, is $\boldsymbol{T}(A)$ bounded in Y.

Definition 3.1 (Compact operator) Let X, Y be Banach spaces, $\boldsymbol{K} : X \to Y$ be a linear operator. We say that $\boldsymbol{K}$ is compact if for each bounded set $A \subset X$ it is true that $\overline{\boldsymbol{K}(A)} \subset Y$ is a compact set. The set of all compact operators is written as $\mathscr{C}(X,Y)$ and $\mathscr{C}(X) \coloneqq \mathscr{C}(X,X)$.

Remark

1. $\mathscr{C}(X,Y) \subset \mathscr{L}(X,Y)$. If $A \subset X$ is bounded, then $\overline{\boldsymbol{K}(A)} \subset Y$ is compact and according to the necessary compactness condition $\overline{\boldsymbol{K}(A)}$ must be a bounded and closed set, i.e. also $\boldsymbol{K}(A) \subseteq \overline{\boldsymbol{K}(A)}$ is bounded, being subset of a bounded set.

2. Recall that boundedness and closeness are sufficient conditions for compactness only in finite-dimensional normed linear space (NLP).

3. For compact sets we can use Weierstrass sub-sequences selection, which we will use further.

Operator characterization using sequences For $\boldsymbol{T} \in \mathscr{L}(X,Y)$ we had:

$$\begin{aligned} &\text{continuity: } x_n \to x \;\Rightarrow\; \boldsymbol{T}x_n \to \boldsymbol{T}x \\ &\text{boundedness: } \{x_n\} \text{ is bounded } \Rightarrow\; \{\boldsymbol{T}x_n\} \text{ is bounded} \end{aligned}$$

The compactness characterization is best summed up in a theorem.

Theorem 3.2 (About characterizing a compact operator by a sequence) *The operator $\boldsymbol{K} : X \to Y$ is compact if for each bounded sequence $\{x_n\}_{n=1}^{\infty} \subset X$ there exists a selected subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and the element $y \in Y$ such that $\boldsymbol{K}(x_{n_k}) \to y$.*

Observation:

If the entire Y space had the property that any constrained sequence in Y could select a convergent subsequence, then

$$\mathscr{L}(X,Y) = \mathscr{C}(X,Y)\,.$$

It suffices to prove

$$\mathscr{L}(X,Y) \subset \mathscr{C}(X,Y)\,,$$

since we have already proved the opposite implication. Let $\boldsymbol{T} \in \mathscr{L}(X,Y)$ and $\{x_n\}_{n=1}^{\infty} \subset X$ be a bounded sequence. Due to the boundedness of $\boldsymbol{T}$, the sequance $\{\boldsymbol{T}x_n\}_{n=1}^{\infty}$ is bounded. By guaranteeing that we can already select a convergent subsequence from each such sequence, we would prove the compactness of each continuous operator. Surely not every metric space can have such a property. In honor of the Bolzano-Weierstrass theorem valid in $\mathbb{R}$, we call this property (just in this textbook) the *B-W property.* $\square$

Lemma 3.3 (About the equivalence condition for the B-W property) Let Y be the Banach space. Pak

$$Y \text{ has the B-W property } \Leftrightarrow \dim Y < \infty.$$

Sketch of a proof:

"$\Leftarrow$" On $\mathbb{R}$, we can use the Bolzano-Weierstrass theorem from the first semester. In the $\mathbb{R}^n$ space we make successive selections by folders. Now since $\dim X = n \in \mathbb{N}$, we can find the base and assign n-tici coordinates to each element of $x \in X$ relative to this base.

"$\Rightarrow$" Let $\dim Y = \infty$. For the sake of better geometrical interpretation we restrict ourselves to Hilbert space and will be working with the "set of infinitely many mutually perpendicular coordinates". We select $y_1 \in Y$, $\|y_1\| = 1$ (the member of a canonical OG basis, lying in the direction of the first coordinate) and then by induction select elements $y_{k+1} \in Y$, $\|y_{k+1}\| = 1$ on the respective perpendicular direction and observe that the distance of the element y_{k+1} from the elements y_j, $j = 1, \ldots, k$ is greater than or equal to one. We therefore have bounded sequence (lying on the unit sphere) from which we are not able to select a convergent subsequence (every element of a sequence is "too far" from any other element.) $\square$

Lemma 3.4 (About identity compactness) For Banach space X ii holds:

$$\boldsymbol{Id} : X \to X \text{ is compact } \Leftrightarrow \ X \text{ has B-W property.}$$

Proof. Obvious. $\square$

From the previous two lemmas, it is interesting to note that in an infinite dimension the identity is not a compact operator. Generally, as a consequence of the previous two lemmas, we have:

$$\mathbf{Id} \in \mathscr{L}(X) \text{ is compact } \Leftrightarrow \ \dim X < \infty.$$

Remark In the theory of partial differential equations, the process of "compact embedding" of one space into another is applied. In this situation we consider two spaces $X \subset Y$, but equipped with different norms $\|\cdot\|_X, \|\cdot\|_Y$. Mapping $\mathbf{Id} : X \to Y$ in this case can be compact. However, this is due to different norms that are not equivalent on infinitesimal spaces.

An example of such a process is the so-called Rellich theorem:

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a smooth border. Define Sobolev space

$$W^{1,2}(\Omega) := \left\{ f : \Omega \to \mathbb{R} : \|f\|_{W^{1,2}} := \left[\int_\Omega \left(|f|^2 + |\nabla f|^2\right) \, \mathrm{d}x \right]^{1/2} < +\infty \right\}.$$

Then $W^{1,2}(\Omega) \subset L^2(\Omega)$ and $\mathbf{Id} : W^{1,2}(\Omega) \to L^2(\Omega)$ is a compact operator.

The practical use of this theorem consists in selecting a convergent subsequence in L^2 from a bounded sequence in the space $W^{1,2}$.

3.1 Properties of compact operators

In this subsection, we formulate seven key properties that greatly simplify the issue of spectral analysis for compact operators.

Lemma 3.5 $\dim Y < \infty \ \Rightarrow \ \mathscr{L}(X, Y) = \mathscr{C}(X, Y)$

Proof. $A \subset X$ is bounded $\xRightarrow{\boldsymbol{T} \in \mathscr{L}(X,Y)}$ $\boldsymbol{T}(A)$ is bounded $\Rightarrow$ $\overline{\boldsymbol{T}(A)}$ is bounded and closed in Y $\xRightarrow{\dim Y < \infty}$ $\overline{\boldsymbol{T}(A)}$ is compact. $\square$

The corollary of this lemma is $\boldsymbol{T} \in \mathscr{L}(X)$, $\dim X = \infty$, $\dim \mathcal{R}(\boldsymbol{T}) < \infty \ \Rightarrow \boldsymbol{T} \in \mathscr{C}(X)$.

Lemma 3.6 $\boldsymbol{S} \in \mathscr{L}(X)$, $\boldsymbol{K} \in \mathscr{C}(X) \ \Rightarrow \boldsymbol{S} \circ \boldsymbol{K} \in \mathscr{C}(X)$, $\boldsymbol{K} \circ \boldsymbol{S} \in \mathscr{C}(X)$.

Proof. Choose the bounded sequence $\{x_n\}_{n=1}^{\infty} \subset X$. Then $\{\boldsymbol{S}x_n\}_{n=1}^{\infty}$ is bounded and due to the compactness of $\boldsymbol{K}$ we can select a convergent subsequence from $\{(\boldsymbol{K} \circ \boldsymbol{S})x_n\}_{n=1}^{\infty}$ therefore $\boldsymbol{K} \circ \boldsymbol{S}$ is compact. Further, from the $\{\boldsymbol{K}x_n\}_{n=1}^{\infty}$ we can select the convergent subsequence $\{\boldsymbol{K}x_{n_k}\}_{k=1}^{\infty}$ and due to the properties of the $\boldsymbol{S}$ operator, the $\{(\boldsymbol{S} \circ \boldsymbol{K})x_{n_k}\}_{k=1}^{\infty}$ is convergent, therefore $\boldsymbol{S} \circ \boldsymbol{K}$ is compact.

□

Lemma 3.7 $\boldsymbol{K} \in \mathscr{C}(X)\ \dim X = \infty\ \Rightarrow 0 \in \sigma(\boldsymbol{K})$

Proof. If $0 \notin \sigma(\boldsymbol{K})\ \Rightarrow \exists \boldsymbol{K}^{-1} \in \mathscr{L}(X)$. Then according to Lemma 3.6 we get that

$$\underset{\in\mathscr{C}}{\boldsymbol{K}} \circ \underset{\in\mathscr{L}}{\boldsymbol{K}^{-1}} \underset{\Rightarrow}{=} \underset{\in\mathscr{C}}{\mathbf{Id}},$$

where identity compactness is in contradiction with Lemma 3.4. □

Lemma 3.8 $\boldsymbol{K} \in \mathscr{C}(X),\ \lambda \neq 0$. Then

1. $\mathcal{R}(\boldsymbol{K} - \lambda\mathbf{Id})$ is a closed set (see Lukeš 5.17).
2. $\boldsymbol{K}$ is surjective (i.e. $\mathcal{R}(\boldsymbol{K} - \lambda\mathbf{Id}) = X$) $\Leftrightarrow$ $\boldsymbol{K} - \lambda\mathbf{Id}$ is injective (see Lukeš 5.27).

Remark The second part of the previous lemma is called "Fredholm's alternative in an infinite dimension".

For $\boldsymbol{K} \in \mathscr{C}(X),\ \lambda \neq 0$, we can modify the spectral table based on our new knowledge. Specifically, we found that

a) the situation $\mathcal{R}(\boldsymbol{K}_\lambda) \neq X$, $\overline{\mathcal{R}(\boldsymbol{K}_\lambda)} = X$ cannot occur because $\mathcal{R}(\boldsymbol{K}_\lambda) = \overline{\mathcal{R}(\boldsymbol{K}_\lambda)}$.

b) $\boldsymbol{K}_\lambda$ is injective $\Leftrightarrow$ $\boldsymbol{K}_\lambda$ is surjective.

So we have

	$\mathcal{R}(\boldsymbol{K}_\lambda) = X$	$\mathcal{R}(\boldsymbol{K}_\lambda) \neq X$, $\overline{\mathcal{R}(\boldsymbol{K}_\lambda)} = X$	$\overline{\mathcal{R}(\boldsymbol{K}_\lambda)} \neq X$
$\boldsymbol{K}_\lambda$ is injective, $\boldsymbol{K}_\lambda^{-1}$ is continuous	λ is regular		~~b)~~
$\boldsymbol{K}_\lambda$ je injective, $\boldsymbol{K}_\lambda^{-1}$ is not continuous		~~a)~~	~~b)~~
$\boldsymbol{K}_\lambda$ is not injective	~~b)~~	~~a)~~	$\lambda \in \sigma_{\mathrm{P}}$ (is eigenvalue)

<u>Summary:</u>

- 0 is always in the spectrum of the compact operator. It is the only element of the spectrum that does not have to be an eigenvalue.
- All non-zero elements of the spectrum are already eigenvalues.

Lemma 3.9 $\boldsymbol{K} \in \mathscr{C}(X)$, $\lambda \neq 0$, $\lambda \in \sigma(\boldsymbol{K})$. Then

- λ is an eigenvalue,
- $\dim \operatorname{Ker}(\boldsymbol{K} - \lambda\mathbf{Id}) < +\infty$,
- $\operatorname{Ker}(\boldsymbol{K} - \lambda\mathbf{Id})$ is the space of all eigenvectors of the respective eigenvalue λ, and is a closed subspace X.

Proof. see Lukeš 5.15. □

Definition 3.10 (Multiplicity of the eigenvalue) The number $\dim \operatorname{Ker}(\boldsymbol{K} - \lambda\mathbf{Id}) \in \mathbb{N}$ is called the multiplicity of the eigenvalue $\lambda \in \sigma_{\mathrm{P}}(\boldsymbol{K})$.

According to the previous lemma, we know that every non-zero eigenvalue has a finite multiplicity - the dimension of space generated by eigenvectors, belonging to the non-zero eigenvalue, is finite.

Lemma 3.11 $\boldsymbol{K} \in \mathscr{C}(X)$. Then $\forall \varepsilon > 0$ the $\sigma(\boldsymbol{K}) \cap \{\lambda \in \mathbb{C} : |\lambda| > \varepsilon\}$ is finite.

As a result, the spectrum of the compact operator is at most countable. Moreover, if the spectrum of a compact operator has a cluster point, it can only be 0.

4 Duality

4.1 Dual and duality

Definition 4.1 (Dual) Let X be a Banach space. Space $X' := \mathscr{L}(X, \mathbb{K})$ is called the (topological) dual to X.

Remark

- The space X' is therefore formed by all continuous linear functionals, where we consider continuity in the sense of

$$x_n \xrightarrow{X} x \implies \boldsymbol{T}x_n \to \boldsymbol{T}x \quad \text{for all } \boldsymbol{T} \in X' .$$

- We know that if X, Y are normed spaces and Y is Banach, then $\mathscr{L}(X, Y)$ is also Banach. Therefore, X' is automatically a Banach space.
- If $\boldsymbol{T} \in X'$, then its norm is naturally $\|\boldsymbol{T}\|_{X'} := \sup\limits_{\|x\|_X \leq 1} |\boldsymbol{T}x|$.
- A topological dual is not the same as a <u>vector dual</u> (only a linear mappings $X \to \mathbb{K}$, we do not require continuity). There are more elements of the vector dual (the "discontinuous" ones). In finite dimension vector dual is always a topological dual.

Definition 4.2 (Duality) Let X be Banach and X' its dual. We call the representation $\boldsymbol{S} : X \times X' \to \mathbb{C}$ duality if it satisfies the properties:

1. sesquilinearity: $\boldsymbol{S}(\alpha x + \beta y, z) = \alpha \boldsymbol{S}(x, z) + \beta \boldsymbol{S}(y, z)$, $\boldsymbol{S}(z, \alpha x + \beta y) = \overline{\alpha} \boldsymbol{S}(z, x) + \overline{\beta} \boldsymbol{S}(z, y)$,
2. continuity: $\boldsymbol{S}(x_n, y_n) \to \boldsymbol{S}(x, y)$ whenever $(x_n, y_n) \to (x, y)$ in $X \times X'$.

Sometimes we write $\boldsymbol{S}(x, \boldsymbol{T}) = \langle x, \boldsymbol{T} \rangle$. In general, $\langle \cdot, \cdot \rangle$ is often the symbol for duality.

Remark

- If in some sense we are often able to identify the spaces X and X' (we will see in the following), then the role of duality is played by the scalar product. By identifying spaces, we mean the following: we write $X \simeq Y$ if there exists a mapping $\boldsymbol{D} : X \to Y$ that is isometric and isomorphic, i.e. it preserves the norm and is bijective.
- The mapping "with swapped components" $\overline{\boldsymbol{S}} : X' \times X \to \mathbb{C}$, which is sesquilinear and continuous in the sense of the previous definition, also has a duality structure. It is a matter of agreement which order of spaces X and X' is chosen in a given situation.

Example Consider the vector space $\mathbb{R}^n$. We know that every linear form $\boldsymbol{T} \in (\mathbb{R}^n)'$ is representable by a scalar product $\boldsymbol{T}[(x_1, \cdots, x_n)] = \sum_{j=1}^n \alpha_j x_j$. This is an isometric and isomorphic representation, so one can write $\mathbb{R}^n \simeq (\mathbb{R}^n)'$. A duality on such a space is, for example, the *scalar product* on $\mathbb{R}^n$. We will see later that the same can be considered in any Hilbert space - in this sense, duality is a generalization of the scalar product.

Theorem 4.3 (On the identification of conjugate Lebesgue spaces) *Let $\Omega \subseteq \mathbb{R}^n$ be an open connected set, $\boldsymbol{T} \in L^p(\Omega)'$, $p \in (1, \infty)$. Let $q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ holds. Then there exists exactly one element $g \in L^q(\Omega)$ such that:*

$$\boldsymbol{T}(f) = \int_\Omega f \overline{g} \, \mathrm{d}x \quad \forall f \in L^p(\Omega) \quad \text{and at the same time} \quad \|\boldsymbol{T}\|_{L^p(\Omega)'} = \|g\|_{L^q(\Omega)} .$$

Proof. We will not provide a proof. ☐

The previous theorem shows that $L^p(\Omega)' \simeq L^q(\Omega)$ holds. In this sense, we identify $\boldsymbol{T}$ and g, and we identify the duality $\langle f, \boldsymbol{T} \rangle \mapsto \boldsymbol{T}(f)$ with the duality

$$\langle f, g \rangle \mapsto \int_\Omega f\overline{g}, \quad f \in L^p,\, g \in L^q \,. \tag{4.1}$$

Note that if we put $p = q = 2$ in the previous theorem, we obtain the property $L^2(\Omega)' \simeq L^2(\Omega)$ and the duality (4.1) has the same form as the scalar product on $L^2(\Omega)$, $\langle f, g \rangle \equiv \big(f, g\big)_{L^2}$. It is natural to wonder if there is something deeper behind this result. The answer is positive.

Theorem 4.4 (Riesz-Fréchet on representation) *Let H be a Hilbert space with scalar product $(\cdot\,,\cdot)_H$, $\boldsymbol{T} \in H'$. Then there exists exactly one element $f \in H$ such that the following satisfies:*

1. $\boldsymbol{T}(x) = \big(x, f\big)_H$ *for each* $x \in H$,
2. $\|\boldsymbol{T}\|_{H'} = \|f\|_H$.

Proof. see Lukeš 2.9 □

Corolarry 4.5 The mapping "$x \mapsto f$" is an isometric isomorphism (it is bijective and norm preserving). Therefore, for all Hilbert spaces H we can make the identification $H' \simeq H$.

Lemma 4.6 (On the inclusion of dual spaces) Let X, Y be Banach and let $X \subset Y$ and $\|x\|_Y = \|x\|_X$ $\forall x \in X$. Then

$$Y' \subset X'$$

applies in the sense of mapping restriction (see proof). However, it is important to be very careful here, it is easy to misinterpret this result!

Proof.

$$\begin{aligned} \text{Let's } \boldsymbol{T} \in Y' &\implies \boldsymbol{T} \text{ continuous and linear (on elements from } Y) \\ &\implies \boldsymbol{T}|_X \text{ continuous and linear (on elements from } X) \implies \boldsymbol{T}|_X \in X' \end{aligned}$$

□

Example Headless application of previous inclusion can drive us into dead ends.
Consider the space $\mathbb{R} \subset \mathbb{R}^2$. According to the previous statement, we can declare $(\mathbb{R}^2)' \subset (\mathbb{R})'$. Since both spaces are Hilbert, they can be equated $(\mathbb{R}^n)' = \mathbb{R}^n$, so $\mathbb{R}^2 \subset \mathbb{R}$ should hold, which is apparently nonsense. But where is the mistake?

$$\begin{array}{ccc} \mathbb{R} \subset \mathbb{R}^2 & \implies & (\mathbb{R}^2)' \subset \mathbb{R}' \\ & & \,\| \qquad \| \\ & & \mathbb{R}^2 \;\subset \mathbb{R} \end{array}$$

In the previous considerations, we made two mistakes: one major, one minor.

1. We made a minor mistake by considering $(\mathbb{R}^n)' \simeq \mathbb{R}^n$ to be an equality, it is actually an *identification.* Each linear mapping on $\mathbb{R}^n$ has the form

$$\boldsymbol{T}(x) = \sum_{j=1}^n \alpha_j x_j$$

and is identified with an n-tition of coefficients

$$T \simeq (\alpha_1, \ldots \alpha_n) \in \mathbb{R}^n \text{ represents } (\mathbb{R}^n)' \,.$$

Representing space $\mathbb{R}^n$ should therefore really be viewed as a space of elements, which represent a linear mapping.

2. We made a big mistake when we considered the inclusion $(\mathbb{R}^2)' \subset \mathbb{R}'$ to be a set inclusion. The statement should be understood in this sense:

"All linear mappings working on $\mathbb{R}^2$ can be restricted to work on $\mathbb{R}$."

If the linear mapping $\boldsymbol{T} \in (\mathbb{R}^2)'$ is representable by the pair $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, then this mapping can indeed be reduced to $\boldsymbol{T}|_{\mathbb{R}}$ represented by $(\alpha_1, 0)$, which we can think of as the space $\mathbb{R}$. This is the true meaning of "inclusion" $Y' \subset X'$.

Remark "Duality" is often manifested by the fact that the formulas containing the elements X and X' show certain symmetries. For example, we know that

$$\|\boldsymbol{T}\|_{X'} = \sup_{\|x\|_X \le 1} |\boldsymbol{T}(x)|.$$

Furthermore, we know that $|\boldsymbol{T}(x)| \le \|\boldsymbol{T}\| \, \|x\|_X$.

Now consider a functional $\boldsymbol{T} \in X'$ such that $\|\boldsymbol{T}\|_{X'} \le 1$. Then it certainly holds $|\boldsymbol{T}x| \le \|x\|_X$, and after passing to the supremum

$$\sup_{\|\boldsymbol{T}\|_{X'} \le 1} |\boldsymbol{T}(x)| \le \|x\|_X \ .$$

The following theorem shows that even *equality* holds:

$$\sup_{\|\boldsymbol{T}\|_{X'} \le 1} |\boldsymbol{T}(x)| = \|x\|_X \ .$$

Theorem 4.7 (Hahn-Banach) *Let X be Banach, $x \in X$ such that $x \neq 0$. Then there exists $\boldsymbol{T} \in X'$ with properties*

$$\boldsymbol{T}(x) = \|x\|_X \ , \qquad \|\boldsymbol{T}\|_{X'} = 1 \, .$$

Proof. see Taylor, p. 181 $\square$

4.2 Dual mapping, dual operator

Definition 4.8 (Dual mapping) Let X, Y be Banach spaces, $\boldsymbol{T} \in \mathscr{L}(X, Y)$, $\boldsymbol{T}' : Y' \to X'$. We say that $\boldsymbol{T}'$ is a dual mapping to $\boldsymbol{T}$ if (written formally)

$$\boldsymbol{T}' \circ \boldsymbol{y}' = \boldsymbol{y}' \circ \boldsymbol{T} \quad \text{for all } \boldsymbol{y}' \in Y' \, ,$$

or more precisely

$$\underbrace{(\boldsymbol{T}'\boldsymbol{y}')}_{\in X'} \underbrace{(x)}_{\in X} = \underbrace{\boldsymbol{y}'}_{\in Y'} \underbrace{(\boldsymbol{T}x)}_{\in Y} \quad \text{for all } \boldsymbol{y}' \in Y' \text{ and all } x \in X \, .$$

In the previous definition, it is crucial to properly identify individual objects.

- $\boldsymbol{y}' \in Y'$ is a map working on Y,
- $\boldsymbol{T}'\boldsymbol{y}' \in X'$ is a mapping working on X,
- $\big(\boldsymbol{T}(\cdot)\big)(\cdot)$ is an object that assigns a number to the elements of $X' \times X$. This corresponds to the structure of duality.

Using the so-called *canonical* duality $\langle \boldsymbol{F}, g \rangle = \boldsymbol{F}(g)$ we can write the definition of the dual mapping in symmetric form:

$$\langle \boldsymbol{T}'\boldsymbol{y}', x \rangle = \langle \boldsymbol{y}', \boldsymbol{T}x \rangle \,.$$

Note that on the left we have the representation $X' \times X \to \mathbb{C}$ and on the right we have the representation $Y' \times Y \to \mathbb{C}$.

Lemma 4.9 (On the linearity and continuity of the dual operator) Let X, Y be Banach spaces and $\boldsymbol{T} \in \mathscr{L}(X, Y)$. Then $\boldsymbol{T}' \in \mathscr{L}(Y', X')$.

Proof. The linearity is obvious. Let's show the continuity. Let us fix $\{\boldsymbol{y}'_n\}_{n=1}^{\infty} \subset Y'$ such that $\boldsymbol{y}'_n \to \boldsymbol{y}'$. We show that the sequence $\{\boldsymbol{T}'\boldsymbol{y}'_n\}_{n=1}^{\infty} \subset X'$ converges to $\boldsymbol{T}'\boldsymbol{y}'$:

$$\begin{aligned} \left\|\boldsymbol{T}'\boldsymbol{y}'_n - \boldsymbol{T}'\boldsymbol{y}'\right\|_{X'} &= \sup_{\|x\|\le 1} \left\|\boldsymbol{T}'\boldsymbol{y}'_n(x) - \boldsymbol{T}'\boldsymbol{y}'(x)\right\|_X = \sup_{\|x\|\le 1} \|\boldsymbol{y}'_n(\boldsymbol{T}x) - \boldsymbol{y}'(\boldsymbol{T}x)\|_X = \sup_{\|x\|\le 1} \|(\boldsymbol{y}'_n - \boldsymbol{y}')(\boldsymbol{T}x)\|_X \le \\ &\le \sup_{\|x\|\le 1} \|\boldsymbol{y}'_n - \boldsymbol{y}'\| \, \|\boldsymbol{T}\| \, \|x\| = \|\boldsymbol{y}'_n - \boldsymbol{y}'\| \, \|\boldsymbol{T}\| \to 0 \,, \end{aligned}$$

which we wanted to show. □

Remark It can be shown that: $\boldsymbol{T}$ is a compact operator if and only if $\boldsymbol{T}'$ is a compact operator (the so-called Schauder theorem). Try to prove yourself that $\|\boldsymbol{T}\| = \|\boldsymbol{T}'\|$.

Furthermore, we will be interested in whether $\boldsymbol{T}$ and $\boldsymbol{T}'$ can be equated in some sense (just as we equate the Hilbert space with its dual proper), that is, whether we could write

$$\underbrace{\boldsymbol{T}}_{X \to Y} \simeq \underbrace{\boldsymbol{T}'}_{Y' \to X'} \,.$$

So something like this would hold:

$$X \simeq Y' \quad \text{a} \quad Y \simeq X' \,.$$

If we now consider the duals of the previous expressions, we get

$$X' \simeq Y'' \quad \text{a} \quad Y' \simeq X'' \,.$$

Combining the two "equations" with a bit of *handwaving* gives us the expression

$$X \simeq X'' \quad \text{a} \quad Y \simeq Y'' \,.$$

This could be true for Hilbert spaces, where even $X' \simeq X$ holds. In general, there may not always be a $\boldsymbol{T}'$ for a given $\boldsymbol{T}$, the above properties only apply if there is. But in the Hilbert space the situation is again better.

Theorem 4.10 (On the dual mapping between Hilbert spaces) *Let H_1, H_2 be Hilbert spaces a $\boldsymbol{T} \in \mathscr{L}(H_1, H_2)$. Then there exists exactly one mapping $\boldsymbol{T}' : H_2 \to H_1$ such that*

$$(\boldsymbol{T}x, y)_{H_2} = (x, \boldsymbol{T}'y)_{H_1} \quad \text{for all } x \in H_1,\ y \in H_2 \,. \tag{4.2}$$

In addition, for such a mapping we have:

1. $\boldsymbol{T}' \in \mathscr{L}(H_2, H_1)$,
2. $\|\boldsymbol{T}'\| = \|\boldsymbol{T}\|$.

Remark Applying the complex association to the equation (4.2), we get

$$\overline{(\boldsymbol{T}x, y)}_{H_2} = \overline{(x, \boldsymbol{T}'y)}_{H_1} \to (\boldsymbol{T}'y, x)_{H_1} = (y, \boldsymbol{T}x)_{H_2} \,.$$

In Hilbert spaces, therefore, the scalar product plays a role of the duality.

Proof. Let us fix $y \in H_2$ and define

$$\boldsymbol{L}_y \in \mathscr{L}(H_1, \mathbb{C}) ,$$
$$\boldsymbol{L}_y(x) \coloneqq \big(\boldsymbol{T}x, y\big)_{H_2} .$$

According to the Riesz-Fréchet theorem (Theorem 4.4), there exists exactly one $z \in H_1$ such that

$$\boldsymbol{L}_y(x) = \big(x, z\big)_{H_1} .$$

Now we define the mapping:

$$\boldsymbol{T}' : H_2 \to H_1 ,$$
$$\boldsymbol{T}'(y) \coloneqq z .$$

This mapping has a property

$$(\boldsymbol{T}x, y)_{H_2} = (x, \boldsymbol{T}'y)_{H_1} \quad \text{for any } x \in H_1, y \in H_2 .$$

With this, we proved the first part of the statement about the existence and uniqueness of the mapping $\boldsymbol{T}'$.

Let us show its linearity. Apparently, for every $x \in H_1$ is satisfied

$$\begin{aligned}\big(\boldsymbol{T}'(\alpha y_1 + \beta y_2), x\big)_{H_1} &= (\alpha y_1 + \beta y_2, \boldsymbol{T}x)_{H_2} = \alpha(y_1, \boldsymbol{T}x)_{H_2} + \beta(y_2, \boldsymbol{T}x)_{H_2} = \\ &= \alpha(\boldsymbol{T}'y_1, x)_{H_1} + \beta(\boldsymbol{T}'y_2, x)_{H_1} = (\alpha\boldsymbol{T}'y_1 + \beta\boldsymbol{T}'y_2, x)_{H_1}\end{aligned}$$

and from there

$$\boldsymbol{T}'(\alpha y_1 + \beta y_2) = \alpha\boldsymbol{T}'y_1 + \beta\boldsymbol{T}'y_2 .$$

Let us show the continuity of the mapping $\boldsymbol{T}'$ using the boundedness of its norm. According to the definition of the mapping $\boldsymbol{T}'$ and the second part of the Riesz-Fréchet theorem (Theorem 4.4), we get

$$\left\|\boldsymbol{T}'y\right\|_{H_1} = \|z\|_{H_1} = \|\boldsymbol{L}_y\| . \tag{4.3}$$

Furthermore, by the definition of $\boldsymbol{L}_y$ and the Cauchy-Schwarz inequality,

$$\|\boldsymbol{L}_y x\| = |(\boldsymbol{T}x, y)_{H_2}| \leq \|\boldsymbol{T}x\|_{H_2} \|y\|_{H_2} \leq \|\boldsymbol{T}\| \|x\|_{H_1} \|y\|_{H_2}$$

and from there using (4.3) follows

$$\left\|\boldsymbol{T}'y\right\|_{H_1} = \|\boldsymbol{L}_y\| = \sup_{\|x\|_{H_1} \leq 1} \|\boldsymbol{L}_y x\|_{H_2} \leq \|\boldsymbol{T}\| \|y\|_{H_2} .$$

$$\left\|\boldsymbol{T}'\right\| = \sup_{\|y\|_Y \leq 1} \left\|\boldsymbol{T}'y\right\| \leq \|\boldsymbol{T}\| \sup_{\|y\| \leq 1} \|y\|_{H_2} = \|\boldsymbol{T}\| < +\infty .$$

We have therefore verified the continuity of $\boldsymbol{T}' \in \mathscr{L}(H_2, H_1)$.

It remains to show the equality of the norms $\left\|\boldsymbol{T}'\right\| = \|\boldsymbol{T}\|$. To that end, we define $\boldsymbol{T}'' = (\boldsymbol{T}')'$. We already know about this representation that $\boldsymbol{T}'' \in \mathscr{L}(H_1, H_2)$, $(\boldsymbol{T}''x, y)_{H_2} = (x, \boldsymbol{T}'y)_{H_1}$ and $\left\|\boldsymbol{T}''\right\| \leq \left\|\boldsymbol{T}'\right\|$. Then, using the previous parts of the proof, we get

$$(\boldsymbol{T}''x, y)_{H_2} = (x, \boldsymbol{T}'y)_{H_1} = \overline{(\boldsymbol{T}'y, x)}_{H_1} = \overline{(y, \boldsymbol{T}x)}_{H_2} = (\boldsymbol{T}x, y)_{H_2} \quad \text{for all } x \in H_1, y \in H_2 ,$$

so

$$\boldsymbol{T}''x = \boldsymbol{T}x \quad \text{for each } x \in H_1 .$$

From here

$$\|\boldsymbol{T}\| = \left\|\boldsymbol{T}''\right\| \leq \left\|\boldsymbol{T}'\right\| \leq \|\boldsymbol{T}\| ,$$

and therefore equalities must apply everywhere in the previous chain. $\square$

Definition 4.11 (Hermitian adjoint operator) Let H_1, H_2 be Hilbert spaces, $\boldsymbol{T} \in \mathscr{L}(H_1, H_2)$. We call the representation $\boldsymbol{T}'$ with the properties from the previous theorem a Hermitian adjoint operator with $\boldsymbol{T}$ (or an adjoint operator to $\boldsymbol{T}$).

Definition 4.12 (Self-adjoint (Hermitian) operator.) Let H be a Hilbert space. We call the operator $\boldsymbol{T} \in \mathscr{L}(H)$ self-adjoint (or Hermitian) if $\boldsymbol{T} = \boldsymbol{T}'$.

Remark In the previous definition, both operators $\boldsymbol{T}, \boldsymbol{T}'$ are defined on the entire Hilbert space. We emphasize here that we are talking about "bounded self-adjoint" operators. In the case of unbounded operators, we will see that the domain of operators completely changes the spectral properties, and the definition of Hermitianity and self-adjointness is more complicated.

4.3 Properties of self-adjoint operators

Lemma 4.13 (On eigenvalues of self-adjoint operators) Let $\boldsymbol{T} \in \mathscr{L}(H)$ be self-adjoint. Then it has only real eigenvalues and the eigenvectors belonging to different eigenvalues are mutually perpendicular.

Proof. The first part of the statement follows from the equality

$$\lambda \|x\|_H^2 = (\lambda x, x)_H = (\boldsymbol{T}x, x)_H = (x, \boldsymbol{T}x)_H = (x, \lambda x)_H = \overline{\lambda}(x, x)_H = \overline{\lambda} \|x\|_H^2 \ .$$

Let $\lambda_1 \neq \lambda_2$ be the eigenvalues corresponding to the eigenvectors y_1, y_2. Then from the equality

$$(\lambda_1 - \lambda_2)(y_1, y_2)_H = (\lambda_1 y_1, y_2)_H - (y_1, \lambda_2 y_2)_H = (\boldsymbol{T}y_1, y_2)_H - (y_1, \boldsymbol{T}y_2)_H = (\boldsymbol{T}y_1, y_2)_H - (\boldsymbol{T}y_1, y_2)_H = 0$$

we see that $(y_1, y_2) = 0$ must hold. $\square$

Remark However, the self-adjoint operator can also have other elements of the spectrum. We generally know nothing about them.

Theorem 4.14 (On spectral properties of self-adjoint operators) *Let H be a Hilbert space and $\boldsymbol{T} \in \mathscr{L}(H)$ be a self-adjoint operator. Let's denote*

$$m(\boldsymbol{T}) = \inf \{(\boldsymbol{T}x, x)_H : \|x\|_H = 1\} \qquad M(\boldsymbol{T}) = \sup \{(\boldsymbol{T}x, x)_H : \|x\|_H = 1\}$$

Then

1. *If $\lambda \in \sigma(\boldsymbol{T})$, then $\lambda \in [m(\boldsymbol{T}), M(\boldsymbol{T})]$.*
2. *It holds that $\rho(\boldsymbol{T}) = \|\boldsymbol{T}\|$. Specially: at least one of the values of $\lambda = \pm \|\boldsymbol{T}\|$ is an eigenvalue of $\boldsymbol{T}$.*

$\boxed{\textbf{Exercise}}$ Let $\mathbb{A} \in \mathcal{M}^{n\times n}$ be a real symmetric matrix, $x \in \mathbb{R}^n$ and $f(x) = (\mathbb{A}x, x) = \sum_{i,j} a_{ij} x_i x_j$. Determine the minimum and maximum of f under the condition $\|x\| = 1$.

4.4 Compact self-adjoint operators on Hilbert spaces

First, we repeat the results we have so far on compact self-adjoint operators. On the Hilbert space H, consider $\boldsymbol{K} \in \mathscr{C}(H)$ such that $(\boldsymbol{K}x, y)_H = (x, \boldsymbol{K}y)_H$ for all $x, y\ inH$. Then

1. $\boldsymbol{K}$ has at most countably many eigenvalues and all are real.
2. The only other element of the spectrum $\sigma(\boldsymbol{K})$ can be the number 0, which may or may not be an eigennumber.
3. For every nonzero eigenvalue, there are at most finitely many linearly independent eigenvectors. Moreover, the eigenvectors corresponding to different eigenvalues are always perpendicular to each other.

The basic question of this chapter is:

"If we take the eigenvectors of all nonzero eigenvalues, do they form a basis of H?"

The so-called Hilbert-Schmidt theorem will give us the answer. But first we have an *intermezzo*, in which we will remember direct sum, orthogonal complement and Fourier series.

Definition 4.15 (Direct sum of subspaces) Let H be a linear vector space and A, B be subspaces of H. We say that H is the direct sum of A and B (we write $A \oplus B = H$) if

1. $A + B = H$, i.e. for every $h \in H$ there exist $a \in A$, $b \in B$ such that $a + b = h$,
2. $A \cap B = \{0\}$.

Example If A and B are two different (not necessarily perpendicular) straight lines crossing the origin, then $\mathbb{R}^2 = A \oplus B$ can be written.

Definition 4.16 (Orthogonal complement) Let H be a Hilbert space and A be a closed linear subspace in H. We define the orthogonal complement $A^\perp \subset H$ by the formula

$$A^\perp := \{y \in H : (x, y) = 0 \text{ for each } x \in A\} \ .$$

Theorem 4.17 (On properties of the orthogonal complement) *Let H be a Hilbert space and let A be a closed linear subspace of H and $A^\perp$ be its orthogonal complement. Then*

1. *$A^\perp$ is a closed linear subspace of H,*
2. *$(A^\perp)^\perp = A$,*
3. *$A \oplus A^\perp = H$.*

Proof.

1. Linearity is obvious. The closure follows from the continuity of the scalar product. For the convergent sequence $\{y_n\}_{n=1}^\infty : y_n \to y$ we have

$$0 = \big(x, y_n\big)_H \to \big(x, y\big)_H = 0\,.$$

2. We leave the second part to the reader as a simple exercise.
3. The third part can best be seen in the context of the so-called *perpendicular projection lemma*:

Lemma 4.18 If A is a linear subspace of H, then for each $x \in H \setminus A$ there exists an element $Px \in A$ such that

$$\big(x - Px, y\big) = 0 \quad \text{for all } y \in A\,,$$

or $x - Px \in A^\perp$.

Now the statement is obvious. For $x \in A$ we have $x = x + 0$. For $x \in H \setminus A$ we can write

$$x = \underbrace{(x - Px)}_{\in A^\perp} + \underbrace{Px}_{\in A}\ .$$

Moreover, for $v \in A \cap A^\perp$ we have $\big(v, v\big) = 0$, hence $v = 0$. This proved the directness of the sum.

□

Recall that a metric space X is said to be separable if there exists a countable dense set in it. An orthogonal system in H is a sequence $\{e_n\}_{n=1}^{\infty} \subset H$ such that $(e_i, e_j)_H = 0$ holds for $i \neq j$. An orthogonal system is said to be complete if and only if

$$y \in H, (y, e_n)_H = 0 \;\; \forall n \implies y = 0\,.$$

Theorem 4.19 (On Fourier series in Hilbert space.) *Let H be a Hilbert space. Then the following statements are equivalent:*

1. *H is a separable space,*
2. *There exists a complete countable orthogonal system $\{e_n\}_{n=1}^{\infty} \subset H$,*
3. *Each element of H is the sum of its Fourier series, i.e. for each $x \in H$ it holds*

$$x = \sum_{n=1}^{\infty} \frac{(x, e_n)_H}{\|e_n\|^2} e_n\,,$$

4. *For every $x \in H$, it holds*

$$\|x\|_H^2 = \sum_{n=1}^{\infty} \frac{(x, e_n)_H^2}{\|e_n\|^2}\,. \qquad \textit{(so-called Parseval's equality)}$$

We are now ready to prove the key theorem of this chapter.

Theorem 4.20 (Hilbert-Schmidt) *Let H be a Hilbert space, $\boldsymbol{K}$ is a compact self-adjoint operator on H. Let Λ denote the closed subspace H generated by all eigenvectors $\boldsymbol{K}$ corresponding to all nonzero eigenvalues $\boldsymbol{K}$. Then*

$$H = \Lambda \oplus \operatorname{Ker} \boldsymbol{K}\,.$$

Proof. We divide the proof of the theorem into several steps.

Step 1: Properties of the Λ space.

Due to the compactness and self-adjointness of $\boldsymbol{K}$, there exists at most a countable sequence $\{\lambda_j\}_{j=1}^{\infty} \subset \sigma_{\mathrm{P}}(K)$ of nonzero eigenvalues. Let us denote the subspace corresponding to the j-th eigenvalue

$$E_j = \operatorname{Ker}\,(\boldsymbol{K} - \lambda_j \boldsymbol{Id}) = \{x \in H : x \neq 0, \boldsymbol{K}x = \lambda_j x\}$$

Due to the compactness of $\boldsymbol{K}$, $\dim E_j := n_j < +\infty$. Let B_j denote the basis of the space E_j. Without loss of generality, we can assume that this basis is orthogonalized (otherwise we can use, for example, Gram-Schmidt orthogonalization). Let's define

$$B := \bigcup_{j=1}^{\infty} B_j\,.$$

Then even B is at most a countable set and is formed by eigenvectors. Let us show that it is also an orthogonal set: we choose $x \neq y \in B$. Then both elements either belong to the same B_j and are perpendicular to each other due to the assumption above, or they are from different bases B_i, B_j and are perpendicular to each other due to the self-adjointness of $\boldsymbol{K}$.

Let us define the space of all at most countable sums of elements from B by the formula

$$\Lambda := \overline{\operatorname{Lin}\, B} = \left\{ z \in H : z = \sum_{j=1}^{\infty} \alpha_j e_j,\; \alpha_j \in \mathbb{C},\; e_j \in B \right\}\,.$$

Then Λ is a closed linear subspace of H, so it is a Hilbert space.

Moreover, Λ is separable. The countable dense set in it is

$$\left\{ z \in H : z = \sum_{j=1}^{N} (r_j + \mathrm{i}q_j)\, e_j,\ r_j, q_j \in \mathbb{Q},\ e_j \in B,\ N \in \mathbb{N} \right\}$$

Step 2: The sets $\boldsymbol{K}(\Lambda)$ and $\boldsymbol{K}(\Lambda^\perp)$.
We first show that $\boldsymbol{K}(\Lambda) \subset \Lambda$. Let's choose $x \in \Lambda$. Then

$$\boldsymbol{K}x = \boldsymbol{K}\left(\sum_{j=1}^{\infty} \gamma_j e_j\right) = \sum_{j=1}^{\infty} \gamma_j \boldsymbol{K} e_j = \sum_{j=1}^{\infty} \gamma_j \lambda_j e_j\ ,$$

thus $\boldsymbol{K}x \in \Lambda$.

Next, we show that $\boldsymbol{K}(\Lambda^\perp) \subset \Lambda^\perp$. Let's choose $x \in \Lambda, y \in \Lambda^\perp$. Then $\boldsymbol{K}x \in \Lambda^\perp$ and from the equality

$$(\boldsymbol{K}y, x)_H = (y, \boldsymbol{K}x)_H = 0$$

we see that $\boldsymbol{K}y \in \Lambda^\perp$ must necessarily hold. Thus, we proved that $\boldsymbol{K}(\Lambda^\perp) \subset \Lambda^\perp$.

Step 3: The validity of the statement $\boldsymbol{K}(\Lambda^\perp) = \{0\}$.
Let us define the restriction of the operator $\boldsymbol{K}$ by the formula

$$\tilde{\boldsymbol{K}} := \boldsymbol{K}|_{\Lambda^\perp}$$

Since $\boldsymbol{K}(\Lambda^\perp) \subset \Lambda^\perp$, $\tilde{\boldsymbol{K}} : \Lambda^\perp \to \Lambda^\perp$. Thus $\tilde{\boldsymbol{K}}$ is also a compact and self-adjoint operator on $\Lambda^\perp$.

We show by contradiction that $\tilde{\boldsymbol{K}}$ has no nonzero eigenvalue. Let the negation of this statement hold and there exists an eigenvalue $\lambda \neq 0$ and an eigenvector $y \in \Lambda^\perp, y \neq 0$ such that $\tilde{\boldsymbol{K}}y = \lambda y$. By self-adjointness, then

$$\boldsymbol{K}y = \tilde{\boldsymbol{K}}y = \lambda y\ ,$$

and therefore λ is also an eigenvalue of the operator $\boldsymbol{K}$. However, according to the second part of the proof, then $y \in \Lambda$. Thus, $y \in \Lambda \cap \Lambda^\perp$, which implies $y = 0$ and this contradicts the assumptions.

Overall, we find that $\tilde{\boldsymbol{K}}$ is a compact operator that does not have a nonzero eigenvalue, that is, $\sigma(\tilde{\boldsymbol{K}}) \subset \{0\}$. According to Theorem 4.14 on spectral properties of self-adjoint operators $\rho(\tilde{\boldsymbol{K}}) = 0$, and therefore $\left\|\tilde{\boldsymbol{K}}\right\| = 0$, which is equivalent to $\tilde{\boldsymbol{K}} \equiv 0$. From the construction of $\tilde{\boldsymbol{K}}$, we can easily see that $\boldsymbol{K}(\Lambda^\perp) = \{0\}$.

Step 4: Completing the proof.
The previous part of the proof gives us $\Lambda^\perp \subset \operatorname{Ker} \boldsymbol{K}$. Obviously, $\Lambda \oplus \Lambda^\perp = H$, so

$$\Lambda + \operatorname{Ker} \boldsymbol{K} = H\ .$$

It remains to show that $\Lambda \cap \operatorname{Ker} \boldsymbol{K} = \{0\}$, thereby proving the directness of the sum of these spaces.

Let's choose $z \in \Lambda \cap \operatorname{Ker} \boldsymbol{K}$. Thanks to its membership in Λ, one can write

$$0 = \boldsymbol{K}z = \boldsymbol{K}\left(\sum_{j=1}^{\infty} \alpha_j e_j\right) = \sum_{j=1}^{\infty} \alpha_j \boldsymbol{K} e_j = \sum_{j=1}^{\infty} \alpha_j \lambda_j e_j\ .$$

This is the Fourier series of the zeroth element. From the theory of uniqueness of Fourier series it follows that

$$\alpha_j \lambda_j = 0 \quad \text{for all } j \in \mathbb{N}\ .$$

Due to the nonzeroness of λ_j, we get

$$\alpha_j = 0 \quad \text{for all } j \in \mathbb{N} \tag{4.4}$$

and thus $z = 0$, which we wanted to show. □

Remark 1. The first part of the proof actually just illuminates the properties of the Λ space. Let's note that we used the separability of the Λ space at the end when we decomposed the zero element into a Fourier series, which we could not have done otherwise.

2. In the proof, we only needed to show that $\Lambda^\perp \subset \mathrm{Ker}\ \boldsymbol{K}$. It can even be shown that $\Lambda^\perp = \mathrm{Ker}\ \boldsymbol{K}$.

   Let us choose $z \in \mathrm{Ker}\ \boldsymbol{K} \setminus \Lambda^\perp$ such that $z \neq 0$. Then there exists $n \in \mathbb{N}$ such that $\big(z, e_n\big) \neq 0$. (Indeed, otherwise $\big(z, \sum_{j=1}^\infty \beta_n e_n\big) = 0$ and hence $z \in \Lambda^\perp$ would hold.) But then $\boldsymbol{K}z = 0$, since $z \in \mathrm{Ker}\ \boldsymbol{K}$. From here we get for all $n \in \mathbb{N}$

$$0 = \big(\boldsymbol{K}z, e_n\big) = \big(z, \boldsymbol{K}e_n\big) = \big(z, \lambda_n e_n\big) = \lambda_n \big(z, e_n\big)\,,$$

   which is in conflict with $\lambda_n \neq 0$. We thereby verified that $\Lambda^\perp = \mathrm{Ker}\ \boldsymbol{K}$.

Corolarry 4.21 (On the decomposition of the Hilbert space using a compact self-adjoint operator) Let H be a Hilbert space, $\boldsymbol{K}$ is a compact self-adjoint operator on H, $\{e_j\}_{j=1}^\infty \subset H$ is an orthogonal system of all eigenvectors belonging to all nonzero eigenvalues of the operator $\boldsymbol{K}$. Then for each element $h \in H$ there exists a sequence $\{\alpha_j\}_{j=1}^\infty$ such that

$$h = \sum_{j=1}^\infty \alpha_j e_j + z\,, \tag{4.5}$$

while $z \in \mathrm{Ker}\ \boldsymbol{K}$. Further

$$\boldsymbol{K}h = \sum_{j=1}^\infty \alpha_j \lambda_j e_j \tag{4.6}$$

and holds

$$\alpha_k = \frac{(h, e_k)}{\|e_k\|^2} \quad \text{for all } k \in \mathbb{N}\,.$$

Remark The Hilbert-Schmidt theorem 4.20 and the corollary 4.21 extend the well-known theorem from the fourth semester of mathematical analysis (Theorem 4.19) also to Hilbert spaces that are not separable. The non-separable part of the space is precisely Ker $\boldsymbol{T}$. While this non-separable part also appears in the equation (4.5), it no longer appears in the equation (4.6). This property then makes it possible to use the properties of decomposition into Fourier series for compact self-adjoint operators even on non-separable spaces.

At the end of the chapter, we say goodbye with a theorem that (among other things) allows constructing compact operators on Hilbert spaces.

Theorem 4.22 *Let H be a separable Hilbert space, $\{e_j\}_{j=1}^\infty \subset H$ be a complete orthonormal system, and $\{\alpha_j\}_{j=1}^\infty \subset \mathbb{C}$ be a bounded sequence. Let's mark*

$$M := \sup_{j\in\mathbb{N}}\{|\alpha_j|\} < +\infty\,.$$

Let us define the operator $\boldsymbol{T}$ by the formula

$$\boldsymbol{T}h = \sum_{j=1}^\infty (h, e_j)_H \cdot \alpha_j e_j\,.$$

Then

1. *$\boldsymbol{T}$ is well defined on the entire H, $\boldsymbol{T} \in \mathscr{L}(H)$ and $\|\boldsymbol{T}\| = M$.*
2. *$\boldsymbol{T}$ is self-adjoint $\iff \alpha_j \in \mathbb{R}$ for all $j \in \mathbb{N}$.*

3. $\boldsymbol{T}$ *is compact* $\iff$ *there exists a permutation of the sequence* $\{\alpha_j\}_{j=1}^{\infty}$ *such that* $\lim_{n\to\infty} \alpha_{j_n} = 0$.

Example

1. Let's choose the sequence $\alpha_n = 1$. Then the operator generated by such a sequence is apparently $\boldsymbol{Id}$. According to the second and third parts of the previous theorem, $\boldsymbol{Id}$ is self-adjoint but not compact.
2. Let's choose the sequence $\alpha_n = \dfrac{1}{n}$. Then, according to the previous theorem, this sequence generates an operator that is self-adjoint and, moreover, is compact due to $\alpha_n \to 0$.

Remark In quantum statistical mechanics, one works with the so-called density operator $\hat{\rho}$ defined by the formula

$$\hat{\rho} := \sum_{j=1}^{\infty} p_j |\psi_j\rangle\langle\psi_j| \,, \quad \text{where } \{|\psi_j\rangle\}_{j=1}^{\infty} \in H \text{ are such that } |\langle\psi_j|\psi_j\rangle| = 1 \text{ and } \sum_{j=1}^{\infty} p_j = 1 \,.$$

Note that in non-Diracian notation we would write such an operator with the formula

$$\boldsymbol{\rho}(h) = \sum_{j=1}^{\infty} p_j (\psi_j, h)_H \psi_j \,.$$

To fulfill the conditions from the previous theorem, it lacks only that $\{|\psi_j\rangle\}_{j=1}^{\infty}$ does not have to form (and usually does not form) an orthonormal sequence. However, it can be shown that the density operator is always compact and self-adjoint. It even belongs to a special class of compact operators, for which the so-called operator trace can be defined by the formula

$$\operatorname{Tr} \hat{\rho} := \sum_{j=1}^{\infty} \langle\psi_j|\hat{\rho}|\psi_j\rangle$$

(so-called *trace-class* operators). More can be found, for example, in the book: BLANK, Jiří, Pavel EXNER and Miloslav HAVLÍČEK. Hilbert Space Operators in Quantum Physics Theoretical and Mathematical Physics. Melville, N.Y.: Springer, 2008. ISBN 9781402088698.

5 Unbounded operators

In the introductory chapter, we showed the characterization of continuous linear operators on Banach spaces: for $\boldsymbol{T}: X \to Y$ linear holds

$$\boldsymbol{T} \text{ is bounded } \Leftrightarrow \|\boldsymbol{T}\| < +\infty \Leftrightarrow \boldsymbol{T} \text{ is continuous} .$$

Note in particular that the chain of these equivalences also gives this chain of equivalences of negated propositions:

$$\boldsymbol{T} \text{ not bounded } \Leftrightarrow \|\boldsymbol{T}\| = +\infty \Leftrightarrow \boldsymbol{T} \text{ not continuous} .$$

In this chapter, we will introduce some properties of linear but unbounded and thus discontinuous operators. We will work only on Hilbert spaces, where we have the properties of the scalar product at our disposal. It turns out that there are problems with the very domain of the relevant self-adjoint operator, and even the $\boldsymbol{L}$ operator itself.

Remark Instead of $\boldsymbol{L}'$ as the notation of an adjoint operator, here we will use the notation $\boldsymbol{L}^*$ to distinguish adjointness from the notation of derivatives on function spaces.

5.1 Symmetry and self-adjointness

Definition 5.1 (Adjoint operator) Let H be a Hilbert space, $\mathcal{D}(\boldsymbol{L}) \subset H$ be a linear subspace, and $\boldsymbol{L} : \mathcal{D}(\boldsymbol{L}) \to H$ be a linear operator.

1. Let us define $\mathcal{D}(\boldsymbol{L}^*) \coloneqq \{y \in H;\ \exists! h^* \in H,\ \big(\boldsymbol{L}x, y\big) = \big(x, h^*\big)\ \forall x \in \mathcal{D}(\boldsymbol{T})\}$
2. If $\mathcal{D}(\boldsymbol{L}^*)$ is a non-empty set, we define the adjoint operator $\boldsymbol{L}^* : \mathcal{D}(\boldsymbol{L}^*) \to H$ by the formula

$$\boldsymbol{L}^* y = h^* .$$

Remark If $\mathcal{D}(\boldsymbol{L}^*) \neq \emptyset$, then from the definition we immediately get

$$(\boldsymbol{L}x, y)_H = (x, \boldsymbol{L}^* y)_H \quad \text{for all } x \in \mathcal{D}(\boldsymbol{L}), y \in \mathcal{D}(\boldsymbol{L}^*) .$$

While for bounded (continuous) operators the previous equality is a consequence of the Riesz-Fréchet theorem (Theorem 4.4), for unbounded operators this equality must be postulated.

Naturally, we also introduce the notion of a self-adjoint operator.

Definition 5.2 (Self-adjoint operator) We call the operator $\boldsymbol{L} : \mathcal{D}(\boldsymbol{L}) \to H$ self-adjoint if

1. $\exists \mathcal{D}(\boldsymbol{L}^*) \neq \emptyset,\ \mathcal{D}(\boldsymbol{L}^*) = \mathcal{D}(\boldsymbol{L})$,
2. $\boldsymbol{L}^* = \boldsymbol{L}$ on $\mathcal{D}(\boldsymbol{L}^*)$.

Remark The equality of domains is very important here. We will see later that if $\mathcal{D}(\boldsymbol{L}) \neq \mathcal{D}(\boldsymbol{L}^*)$ and $\boldsymbol{L} = \boldsymbol{L}^*$ on $\mathcal{D}(\boldsymbol{L}) \cap \mathcal{D}(\boldsymbol{L}^*)$, we get different spectral properties for $\boldsymbol{L}$ than if it were self-adjoint.

The following lemma follows directly from the definition.

Lemma 5.3 $\mathcal{D}(\boldsymbol{L}^*) \neq \emptyset \Rightarrow \boldsymbol{L}^*$ is linear.

Important question: When is $\mathcal{D}(\boldsymbol{T}^*) \neq \emptyset$?

Theorem 5.4 (On the existence of an adjoint operator) *$\boldsymbol{L}$ is a linear operator with domain $\mathcal{D}(\boldsymbol{L})$. Then*

$$\mathcal{D}(\boldsymbol{L}^*) \neq \emptyset \Leftrightarrow \overline{\mathcal{D}(\boldsymbol{L})} = H .$$

Proof. Can be found in Lukeš textbook, 11.6. □

Remark: The simplest implementation of $\overline{\mathcal{D}(\boldsymbol{L})} = H$ appears to be $\mathcal{D}(\boldsymbol{L}) = H$ directly. We will get to the explanation of why this situation cannot occur later.

Definition 5.5 (Symmetric operator) $\boldsymbol{L} : \mathcal{D}(\boldsymbol{L}) \to H$ is a linear operator. We say that $\boldsymbol{L}$ is symmetric if

$$\big(\boldsymbol{L}x, y\big)_H = \big(x, \boldsymbol{L}y\big)_H \quad \forall x, y \in \mathcal{D}(\boldsymbol{L}) \,.$$

Which is not the same as self-adjointness for unbounded operators.[4]

Lemma 5.6

$$\boldsymbol{L} \text{ is symmetric } \Leftrightarrow \begin{cases} & \mathcal{D}(\boldsymbol{L}) \subseteq \mathcal{D}(\boldsymbol{L}^*) \\ & \boldsymbol{L} = \boldsymbol{L}^* \text{ on } \mathcal{D}(\boldsymbol{L}) \end{cases}$$

This implies $\underline{\boldsymbol{L} \text{ is self-adjoint} \Rightarrow \boldsymbol{L} \text{ is symmetric}}$. In particular, $\underline{\boldsymbol{L} \text{ is not symmetric} \Rightarrow}$ $\underline{\boldsymbol{L} \text{ is not self-adjoint}}$. This is often used in practice to prove that $\boldsymbol{L}$ is not self-adjoint, without looking for $\mathcal{D}(\boldsymbol{L}^*)$.

Theorem 5.7 (On the boundedness of a symmetric operator on the entire space)

$$\left.\begin{array}{l} \mathcal{D}(\boldsymbol{L}) = H \\ \boldsymbol{L} \textit{ is linear and symmetric} \end{array}\right\} \Rightarrow \boldsymbol{L} \textit{ is bounded}$$

Proof. Can be found again in the textbook Lukeš, 11.10. □

It flows from there

$$\left.\begin{array}{l} \mathcal{D}(\boldsymbol{L}) = H \\ \boldsymbol{L} \text{ is linear and self-adjoint} \end{array}\right\} \Rightarrow \boldsymbol{L} \text{ is bounded,}$$

thus, the unbounded, self-adjoint operator has $\mathcal{D}(\boldsymbol{L}) \neq H$.

The only possible situation for self-adjoint unbounded operators on H is

$$\mathcal{D}(\boldsymbol{L}) \neq H,\ \overline{\mathcal{D}(\boldsymbol{L})} = H$$
$$\mathcal{D}(\boldsymbol{L}) \text{ is lin. subspace.}$$

Such an operator $\boldsymbol{L}$ is said to be *densely defined* on H.

[4] The terminology used in the literature varies. Some authors use the terms "symmetric" and "self-adjoint" (in addition to this text, e.g. Lukeš, Formánek) and others "Hermitian" (for what we call here symmetric) and "self-adjoint" (e.g. Černý+Pokorný, Čihák). Many authors do not even distinguish between the terms, mostly because of working with bounded operators, where all the terms really merge.

Exercise (An operator of differentiation) $H = L^2\left((0,1)\right)$, $\mathcal{D}(\boldsymbol{L}) = \mathcal{C}^1\left([0,1]\right)$. Let us define $\boldsymbol{L}f = f'$ on this space.
We know that $\overline{\mathcal{C}^1\left([0,1]\right)} = L^2((0,1))$. The operator is apparently linear and unbounded.
Let us examine symmetry as a necessary condition for self-adjointness. Apparently it is

$$\left(\boldsymbol{L}f, g\right)_{L^2} = \left(f', g\right)_{L^2} = \int_0^1 f'(x)\overline{g}(x)\,\mathrm{d}x\,, \qquad \left(f, \boldsymbol{L}g\right)_{L^2} = \int_0^1 f(x)\overline{g}'(x)\,\mathrm{d}x\,.$$

Applying the per-partes method, we get

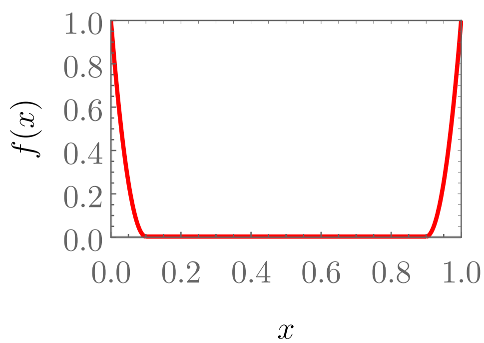
$$\int_0^1 f'(x)\overline{g(x)}\,\mathrm{d}x = \Big[f(x)\overline{g(x)}\Big]_0^1 - \int_0^1 f'(x)\overline{g(x)}\,\mathrm{d}x\,.$$

Even if we get rid of the boundary terms using appropriate boundary conditions, the derivative operator $\boldsymbol{L}$ can never be symmetric, i.e. not even self-adjoint, due to the change of sign before the integral.
We now determine $\mathcal{D}(\boldsymbol{L}^*)$ by examining the set

$$\left\{g \in C^1\left([0,1]\right) : \exists! h^* \in L^2\left((0,1)\right) \text{ such that } \overbrace{\left(\boldsymbol{L}f, g\right)_{L^2} = \left(f, h^*\right)_{L^2}}^{[f\overline{g}]_0^1 - \int_0^1 f\overline{g'} = \int_0^1 f\overline{h^*} \quad (\star)} \ \forall f \in C^1\left([0,1]\right)\right\},$$

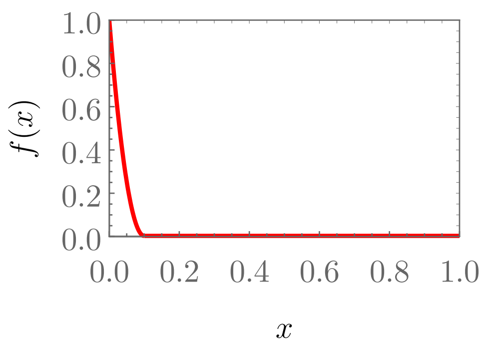
By choosing special functions f, we find the necessary conditions for $\mathcal{D}(\boldsymbol{L}^*)$.

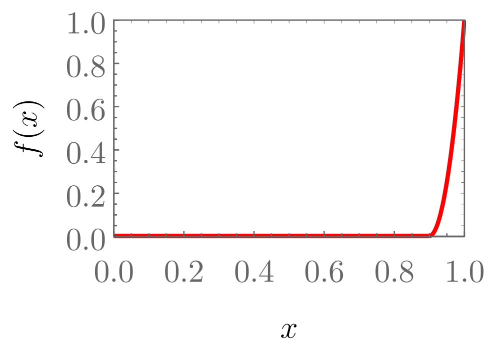
1. We choose the functions f_ε as in the picture:



   Substituting these f_ϵ into $(\star)$ and performing the $\epsilon \to 0^+$ limit, we get $[f\overline{g}]_0^1 = 0$.

2. We choose functions f_ε according to one of the images





   we get the condition $g(0) = g(1) = 0$. This is the first refinement from which

$$\mathcal{D}(\boldsymbol{L}^*) \subseteq \{g \in \mathcal{C}^1([0,1]),\ g(0) = g(1) = 0\}$$

3. Thus, $(\star)$ reduces to

$$-\int_0^1 f\overline{g'} = \int_0^1 f\overline{h^*}$$

$$\int_0^1 f\overline{(g'+h^*)} = 0 \;\; \forall f \in \mathcal{C}^1([0,1]).$$

   It follows (from the du Bois-Reymond lemma) that $g' + h^*$ is s.v. equal to zero, and thus h^* is s.v. equally continuous function $-g'$. Thus, after redefining h^* at the points of measure zero (by the values of the $-g'$ function), h^* can be considered continuous.

We have found h^*, so there is no need to modify $\mathcal{D}(\boldsymbol{L}^*)$ further. We have

$$\mathcal{D}(\boldsymbol{L}^*) = \{g \in \mathcal{C}^1([0,1]),\; g(0) = g(1) = 0\}$$
$$\boldsymbol{L}^* g = -g'$$

Evidently $\boldsymbol{L} \neq \boldsymbol{L}^*$, moreover $\mathcal{D}(\boldsymbol{L}^*) \subsetneqq \mathcal{D}(\boldsymbol{L})$. So it is definitely not a self-adjoint operator.

$\boxed{\textbf{Exercise}}$ (Modified operator of differentiation) For the self-adjointness of this operator, it is necessary to modify both $\boldsymbol{L}$ (so that $\boldsymbol{L}^* = \boldsymbol{L}$) and $\mathcal{D}(\boldsymbol{L})$ (so that $\mathcal{D}(\boldsymbol{L}^*) = \mathcal{D}(\boldsymbol{L})$). The idea for the modification comes from an observation

$$\boldsymbol{L}f = f' \;\Rightarrow\; \boldsymbol{L}^* f = -f',$$

which says that the minus sign must be "halved between $\boldsymbol{L}$ and $\boldsymbol{L}^*$". So let's define

$$\boldsymbol{L}f := if' \quad \text{ on } L^2\left((0,1)\right).$$

Since a necessary condition for self-adjointness is symmetry, it will be necessary for symmetry to have properly captured boundary conditions in $\mathcal{D}(\boldsymbol{L})$. Let's consider three options

1. $\mathcal{D}(\boldsymbol{L}_1) = L^2\left((0,1)\right) \cap C^1\left([0,1]\right)$,
2. $\mathcal{D}(\boldsymbol{L}_2) = L^2\left((0,1)\right) \cap C^1\left([0,1]\right) \cap \{f(x) : f(0) = f(1)\}$,
3. $\mathcal{D}(\boldsymbol{L}_3) = L^2\left((0,1)\right) \cap C^1\left([0,1]\right) \cap \{f(x) : f(0) = f(1) = 0\}$

and three restrictions

$$\boldsymbol{L}_1 = \boldsymbol{L}|_{\mathcal{D}(\boldsymbol{L}_1)}\,, \qquad \boldsymbol{L}_2 = \boldsymbol{L}|_{\mathcal{D}(\boldsymbol{L}_2)}\,, \qquad \boldsymbol{L}_3 = \boldsymbol{L}|_{\mathcal{D}(\boldsymbol{L}_3)}\,.$$

Let's explore symmetry again using per partes:

$$\big(\boldsymbol{L}f, g\big) = \int_0^1 if'\overline{g} = \underbrace{[if\overline{g}]_0^1}_{(\star\star)} - i\int_0^1 f\overline{g}' = \underbrace{[if\overline{g}]_0^1}_{(\star\star)} + \int_0^1 f\overline{ig'} = (\star\star) + \big(f, \boldsymbol{L}g\big)$$

$$(\star\star)\begin{cases} = 0 \;\text{ for } f, g \in \mathcal{D}(\boldsymbol{L}_2), \text{ or } f, g \in \mathcal{D}(\boldsymbol{L}_3) \\ \neq 0 \;\text{ for } f, g \in \mathcal{D}(\boldsymbol{L}_1), \end{cases}$$

from which it follows that only $\boldsymbol{L}_2, \boldsymbol{L}_3$ are symmetric, and $\boldsymbol{L}_1$ is not symmetric. As an exercise, try to show yourself

- $\mathcal{D}(\boldsymbol{L}_1^*) = \mathcal{D}(\boldsymbol{L}_3) \subsetneqq \mathcal{D}(\boldsymbol{L}_1)$ (another confirmation that $\boldsymbol{L}_1$ is not symmetric)
- $\mathcal{D}(\boldsymbol{L}_2^*) = \mathcal{D}(\boldsymbol{L}_2)$ (i.e. it is symmetric and can also be self-adjoint if $\boldsymbol{L}_2^* = \boldsymbol{L}_2$)
- $\mathcal{D}(\boldsymbol{L}_3^*) = \mathcal{D}(\boldsymbol{L}_1) \supsetneqq \mathcal{D}(\boldsymbol{L}_3)$ (i.e. confirmation of symmetry, but at the same time proof that $\boldsymbol{L}_3$ *is not* self-adjoint.)

The only candidate for self-adjointness is $\boldsymbol{L}_2$, which is symmetric and satisfies $\mathcal{D}(\boldsymbol{L}_2^*) = \mathcal{D}(\boldsymbol{L}_2)$. It remains to verify that $\boldsymbol{L} = \boldsymbol{L}^*$ on this domain of definition. This flows similarly to the previous example. From symmetry we have

$$\big(\boldsymbol{L}f, g\big) = \big(f, \boldsymbol{L}g\big) \overset{\text{on } \mathcal{D}(\boldsymbol{L}_2^*)}{=} \big(f, h^*\big) = \big(f, \boldsymbol{L}^* g\big) \quad \forall f \in \mathcal{C}^1([0,1])$$

and we proceed in the same way using the du Bois-Reymond lemma.
<u>Conclusion:</u> $\boldsymbol{L}_1$ is not symmetric (nor self-adjoint), $\boldsymbol{L}_3$ is symmetric (but not self-adjoint), and $\boldsymbol{L}_2$ is self-adjoint. We see that even in the case of $\mathcal{D}(\boldsymbol{L})$ the boundary conditions is "split" between $\mathcal{D}(\boldsymbol{L}_2)$ and $\mathcal{D}(\boldsymbol{L}_2^*)$.

Remark Recall the quantum mechanical momentum operator

$$\hat{p} = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \quad \text{defined on } L^2(\mathbb{R}) \cap C^1(\mathbb{R})\,.$$

By modifying the previous example, we easily see that the operator defined on such a set is symmetric, but not generally self-adjoint. It is necessary to realize from which space we take the eigenvectors. If we were trying to find a solution to the equation

$$-i\hbar \frac{\mathrm{d}\psi}{\mathrm{d}x} = k\psi(x)\,,$$

we would find that solving the equation $\psi(x,k)$ for the eigenfunctions

$$\psi(x,k) = Ce^{\frac{ik}{\hbar}x},$$

these do not belong to $L^2(\mathbb{R})$. Thus, the momentum operator on $L^2(\mathbb{R})$ has no eigenvectors. In the following chapter, we will be able to deal with this deficiency using the so-called weighted Hilbert spaces L^2_ϱ.

5.2 Spectrum of unbounded Ooperators

In the case of bounded operators, the following classes of operators played a crucial role in the character of the spectrum:

- Self-adjoint operators: they are also introduced for unbounded operators, but in a more complicated way.
- Compact operators: however, no unbounded operator can be compact because every compact operator is already bounded. The role of compact operators for unbounded operators will be partially taken over by the class of so-called closed operators, which we will now define.

Definition 5.8 (Closed operator) $\boldsymbol{L} : \mathcal{D}(\boldsymbol{L}) \to H$ is a linear operator densely defined on H. We say that $\boldsymbol{L}$ is closed if for $x_n \in \mathcal{D}(\boldsymbol{L})$:

$$\left.\begin{array}{rcl} x_n & \to & x \in H \\ \boldsymbol{L}x_n & \to & g \in H \end{array}\right\} \Rightarrow x \in \mathcal{D}(\boldsymbol{L}),\ \boldsymbol{L}x = g,$$

or if $\boldsymbol{L}$ has a *closed graph*: $[x_n, \boldsymbol{L}x_n] \to [x,g] \ \Rightarrow g = \boldsymbol{L}x$ and $[x, \boldsymbol{L}x] \in$ graph.

Next, for bounded operators, we studied whether they are injective and surjective. This question makes sense even with unbounded operators. Finally, we studied the continuity of inverse operators. Quite surprisingly, it makes sense to deal with this question in this chapter as well. It turns out that discontinuous linear operators in infinite dimension can b closed even though they are discontinuous and can have a continuous inversion even though they are themselves discontinuous. This just goes to show how little intuition we have in infinite dimensional spaces.

Theorem 5.9 (On spectral properties of unbounded operators) $\boldsymbol{L} : \mathcal{D}(\boldsymbol{L}) \to H$ *is a linear operator densely defined on* H*. Then*

1. $\overline{\mathcal{R}(\boldsymbol{L})} = H \ \Rightarrow\ \boldsymbol{L}$ *is injective and surjective* $\mathcal{R}(\boldsymbol{L})$.
2. $\mathcal{R}(\boldsymbol{L}) = H \ \Rightarrow\ \boldsymbol{L}$ *is injective and surjective on* H*, self-adjoint, and* $\boldsymbol{L}^{-1}$ *is continuous.*
3. $\boldsymbol{L}^{-1}$ *is continuous* $\Leftrightarrow$ $\boldsymbol{L}$ *is injective and surjective on* H *and closed.*

Proof. See e.g. Rudin: Functional analysis, 13.11 et seq. □

Definition 5.10 (Resolvent, operator spectrum)

- Resolvent $\boldsymbol{L} \equiv \mathrm{Res}\,(\boldsymbol{L}) := \{\lambda \in \mathbb{C},\ \boldsymbol{L}_\lambda$ injective and surjective on $H,\ \boldsymbol{L}_\lambda^{-1}$ continuous$\}$.

- Spectrum $\boldsymbol{L} \equiv \sigma(\boldsymbol{L}) := \mathbb{C} \backslash \mathrm{Res}\ \boldsymbol{L} = \begin{cases} \text{point spectrum (eigenvalues)} := \{\lambda \in \mathbb{C},\ \exists x \in \mathcal{D}(\boldsymbol{L}),\ x \neq 0,\ \boldsymbol{L}x = \lambda x\} \\ \text{the rest} \end{cases}$

Remark The spectrum of an unbounded operator can be any unbounded subset of $\mathbb{C}$, including the entire $\mathbb{C}$.

Spectrum properties of linear unbounded operators $\boldsymbol{L}$ on H

1. $\boldsymbol{L}$ is closed $\Rightarrow \sigma(\boldsymbol{L})$ is a closed set in $\mathbb{C}$.
2. $\boldsymbol{L}$ is closed and symmetric, then exactly one of the following situations occurs:
   (a) $\sigma(\boldsymbol{L}) = \mathbb{C}$,
   (b) $\sigma(\boldsymbol{L}) = \{\lambda \in \mathbb{C},\ \mathrm{Im}\,\lambda \geq 0\}$,
   (c) $\sigma(\boldsymbol{L}) = \{\lambda \in \mathbb{C},\ \mathrm{Im}\,\lambda \leq 0\}$,
   (d) $\sigma(\boldsymbol{L})$ is a closed subset of $\mathbb{R}$.

   Only in d) is $\boldsymbol{L}$ self-adjoint, otherwise it is only symmetric.
3. If $\boldsymbol{L}$ is symmetric and has real eigenvalues (or if it is closed and self-adjoint), then the eigenvectors belonging to different eigenvalues are mutually perpendicular.

Remark

1. The second part of the previous proprosition shows a big difference between the self-adjoint and the symmetric operator. An unbounded symmetric operator can contain elements of spectrum that are not real. Such a situation cannot occur with a self-adjoint operator.
2. An unbounded operator can generally have an uncountable number of eigenvalues and vectors. The branch of mathematics dealing with this issue is called continuous functional calculus, which uses a more general integral instead of summation when decomposing elements into eigenvectors.
3. In the theory of unbounded operators, we do not have a priori theorems about the completeness of the specified system. In specific cases, their completeness must be proven on a case-by-case basis.

6 Linear differential operators

In this chapter, we will introduce a special case of linear unbounded operators, linear differential operators, for which we will generate bases in polynomial form.

6.1 Expressions in self-adjoint form

Definition 6.1 (Linear differential expression of the nth order) Let $-\infty \leq a < b \leq \infty$, $y \in \mathcal{C}^n(a,b)$. We call the expression a linear differential expression of the nth order

$$\ell(y) = \sum_{k=0}^{n} p_k(x) y^{(k)}, \tag{6.1}$$

where $y = y(x)$, $\forall k$ is $p_k \in \mathcal{C}^n(a,b)$ and it holds $p_n \not\equiv 0$ on (a,b).

Remark In this definition, we require the existence of a larger number of continuous derivatives for the functions p_k than is necessary for the correctness of the definition. However, we do so with respect to (6.2). Terminologically: in the previous definition, we do not use the term linear differential operator to distinguish the fact that to define the operator we also need to have a suitable domain – not only the space of all functions for which (6.1) makes sense, but such a domain that from ℓ will at least make a symmetric operator.

Definition 6.2 (Linear differential operator) By the term linear differential operator $\boldsymbol{L}$ of the nth order, we mean the corresponding linear one the differential expression $\ell(y)$ on some pre-specified domain of definition, that is

$$\boldsymbol{L} = \ell|_{\mathcal{D}(\boldsymbol{L})}$$

In accordance with the considerations of the previous chapter, we will want $\boldsymbol{L}$ to be densely defined on the Hilbert space H, i.e. $\mathcal{D}(\boldsymbol{L}) \neq H$, $\overline{\mathcal{D}(\boldsymbol{L})} = H$. Typically we will have $\mathcal{D}(\boldsymbol{L}) \subsetneqq (H \cap \mathcal{C}^n(a,b))$ so that $\mathcal{D}(\boldsymbol{L})$ forms a linear subspace H.

We know that symmetry is a necessary condition for self-adjointness. We will continue to look for more necessary conditions of self-adjointness, or symmetry and we will work on a much smaller space $\mathcal{C}^\infty_{cpt}(a,b)$, i.e. infinitely differentiable functions with a compact support.

Remark Thanks to this approach, we do not have to deal with boundary terms when performing per partes since all functions from the considered space $\mathcal{C}^\infty_{cpt}(a,b)$ (and their derivatives) are zero on some neighborhood of the extreme points. We can afford this simplification, since we are looking for the necessary conditions of self-adjointness, or symmetry. If an operator will not satisfy the conditions for such "nice" functions, it cannot satisfy them for any other scope. When examining or constructing $\mathcal{D}(\boldsymbol{L})$, however, we must always keep in mind (or make sure) that $\mathcal{C}^\infty_{cpt}(a,b)$ is a dense space in $\mathcal{D}(\boldsymbol{L})$ so that the results obtained on $\mathcal{C}^\infty_{cpt}(a,b)$ can be transferred to $\mathcal{D}(\boldsymbol{L})$ using the density argument. For example, note the end of the proof of Lemma 6.4.

Definition 6.3 (Adjoint expression to $\ell(y)$) We denote the adjoint expression to the expression $\ell(y)$ by $\ell^*(y)$ and define it as

$$\ell^*(y) = \sum_{k=0}^{n} (-1)^k \left(\overline{p_k(y)}\ y\right)^{(k)} \tag{6.2}$$

Lemma 6.4 Given ℓ, ℓ^* is the only linear differential expression for which

$$(\ell(y), z) = (y, \ell^*(z)) \qquad \forall y, z \in \mathcal{C}^\infty_{cpt}(a,b).$$

Proof. In each of the members of the type $\int_a^b p_k(x) y^{(k)}(x)\overline{z(x)}$, $k = 1, \ldots, n$, we first perform k- times per partes and we get:

$$\int_a^b p_k(x) y^{(k)}(x)\overline{z(x)} = (-1)\int_a^b \left(p_k(x)\overline{z(x)}\right)' y^{(k-1)} = (-1)^{(k)} \int_a^b \left(p_k(x)\overline{z(x)}\right)^{(k)} y.$$

the boundary terms are zero due to which spaces we take y, z from. In total, we then have

$$\big(\ell(y), z\big) = \sum_{k=0}^{n} \int_a^b p_k(x) y^{(k)}(x)\overline{z(x)} = \sum_{k=0}^{n} (-1)^{(k)} \int_a^b \Big(p_k(x)\overline{z(x)}\Big)^{(k)} y = \big(y, \ell^*(z)\big).$$

The last equality holds due to $p_k(x)\overline{z(x)} = \overline{\overline{p_k(x)} z(x)}$.

We can prove the uniqueness by contradiction, suppose there are two adjoint operators ℓ^* and $\tilde{\ell}$ satisfying the equality. We have

$$\big(\ell(y), z\big) = \big(y, \ell^*(z)\big) = \big(y, \tilde{\ell}(z)\big) \qquad \forall y, z \in \mathcal{C}^{\infty}_{cpt}(a, b).$$

But then necessarily from the last equality, which must hold for all y from the dense subset in L^2, we get

$$\ell^*(z) = \tilde{\ell}(z) \qquad \forall z \in \mathcal{C}^{\infty}_{cpt}(a, b).$$

Indeed, if we choose a fixed z, then due to the continuity of the scalar product, the elements in the second component of the scalar product must be the same. However that means that $\ell^* = \tilde{\ell}$ on $\mathcal{C}^{\infty}_{cpt}(a, b)$. However, this feature set is dense in $\mathcal{D}(\boldsymbol{L})$ and therefore

$$\ell^* = \tilde{\ell} \text{ on } \mathcal{D}(\boldsymbol{L}).$$

□

Another condition we need for self-adjointness is $\ell = \ell^*$, which imposes a condition on the shape of the individual coefficients p_k

$$\sum_{k=0}^{n} p_k\, y^{(k)} = \sum_{k=0}^{n} (-1)^k \left(\overline{p_k}\, y\right)^{(k)} = \sum_{k=0}^{n} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \overline{p_k}^{(k-j)} y^{(j)}.$$

Let's compare the coefficients of $y^{(n)}$

$$\begin{aligned}
&p_n = (-1)^n \overline{p_n} \\
&n \text{ even} : p_n = \overline{p_n} \implies p_n \text{ is real} \\
&n \text{ odd} : p_n = -\overline{p_n} \implies p_n + \overline{p_n} = 2Re(p_n) = 0 \implies p_n \text{ is purely imaginary}
\end{aligned}$$

and we proceed in the same way. We now define elementary differential expressions, with the help of which we subsequently write the linear differential expression $\ell = \ell^*$.

Definition 6.5 (Elementary differential expression) We call the elementary differential expression in the form

$$\begin{aligned}
E_{2k} &= (-1)^k \left(p\, y^{(k)}\right)^{(k)} \\
E_{2k-1} &= \frac{i}{2} \left[\left(p\, y^{(k-1)}\right)^{(k)} + \left(p\, y^{(k)}\right)^{(k-1)} \right],
\end{aligned} \tag{6.3}$$

where p is a real function.

Theorem 6.6 (On the combination of elementary differential expressions) $\ell(y) = \ell^*(y)\ \forall y \in \mathcal{C}^{\infty}_{cpt} \iff \ell(y)$ *is a finite linear combination of elementary differential expressions* E_{2k} *and* E_{2k-1}*. For more, see* CIHÁK*, p. 210.*

$\boxed{\textbf{Exercise}}$ Let's see how the elementary differential expression works out for $k = 1$ and $k = 2$.

$E_1 = \frac{i}{2}\left((p\, y)' + p\, y'\right) = \frac{i}{2}(p'\, y + 2p\, y') = ip\, y' + \frac{i}{2} p'\, y.$ For $p = 1$ we get iy'.

$E_2 = -(p\, y')' \ldots$ the so-called differential expression of the 2nd order in self-adjoint form.

Remark If we were to work in a higher dimension, E_2 will correspond to the basic form for the second-order equation for the function u

$$-\mathrm{div}(p\nabla u).$$

For $p \equiv 1$, we then get the Laplace operator (when restricted to the relevant domain)

$$-\Delta u$$

which is therefore in self-adjoint form.

6.2 Orthogonal bases in L^2_ρ composed of polynomials

We will consider space

$$H = L^2_\rho(a,b) \equiv \Big\{ f : (a,b) \to \mathbb{C};\ \int_a^b \rho\,|f|^2 < \infty,\ \text{where } \rho : (a,b) \to \mathbb{R} \text{ is the so-called } \underline{\text{weight}}$$
$$\text{satisfying } \rho > 0, \rho \in \mathcal{C}(a,b), \rho \in L^1 \Big\}.$$

Remark A more general weight than $\rho \in \mathcal{C}(a,b)$ is often considered.

It can be shown that $L^2_\rho(a,b)$ is a Hilbert space with scalar product

$$\big(y,z\big)_{2,\rho} \equiv \int_a^b \rho y \overline{z}$$

and norm

$$\|y\|_{2,\rho} = \int_a^b \rho\,|y|^2\,.$$

Remark We consider the space L^2_ρ, for example, because we want to work with polynomials on $\mathbb{R}$. At the same time, no polynomial P is an element of $L^2(\mathbb{R})$, but all polynomials are an element of $L^2_{e^{-x^2}}$.

Now consider $\boldsymbol{T} : \mathcal{D}(\boldsymbol{T}) \to L^2_\rho$, symmetric on $\mathcal{D}(\boldsymbol{T})$, while let $\mathcal{D}(\boldsymbol{T})$ be such that $\mathcal{D}(\boldsymbol{T}) \subsetneqq L^2_\rho$, $\overline{\mathcal{D}(\boldsymbol{T})} = L^2_\rho$ and $\mathcal{D}(\boldsymbol{T}) \subset L^2_\rho \cap L^2$.

Definition 6.7 (Eigennumber and eigenfunction with weight ρ) $\boldsymbol{T}$ is densely defined on $L^2_\rho(a,b)$. We call the number λ an eigenvalue with weight ρ if $\exists y \neq 0, y \in L^2_\rho$ such that

$$\boldsymbol{T}y = \lambda\rho y \tag{6.4}$$

Let $\boldsymbol{T}$ be at least symmetric on $\mathcal{D}(\boldsymbol{T})$ in the sense $\big(\boldsymbol{T}y,z\big)_2 = \big(y,\boldsymbol{T}z\big)_2$ Note that here we are working with the scalar product <u>unweighted</u>, even though we consider the eigenvectors and eigenvalues <u>weighted</u>. Then

$$\begin{aligned} &\big(\boldsymbol{T}y,y\big)_2 = \big(\lambda y\rho,y\big)_2 = \lambda\big(\rho y,y\big)_2 = \lambda\int_a^b \rho|y|^2 = \lambda\,\|y\|^2_{2,\rho} \\ &\quad\| \\ &\big(y,\boldsymbol{T}y\big)_2 = \cdots = \overline{\lambda}\,\|y\|^2_{2,\rho}\,. \end{aligned}$$

Thus, if $y \in \mathcal{D}(\boldsymbol{T})$ is $\lambda \in \mathbb{R}$.

Furthermore, for $\boldsymbol{T}y_j = \lambda_j\rho y_j$, $j = 1,2\ldots$, ($\boldsymbol{T}$ is symmetric, so both of these eigenvalues are real), $\lambda_1 \neq \lambda_2$, we have

$$\lambda_1\big(y_1,y_2\big)_{2,\rho} = \lambda_1\big(y_1\rho,y_2\big)_2 = \big(\boldsymbol{T}y_1,y_2\big)_2 = \big(y_1,\boldsymbol{T}y_2\big) = \cdots = \lambda_2\big(y_1,y_2\big)_{2,\rho}. \tag{6.5}$$

Since we have different eigenvalues $\lambda_1 \neq \lambda_2$, we get perpendicularity in L^2_ρ: $\big(y_1,y_2\big)_{2,\rho} = 0$.

<u>Conclusion:</u> In the case of the symmetric operator $\mathcal{T}$, the weight eigenvalues are real and the eigenvectors then form an orthogonal system in L^2_ρ. Generally, however, in this case it is <u>not available</u>

- statement about the multiplicity of the OG function system created in this way,
- a statement about the completeness of this system, i.e. about the fact that it would form a basis (must be proved for individual cases separately).

However, we know that we are generating OG sets. The situation of function spaces on compact sets is simpler, where we have the Weistrass theorem that polynomials are dense in $\mathcal{C}(K)$ (for K compact) if the given system contains polynomials of all degrees. So it seems reasonable to focus on this special OG set. Since $\mathcal{C}(K)$ is dense in $L^2_\rho(K)$, the completeness of the OG set of polynomials (containing polynomials of all degrees) in $L^2_\rho(K)$ follows from the Weistrass sentences. On non-compacts, the completeness of any OG feature set is much more difficult to prove.

Based on this remark, we will now consider the OG sets of eigenfunctions that are polynomials.

Theorem 6.8 (Recurrence Formula Theorem) *Let us have a space $L^2_\rho(a,b)$, $-\infty \le a < b \le +\infty$, ρ such that $\|P\|_{2,\rho} < \infty$ $\forall$ polynomials P. Let $\{\varphi_n\}_{n=0}^\infty$ be a system of real OG polynomials in $L^2_\rho(a,b)$, degree$(\varphi_n) = n$, $n = 0,1,2,3,\dots$ Then $\forall n \in \mathbb{N}$ $\exists A_n, B_n, C_n \in \mathbb{R}$ that*

$$x\varphi_n = A_n\varphi_{n+1} + C_n\varphi_n + B_n\varphi_{n-1}. \tag{6.6}$$

Remark

$$n = 0 \implies \varphi_0 = c \ne 0, \text{ then } x\varphi_0 = cx = \frac{c}{a}\underbrace{(ax+b)}_{\varphi_1,\ a\ne 0} - \frac{b}{a}\underbrace{c}_{\varphi_0} \implies x\varphi_0 = \frac{c}{a}\varphi_1 - \frac{b}{a}\varphi_0$$

Proof. $n \in \mathbb{N}:$ degree$(x\varphi_n) = n+1 \implies \exists \gamma_{n,k} \in \mathbb{R}$ that

$$x\varphi_n = \sum_{k=0}^{n+1} \gamma_{n,k}\varphi_k \tag{6.7}$$

The relation (6.7) applies in general to any polynomials, degree$(\varphi_n) = n$, these polynomials need not be OG. Now we perform the scalar product "with weight" on the (6.7): $\left(\bullet, \varphi_j\right)_{2,\rho}$ $\forall j = 0,1,\dots$ and we get

$$\left(x\varphi_n, \varphi_j\right)_{2,\rho} = \sum_{k=0}^{n+1} \gamma_{n,k} \underbrace{\left(\varphi_k, \varphi_j\right)_{2,\rho}}_{\delta_{kj}\|\varphi_k\|^2_{2,\rho}} = \begin{cases} \gamma_{n,j}\,\|\varphi_j\|^2_{2,\rho} & j \le n+1 \\ 0 & \text{elsewhere,} \end{cases} \tag{6.8}$$

since $\gamma_{n,j} = 0$ for $j > n+1$ (the sum is equal to zero for $j > n+1$ because $\varphi_k \perp \varphi_j$ for $k \in \{0,\dots,n+1\}$ and $j > n+1$).

Due to the reality of φ_n is

$$\left(x\varphi_n, \varphi_j\right)_{2,\rho} = \left(\varphi_n, x\varphi_j\right)_{2,\rho} = \left(\varphi_n, \sum_{p=0}^{j+1}\gamma_{j,p}\varphi_p\right)_{2,\rho} = \sum_{p=0}^{j+1}\gamma_{j,p}\left(\varphi_n, \varphi_p\right)_{2,\rho}. \tag{6.9}$$

We will compare this relation with (6.8), where we will read that "the resulting expression is zero if the index of φ, which is not an addition index, is greater than the upper bound of the sum". Here it means that the expression (6.9) is zero for $n > j+1$ and therefore nonzero only for $j \ge n-1$. However, the last expression in (6.9) is still equal to the last expression in (6.8), so

$$\gamma_{n,j}\,\|\varphi_j\|^2_{2,\rho}, \tag{6.10}$$

while we find that this expression is nonzero only for $n-1 \le j \le n+1$. Since the norms of all polynomials φ_j are non-zero (they are polynomials of degree j), the coefficient $\gamma_{n,j}$ determines the nullity of the expression (6.10). This means that the only nonzero coefficients are $\gamma_{n,n-1}$, $\gamma_{n,n}$, and $\gamma_{n,n+1}$. It then follows from (6.7)

$$x\varphi_n = \underbrace{\gamma_{n,n-1}}_{=:B_n}\varphi_{n-1} + \underbrace{\gamma_{n,n}}_{=:C_n}\varphi_n + \underbrace{\gamma_{n,n+1}}_{=:A_n}\varphi_{n+1}.$$

□

This recurrent formula can be used, for example, to calculate systems of OG polynomials and calculate their norms. We know that:

$$x\varphi_n = A_n\varphi_{n+1} + C_n\varphi_n + B_n\varphi_{n-1}, \quad n = 1, 2, 3, \ldots. \tag{6.11}$$

Above all, we see that

- $A_n \neq 0\ \forall n$, otherwise the degree of the polynomial on the right would be n, while the degree of the polynomial on the left is $n+1$.
- By multiplying the equality (6.11) first $\left(\bullet, \varphi_{n+1}\right)_{2,\rho}$ and then $\left(\bullet, \varphi_{n-1}\right)_{2,\rho}$ we get using the orthogonality of the system $\{\varphi_n\}$ that

$$\begin{aligned} \left(x\varphi_n, \varphi_{n+1}\right)_{2,\rho} &= A_n \left\|\varphi_{n+1}\right\|_{2,\rho}^2 \\ \left(x\varphi_n, \varphi_{n-1}\right)_{2,\rho} &= B_n \left\|\varphi_{n-1}\right\|_{2,\rho}^2 \\ \| \qquad & \\ \left(x\varphi_{n+1}, \varphi_n\right)_{2,\rho} &= B_{n+1} \left\|\varphi_n\right\|_{2,\rho}^2 . \end{aligned}$$

  Since all polynomials are real, $\left(x\varphi_n, \varphi_{n+1}\right)_{2,\rho} = \left(x\varphi_{n+1}, \varphi_n\right)_{2,\rho}$, and it follows by comparing the first and third lines of the previous calculation $A_n \left\|\varphi_{n+1}\right\|_{2,\rho}^2 = B_{n+1} \left\|\varphi_n\right\|_{2,\rho}^2$, from which we get the recurrence relation for the norms

$$\left\|\varphi_{n+1}\right\|_{2,\rho}^2 = \frac{B_{n+1}}{A_n} \left\|\varphi_n\right\|_{2,\rho}^2 \quad n = 1, 2, 3, \ldots \tag{6.12}$$

  Of course, we need to know $\|\varphi_0\|$ and $\|\varphi_1\|$. Moreover, from the (6.12) and from the fact that $A_n \neq 0\ \forall n \in \mathbb{N}$ follows that also $B_n \neq 0\ \forall n$.

Remark We have convinced ourselves that the coefficients A_n and B_n in (6.11) are always non-zero. However, the coefficients C_n can be zero in a special case. Specifically, if $b = -a$, and the weight ρ is even on the interval (a, b), then $C_n = 0\ \forall n \in \mathbb{N}$.

Bonus The proof of this remark can be done, for example, as follows: Again, by the general property of polynomials, we have

$$\varphi_n(-x) = \sum_{k=0}^{n} \beta_{n,k}\varphi_k(x) \quad \Big/ \big(\bullet, \varphi_j(x)\big)_{2,\rho} \text{ for } j = 0, \dots, n, \text{ otherwise the product is } 0$$

$$\beta_{n,j}\,\|\varphi_j\|^2_{2,\rho} = \big(\varphi_n(-x), \varphi_j(x)\big)_{2,\rho} = \int_{-a}^{a} \varphi_n(-x)\varphi_j(x)\rho(x)\,\mathrm{d}x \overset{t:=-x}{=} -\int_{a}^{-a} \varphi_n(t)\varphi_j(-t)\rho(t)\,\mathrm{d}t$$

$$= \big(\varphi_n(x), \varphi_j(-x)\big)_{2,\rho} = \Big(\varphi_n(x), \sum_{m=0}^{j} \beta_{j,m}\varphi_m(x)\Big) = 0 \text{ for } n > j.$$

Thus $\beta_{n,j}$ are non-zero only for $j = n$ and from the first relation we have $\varphi_n(-x) = \beta_{n,n}\varphi_n(x)$. Now let's compare the coefficients of x^n in the polynomial $\varphi_n(x) = \sum_{k=0}^{n} a_k x^k$:

$$a_n(-x)^n = \beta_{n,n}a_n x^n \implies \beta_{n,n} = (-1)^n.$$

Therefore $\varphi_n(-x) = (-1)^n\varphi_n(x) \implies (\varphi_n(-x))^2 = (\varphi_n(x))^2$, i.e. $|\varphi_n|^2$ is <u>even</u>.
In conclusion, we have

$$x\varphi_n = A_n\varphi_{n+1} + C_n\varphi_n + B_n\varphi_{n-1} \quad \Big/ \big(\bullet, \varphi_n\big)_{2,\rho}$$

$$\underbrace{\big(x\varphi_n, \varphi_n\big)}_{\overbrace{\int_{-a}^{a} x|\varphi_n|^2\rho(x)}^{\parallel}} = C_n\,\|\varphi_n\|^2_{2,\rho}$$

$$\int_{-a}^{a} x|\varphi_n|^2\rho(x) = 0, \qquad \text{since } x \text{ are odd and } |\varphi_n|^2_\rho \text{ even,}$$

from which $C_n = 0$ follows.

6.3 Gauss's reduced equation and orthogonal systems of polynomials

Consider the so-called Gaussian Reduced Equation (GRR)

$\boxed{xy'' + (s + 1 - x)y' - \alpha y = 0}$, $\quad x \neq 0;\ \alpha, s \in \mathbb{C};\ s \neq -1, -2, -3, \dots$ (no spoilers, we'll find out the reason soon).

1. We first show that this equation can be written in the form "eigenvector and eigennumber with weight", i.e. in the form

$$\boldsymbol{T}y = \lambda\rho y \quad \text{for } \lambda \in \mathbb{C} \text{ with suitable weight } \rho,$$

where $\boldsymbol{T}y$ has the form of a differential expression in self-adjoint form, i.e., $\boldsymbol{T}y = (-py')'$. So,

$$(-py')' = \lambda\rho y, \quad p \not\equiv 0$$
$$-p'y' - py'' - \lambda\rho y = 0$$
$$\boxed{y'' + \frac{p'}{p}y + \lambda\frac{\rho}{p}y = 0}$$

Comparing the last equality with the GRR entry and adjusting for $x \neq 0$, we get

$$y'' + \left(\frac{s+1}{x} - 1\right)y' - \frac{\alpha}{x}y = 0.$$

Since we only need to find an arbitrary, non-trivial solution, we can write

$$\begin{aligned}
\frac{p'}{p} &= \frac{s+1}{x} - 1 \\
\big(\ln|p|\big)' &= (s+1)\big(\ln|x|\big)' - 1 \\
|p| &= |x|^{s+1}e^{-x}K\,, \quad K>0 \\
p &= x^{s+1}e^{-x}\,, \quad \text{we chose the branch } x>0.
\end{aligned}$$

For $x>0$, we require $\rho \in L^1(0,\infty)$, so $\underline{s>-1}$. Moreover, we have $\lambda = -\alpha$, $\frac{\rho}{p} = \frac{1}{x} \implies \rho = \frac{p}{x}$, $\rho = x^s e^{-x}$. So we got the equation

$$\boxed{\Big(-\underbrace{x^{s+1}e^{-x}}_{p}y'\Big)' = (-\alpha)\underbrace{x^s e^{-x}}_{\rho}y\,,} \tag{6.13}$$

the solution of which solves by GRR. It is important to realize that we are working on the space $L^2_\rho(0,\infty) = L^2_{x^s e^{-x}}(0,\infty),\ s>-1$

2. We will look for a GRR solution in the form of a series. However, for this we must make the following considerations.

   - For $x=0$ the GRR degenerates (loses order), it needs to be investigated separately at $(-\infty,0)$ and at $(0,\infty)$.
   - We can assume that these two separate solutions will be possible to "glue" together at the point $x=0$ so that a solution is created on some interval $(-a,a)\subset\mathbb{R}$. If we are looking for GRR solutions in the class of such "glueable" solutions, which are also analytic in the neighborhood of zero (they can be expressed there by a Taylor series), we can also look for them in the form of a Taylor series centered at zero. With the risk of not finding a solution in this form, which would lead us to conclude that the problem has no "glueable" solutions in the form of the Taylor series.

Under this condition, we put $y =: \sum_{n=0}^{\infty} c_n x^n$ and substituting into GRR we get

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}x + (s+1)\sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=1}^{\infty} c_n n x^n - \alpha\sum_{n=0}^{\infty} c_n x^n = 0$$
$$\sum_{n=1}^{\infty} c_{n+1}(n+1)nx^x + \sum_{n=0}^{\infty}(s+1)c_{n+1}(n+1)x^n - \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n\alpha x^n = 0$$

Now let's compare the coefficients:

$$x^0: \quad (s+1)c_1 = c_0\alpha \implies c_1 = c_0\frac{\alpha}{s+1} \quad (\text{hence } s\neq -1)$$

for $n\geq 1$ we have x^n : $\quad c_{n+1}\left[(n+1)n + (s+1)(n+1)\right] = c_n(n+\alpha)$

$$c_{n+1} = c_n\frac{n+\alpha}{(n+1)(s+n+1)} \quad (\text{hence } s\neq -2,-3,\dots)\ ,$$

while the obtained recurrence relation is also valid for $n=0$. Since each multiple of the GRR solution is in turn its solution, one can WLOG choose the basic solution for $c_0=1$. We get that the coefficients of the series that define the solution would have to have the form

$$\begin{aligned}
c_0 &= 1 \\
c_{n+1} &= \frac{n+\alpha}{n+s+1}\frac{c_n}{n+1}\,, \quad \text{for } n=0,1,2,\dots; s\neq -1,-2,\dots\,.
\end{aligned}$$

However, we still have to show that the series with coefficients $c_0,\ c_1$ given in this way converges somewhere. Since series with coefficients of this type form one very important class of series, we devote the following intermezzo to them.

Intermezzo: Hypergeometric Series

Definition 6.9 (Hypergeometric series) We call a hypergeometric series a power series of the form

$$\sum_{n=0}^{\infty} c_n x^n,$$

for coefficients satisfying

- $\exists$ polynomials P, Q with coefficients at the highest power equal to 1, st $P = p \geq 0$, st $Q = q \geq 0$, Q has no roots between $\mathbb{N} \cup \{0\}$,
- $\frac{c_{n+1}}{c_n} = \frac{P(n)}{Q(n)} \frac{1}{n+1}$, $\quad n = 0, 1, 2, \dots$; $c_0 = 1$

Remark For $P(n) = Q(n)(n+1)$ we have $\frac{c_{n+1}}{c_n} = 1$, $\left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right| = |x|$, i.e. a geometric series with the quotient x. Member $\frac{1}{n+1}$ is here only for historical reasons.

Let us now decompose P and Q into root factors in $\mathbb{C}$ (unlike standard practice, we denote the roots by $-a_j$ and $-b_j$). We will get

$$\frac{c_{n+1}}{c_n} = \frac{(a_1+n)(a_2+n)\dots(a_p+n)}{(b_1+n)(b_2+n)\dots(b_q+n)} \frac{1}{n+1} \,. \tag{6.14}$$

The following hypergeometric series notation is introduced for this situation:

$$\sum_{n=0}^{\infty} c_n x^n = {}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q](x).$$

From the equation 6.14 we immediately see:

- $p < q+1 \implies \left|\frac{c_{n+1}}{c_n}\right| \to 0 \implies R = +\infty \implies \sum c_n x^n$ defines a holomorphic ($\mathcal{C}^\infty$) function on the entire $\mathbb{C}$.
- $p = q+1 \implies \left|\frac{c_{n+1}}{c_n}\right| \to 1 \implies R = 1 \implies \sum c_n x^n$ defines a holomorphic ($\mathcal{C}^\infty$) function on $\mathcal{U}^1(0)$.
- $p > q+1 \implies \left|\frac{c_{n+1}}{c_n}\right| \to \infty \implies R = 0$, which defines no differentiable function.

Definition 6.10 (Pochhammer symbol) For $a \in \mathbb{C}$, let us define

$$(a)_0 \coloneqq 1\,, \quad (a)_n \coloneqq \underbrace{a(a+1)\dots(a+n-1)}_{n\in\mathbb{N}\text{ members}}$$

The Pochhammer symbol is sometimes also called "Rising factorial" and the notation $\langle a\rangle_n$ is also used. We will read it "a Pochhammer n", or "a down n". Finally, notice that, for example, $(1)_n = n!$, or $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

With this notation, we rewrite the equation 6.14

$$\begin{aligned} c_n &= \frac{(a_1+n-1)(a_2+n-1)\dots(a_p+n-1)}{(b_1+n-1)(b_2+n-1)\dots(b_q+n-1)} \frac{1}{n} c_{n-1} \\ &= \frac{[(a_1+n-1)(a_2+n-2)]\dots[(a_p+n-1)(a_p+n-2)]}{\underbrace{[(b_1+n-1)(b_2+n-2)]}_{\text{next steps go to }(b_1+n-1)\dots(b_1)=(b_1)_n} \quad \dots[(b_q+n-1)(b_q+n-2)]} \underbrace{\frac{1}{n}\frac{1}{n-1}}_{\text{goes to }\frac{1}{n!}} c_{n-2} = \end{aligned}$$

$$\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{1}{n!}\underbrace{c_0}_{=1} = \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{k=1}^{q}(b_k)_n}\frac{1}{n!}\,.$$

This leads us to explicit expression of the hypergeometric series

$$\boxed{{}_pF_q[a_1,\dots,a_p;b_1,\dots,b_q](x) = \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(a_j)_n}{\prod\limits_{k=1}^{q}(b_k)_n} \frac{x^n}{n!}}$$

The reason for adding that historical term $\frac{1}{n+1}$ stands out in the hypergeometric series ${}_0F_0[;](x)$. According to the previous conclusion, it is equal

$$ {}_0F_0[;](x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Exercise

- Show:

$$ {}_0F_1\left[;\frac{1}{2}\right]\left(-\frac{x^2}{4}\right) = \cos x.$$

The series on the left has $p=0$, $q=1 \implies p<q+1 \implies$ the series defines a smooth (and holomorphic) function in $\mathbb{C}$.

Solution:

$$\begin{aligned} {}_0F_1\left[;\frac{1}{2}\right]\left(-\frac{x^2}{4}\right) &= \sum_{n=0}^{\infty} \frac{1}{(1/2)_n}\frac{1}{n!}\left(-\frac{x^2}{4}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n x^{2n} \underbrace{\frac{1}{n!4^n\frac{1}{2}(\ frac12+1)\dots(\frac{1}{2}+n-1)}}_{\frac{2^n}{n!4^n(1\cdot 3\dots(2n-1))}=\frac{1}{(2n)!}} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \end{aligned}$$

- Show

$$\frac{2x}{\sqrt{\pi}}\,{}_1F_1\left[\frac{1}{2};\frac{3}{2}\right]\left(-x^2\right) = \frac{2}{\sqrt{\pi}}\int\limits_0^x e^{-t^2}\,\mathrm{d}t = \mathrm{erf}\,(x)\,,$$

where the row on the left makes sense for $\forall x\in\mathbb{C}$, but the row on the right only makes sense for $x\in\mathbb{R}$. Thus, the series on the left can be understood as an extension of erf (x) to complex numbers.

end of intermezzo. Let's go back to GRR.

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3. The solution of GRR is the series $\sum_{n=0}^{\infty} c_n x^n$, where

$$c_0 = 1\,,\quad c_{n+1} = \frac{n+\alpha}{n+s+1}\frac{1}{n+1}c_n\,,\quad n=0,1,2,\dots\,.$$

It is therefore a hypergeometric series for $p=1$, $q=1$, i.e. $p<q+1$ and we have

$$ {}_1F_1\left[\alpha;\beta+1\right](x) = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(s+1)_k}\frac{x^k}{k!} \in \mathcal{C}^{\infty}(\mathbb{R})\,,\quad \alpha\in\mathbb{C};\ s\in\mathbb{C}\setminus\{-1,-2,-3,\dots\}$$

${}_1F_1[\alpha,s+1](x)$ is a polynomial if and only if the series on the right contains a finite number of terms, which happens if and only if $\exists n\in\mathbb{N},\ (\alpha)_k = 0\ \forall k>n$. Moreover, since $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$, we get for $\alpha=-n$ the equivalence

$$(\alpha)_k = 0 \ \Leftrightarrow\ k>n \text{ for } n\in\mathbb{N}\cup\{0\}\,.$$

Definition 6.11 (Laguerre polynomial) A Laguerre polynomial of order s and degree n is a polynomial defined for $x, s \in \mathbb{R}$, $s > -1$, as

$$\boxed{L_n^s(x) := \frac{(s+1)_n}{n!}\,{}_1F_1[-n; s+1](x) = \frac{(s+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(s+1)_k}\frac{x^k}{k!}\,.}$$

Remark

– $L_n^s(x)$ solves the GRR $\forall n \in \mathbb{N} \cup \{0\}$ if we put $\alpha = -n$ in it.

– Referring to the form of the equation 6.13, we perform the following restrictions:

- We consider $x > 0$, i.e. $x \in (0, \infty)$
- We consider $s \in \mathbb{R}$, $s > -1$, $\rho(x) = x^s e^{-x}$. Then $\rho > 0$ on $(0, \infty)$, $\rho \in \mathcal{C}(0, \infty) \cap L^1(0, \infty)$, which shows the validity choices ρ, as weights.
- So we consider the Hilbert space $L^2_{x^s e^{-x}}(0, \infty)$.
- $\alpha = -n$, $n \in \mathbb{N} \cup \{0\}$.

We have shown that the GRR can be written in self-adjoint form (equation (6.13))

$$\boldsymbol{T}y = n\rho y\,,$$

where $\boldsymbol{T}y = -(py')'$, $p(x) = x^{s+1}e^{-x}$.

Then $n = 0, 1, 2, \ldots$ are eigenvalues $\boldsymbol{T}$ with weight ρ (on $L^2_\rho(0,\infty)$) and their corresponding eigenfunctions are Laguerre polynomials L_n^s.

According to the calculation made earlier (see considerations behind the equation 6.5), the Laguerre polynomials (for fixed $s > -1$ and for $n \in \mathbb{N} \cup \{0\}$) form an OG system of polynomials in $L^2_{x^s e^{-x}}(0, \infty)$. There is therefore a recurrent formula for their generation (we derive it further).

In addition, Laguerre polynomials are a complete system, i.e. every function from $L^2_{x^s e^{-x}}(0, \infty)$ can be written in the form $\sum_{n=0}^{\infty} c_n L_n^s(x)$. The proof is beyond the scope of this lecture notes and can be found, for example, in Čihák et al: MA for physics V. (Theorem 31, p. 196).

6.4 Some important properties of Laguerre polynomials

Explicit statement

It applies

$$\boxed{L_n^s(x) = \frac{1}{n!}x^{-s}e^x\left(x^{s+n}e^{-x}\right)^{(n)}} \tag{6.15}$$

It follows that

$$\begin{aligned}
L_0^s(x) &= x^{-s}e^x x^s e^x = 1\\
L_1^s(x) &= x^{-s}e^x(x^{s+1}e^{-x})' = x^{-s}e^x(s+1)x^s e^{-x} + x^{-s}e^x x^{s+1}(-e^{-x}) = (s+1) - x \text{ etc.}
\end{aligned}$$

Another use of the explicit expression 6.15 is offered in the calculations of integrals of the type $\int\limits_0^\infty L_n^s(x) f(x)\,\mathrm{d}x\,,$ because it allows the per-partes method to be used. We will perform such a calculation later.

Proof. We start the proof of the expression 6.15 by modifying GRR

$$xy'' + (s+1-x)y' - \alpha y = 0$$
$$\left(s^{s+1}e^{-x}y'\right)' = \alpha x^s e^{-x} y \,, \text{ denote the equality as } \mathrm{GRR}(y, s+1, \alpha)\,.$$

By derivation we get

$$xy''' + y'' + (s+1-x)y'' - y' - \alpha y' = 0$$
$$xy''' + (s+2-x)y'' - (\alpha+1)y' = 0\,, \text{ denote equality: } \mathrm{GRR}(y', s+2, \alpha+1)\,.$$

□

We thus get $n-1$ derivatives

$$\mathrm{GRR}(y^{(n-1)}, s+n, \alpha+n-1) \equiv (\underbrace{x^{s+n}e^{-x}y^{(n)}}_{=:V_n})' = (\alpha+n-1)\underbrace{x^{s+n-1}e^{-x}y^{(n-1)}}_{V_{n-1}}\,,$$

thus $V_n' = (\alpha+n-1)V_{n-1}$ and by further differentiation we have

$$V_n'' = (\alpha+n-1)V_{n-1}' = (\alpha+n-1)(\alpha+n-2)V_{n-2}.$$

We then gradually have

$$V_n^{(n)} = (\alpha)_n V_0 = (\alpha)_n x^s e^{-x} y\,,$$

from which we have

$$\left(x^{s+n}e^{-x}y^{(n)}\right)^{(n)} = (\alpha)_n x^s e^{-x} y$$

and by substituting $\alpha \coloneqq -n$ we get

$$y = \frac{1}{(-n)_n} x^{-s} e^x \left(x^{s+n}e^{-x}y^{(n)}\right)^{(n)}\,. \tag{6.16}$$

If $\alpha = -n$, the GRR solution is L_n^s , which is a polynomial of degree n . Its n-th derivative is therefore a constant, $(L_n^s)^{(n)} = a_n n!$, where a_n denotes the coefficient of x^n. A polynomial can be written as a series

$$L_n^s(x) = \frac{(s+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(s+1)_k}\frac{x^k}{k!}\,,$$

so $a_n = \frac{(s+1)_n}{n!}\frac{(-n)_n}{(s+1)_n}\frac{1}{n!}$, therefore $(L_n^s)^{(n)} = \frac{(-n)_n}{n!}$.

Substituting 6.16 into the equation, we have

$$L_n^s(x) = \frac{1}{(-n)_n} x^{-s} e^x \left(x^{s+n}e^{-x}\frac{(-n)_n}{n!}\right)^{(n)}$$

$$L_n^s(x) = \frac{1}{n!} x^{-s} e^x \left(x^{s+n}e^{-x}\right)^{(n)} \tag{6.17}$$

Recurrence Formula

We start from the equation 6.17

$$L_n^s(x) = \frac{1}{n!} x^{-s} e^x \underbrace{\left(x^{s+n}e^{-x}\right)^{(n)}}_{=:E_n}\,.$$

Then by direct differentiation we get

$$E_{n+1} = \left(\left(x^{s+n+1}e^{-x}\right)'\right)^{(n)} = (s+n+1)\underbrace{\left(x^{s+n}e^{-x}\right)^{(n)}}_{E_n} - \underbrace{\left(x^{s+n+1}e^{-x}\right)^{(n)}}_{=:I_n}\,. \tag{6.18}$$

Our goal now is to express I_n in terms of E_n.

$$I_n = \left(x \cdot x^{s+n} e^{-x}\right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} (x^{s+n} e^{-x})^{(n-k)} = /\text{is nonzero only for } k \in \{0,1\}/$$
$$= x\left(x^{s+n} e^{-x}\right)^{(n)} + n\left(x^{s+n} e^{-x}\right)^{(n-1)} = xE_n + n\underbrace{(x^{s+n} e^{-x})^{(n-1)}}_{I_{n-1}},$$

i.e. $I_n = xE_n + nI_{n-1}$.

If we additionally take 6.18 for $n-1$, i.e. $E_n = (s+n)E_{n-1} - I_{n-1}$, we get

$$I_n = xE_n + n(s+n)E_{n-1} - nE_n \,.$$

We substitute this equation into 6.18, which gives

$$E_{n+1} = (s+n+1)E_n - xE_n - n(x+n)E_{n-1} + nE_n$$
$$xE_n = (s+2n+1)E_n - E_{n+1} - n(s+n)E_{n-1} \quad \Big/ \cdot \frac{1}{n!} x^{-s} e^{x}$$
$$\boxed{xL_n^s(x) = (s+2n+1)L_n^s(x) - (n+1)L_{n+1}^s(x) - (s+n)L_{n-1}(x)\,,}$$

which is the recurrent pattern sought. From the knowledge of the first two terms ($L_0^s = 1$, $L_1^s = (s+1) - x$) all L_n^s can be generated.

Standards

We know from before

$$\|\varphi_{n+1}\|_{2,\rho}^2 = \frac{B_{n+1}}{A_n} \|\varphi_n\|_{2,\rho}^2 \;, \quad n = 1, 2, \ldots\,,$$

if

$$x\varphi_n = A_n\varphi_{n+1} + C_n\varphi_n + B_n\varphi_{n-1}\,.$$

Here we have $A_n = -(n+1)$, $B_n = -(s+n)$, that is

$$\|L_{n+1}\|_{2,\rho}^2 = \frac{s+n+1}{n+1} \|L_n^0\|_{2,\rho}^2 \;, \quad n = 1, 2, 3, \ldots\,.$$

We calculate the first two norms directly: We have $\|L_0^s\|_{2,\rho}^2 = \int\limits_0^\infty 1 \cdot x^s e^{-x} = \Gamma(s+1)$.

$$\|L_1^s\|_{2,\rho}^2 = \int\limits_0^\infty \left((s+1) - x\right)^2 x^s e^{-x} = (s+1)^2\Gamma(s+1) - 2(s+1)\Gamma(s+2) + \Gamma(s+3)$$
$$= (s+1)\Gamma(s+2) - 2(s+1)\Gamma(s+2) + \Gamma(s+3)$$
$$= \Gamma(s+3) - (s+1)\Gamma(s+2)$$
$$= (s+2)\Gamma(s+2) - (s+1)\Gamma(s+2) = \Gamma(s+2)$$

and using recursion we get

$$\|L_n^s\|_{2,\rho}^2 = \frac{s+n}{n}\frac{s+n-1}{n-1} \cdots \frac{s+2}{2} \underbrace{\|L_1^s\|_{2,\rho}^2}_{\Gamma(s+2)} = \frac{1}{n!}\Gamma(s+n+1)\ \text{, which also applies to } n \in \{0,1\}$$

$$\boxed{\|L_n^s\|_{2,\rho}^2 = \frac{1}{n!}\Gamma(s+n+1) \quad \forall n = 0, 1, 2\ dots}$$

Genereting Functions

Definition 6.12 (Generating Functions) Generating functions for a given system $\{\varphi_n\}_{n=0}^{\infty}$, $\varphi = \varphi_n(x)$, I will call such a function $F = F(x,t)$ which is analytic in the neighborhood of $t = 0$ (for all fixed x) and whose expansion into the Taylor series according to t in $\mathcal{U}(0)$ generates $\varphi_n(x)$ coefficients. I mean

$$F(x,t) = \sum_{n=0}^{\infty} \varphi_n(x) t^n \,.$$

So here we are looking for such F for which $F(x,t) = \sum\limits_{n=0}^{\infty} L_n^s(x) t^n$.

We will proceed by expanding the appropriate function $f \in L^2_\rho(0,\infty)$ with parameter t into a series in Laguerre polynomials. This gives us a series of the type $\sum_{n=0}^{\infty} c_n(t) L_n^s(x)$ and we will tend to make $c_n \approx t^n$.

The theory says that if $f \in L^2_{x^s e^{-x}}(0,\infty)$ and $L_n^s(x)$ is complete in $L^2_{x^s e^{-x}}(0,\infty)$, so from the general theory of Fourier series it follows that $\exists c_n \in \mathbb{C}$ of the form

$$c_n = \frac{1}{\|L_n^s\|_{2,\rho}^2} \left(f, L_n^s\right)_{2,\rho} \,, \text{ that } f = \sum_{n=0}^{\infty} c_n L_n^s \,,$$

while the equality of the sum is meant in $L^2_{x^s e^{-x}}(0,\infty)$.

We will expand the function e^{-ax} (then we will look for $a = a(t)$).

<u>The first step</u> is to find a condition for $a \in \mathbb{R}$ where $e^{-ax} \in L^2_{x^s e^{-x}}(0,\infty)$, i.e. when

$$\int_0^\infty (e^{-ax})^2 x^s e^{-x} \,\mathrm{d}x < \infty$$

$$\int_0^\infty x^s e^{-(2a+1)x} \,\mathrm{d}x < \infty \text{ for } s > -1 \text{ if } 2a+1 > 0 \Leftrightarrow a > -\frac{1}{2} \,.$$

For this a we calculate

$$\begin{aligned}
c_n &= \frac{1}{\|L_n^s\|_{2,\rho}^2} \int_0^\infty e^{-ax} x^s e^{-x} L_n^s(x) \,\mathrm{d}x = /\text{explicit expression } L_n^s/ \\
&= \frac{n!}{\Gamma(s+n+1)} \int_0^\infty e^{-ax} x^s e^{-x} \left(\frac{1}{n!} x^{-s} e^x (x^{s+n} e^{-x})^{(n)}\right) \,\mathrm{d}x \\
&= \frac{1}{\Gamma(s+n+1)} \int_0^\infty e^{-ax} (x^{s+n} e^{-x})^{(n)} \,\mathrm{d}x = /n \times \text{ per partes, boundary terms are zero}/ \\
&= \frac{a^n}{\Gamma(s+n+1)} \int_0^\infty \underbrace{e^{-ax} x^{s+n} e^{-x}}_{(a+1)x=y} \,\mathrm{d}x \\
&= \frac{a^n}{\Gamma(s+n+1)} \int_0^\infty e^{-y} \left(\frac{y}{a+1}\right)^{s+n} \frac{1}{a+1} \,\mathrm{d}y \\
&= \frac{a^n}{\Gamma(s+n+1)} \frac{1}{(a+1)^{s+n+1}} \Gamma(s+n+1) = \frac{1}{(a+1)^{s+1}} \left(\frac{a}{a+1}\right)^n \,.
\end{aligned}$$

From there we get

$$\boxed{e^{-ax} \overset{\text{s.v.}}{=} \frac{1}{(a+1)^{s+1}} \sum_{n=0}^{\infty} \left(\frac{a}{a+1}\right)^n L_n^s(x) \,, \quad a > -\frac{1}{2} \,.} \tag{6.19}$$

Remark • In general, the equality applies in the sense of the space in which it was derived, i.e. in $L^2_{x^s e^{-x}}(0,\infty)$, or almost everywhere (s.v.). However, if there are continuous functions on both sides (e.g. the series on the right converges at least locally uniformly in $\mathbb{R}$), the equality holds in all $x \in \mathbb{R}$.

- By substituting $a = 0$ into the equation 6.19, all terms for $n \geq 1$ drop out and we get $1 = L_0^s(x)$.

- For $a = 1$, the equation gives 6.19
$$e^{-x} = \frac{1}{2^{s+1}} \sum_{n=0}^{\infty} \frac{L_n^s(x)}{2^n} \,,$$
especially for $s = 0$ we have
$$e^{-x} = \sum_{n=0}^{\infty} \frac{L_n^0(x)}{2^{n+1}} \,.$$

The second step is to build the generating function.

Let's put $t = \frac{a}{a+1}$ ($\Leftrightarrow a = \frac{t}{1-t}$) in the equation 6.19. The derivative of this expression $\frac{\mathrm{d}t}{\mathrm{d}a} = \frac{1}{(a+1)^2} > 0$ and therefore $t(a)$ is simple. It holds $a > -\frac{1}{2}$ just for $t \in (-1,1)$. Modifying the equation 6.19 then we have

$$(a+1)^{s+1} e^{-ax} = \sum_{n=0}^{\infty} L_n^s(x) \left(\frac{a}{a+1}\right)^n \quad \Big/ a \to t \tag{6.20}$$

$$\underbrace{\frac{1}{(1-t)^{s+1}} e^{-\frac{tx}{1-t}}}_{\text{create. fce for Laguerre p.}} = \sum_{n=0}^{\infty} L_n^s(x) t^n \,, \quad t \in (-1,1) \,, \tag{6.21}$$

The table "Orthogonal systems of polynomials" in the appendix lists the Laguerre, Hermite, Legendre, Chebyshev and Gegenbauer systems of polynomials. We always state the generating function, the series expression, the explicit form, the recurrent relation and the magnitudes of the norms, the generating function and especially the space in which it forms the basis.