7. cvičení (28. 3. 2023)

(1) (a) $u_1, \dots, u_m \in \mathbb{R}^n$

$$u_1 \wedge \dots \wedge u_m = \det (u_{i,j})_{i,j=1}^{m} \; e_1 \wedge \dots \wedge e_m$$

(b) $u, v \in \mathbb{R}^n$

$$u \wedge v = \sum_{i<j} (u_i v_j - u_j v_i) \, e_i \wedge e_j$$

(c) $u_1, \dots, u_k \in \mathbb{R}^n$

$$u_1 \wedge \dots \wedge u_k = \sum_{|I|=k} \det (u_{i,j})_{i=1,\, j\in I}^{k} \; e_I$$

(2) Spočtěte $d\omega$:

(a) $\omega = x\,dx + y\,dy + z\,dz$

(b) $\omega = yz\,dx + xz\,dy + xy\,dz$

(c) $\omega = x^2\,dy \wedge dz + y^2\,dz \wedge dx + z^2\,dx \wedge dy$.

(3) Najděte $d\omega$:

(a) $\omega = F_1\,dx + F_2\,dy + F_3\,dz$,

(b) $\omega = T_1\,dy \wedge dz + T_2\,dz \wedge dx + T_3\,dx \wedge dy$.

(4) $\Omega \subset \mathbb{R}^3$ otevř., označme operátory:

$$A_0 : C^\infty(\Omega) \to \mathcal{E}^0(\Omega), \quad f \mapsto f$$

$$A_1 : C^\infty(\Omega, \mathbb{R}^3) \to \mathcal{E}^1(\Omega), \quad F \mapsto F_1\,dx + F_2\,dy + F_3\,dz$$

$$A_2 : C^\infty(\Omega, \mathbb{R}^3) \to \mathcal{E}^2(\Omega), \quad F \mapsto F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$$

$$A_3 : C^\infty(\Omega) \to \mathcal{E}^3(\Omega), \quad f \mapsto f\,dx \wedge dy \wedge dz.$$

Dále pro vektorové pole $F \in C^\infty(\Omega, \mathbb{R}^3)$

$$\operatorname{rot} F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\cancel{\operatorname{grad} F = \left(\frac{\partial F_1}{\partial x}, \partial F_2 \right.}$$

a pro $f \in C^\infty(\Omega)$

$$\operatorname{grad} f := \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

Ukažte, že komutuje diagram:

$$\begin{tikzcd}
C^\infty(\Omega) \arrow[r, "\operatorname{grad}"] \arrow[d, "A_0"] & C^\infty(\Omega, \mathbb{R}^3) \arrow[r, "\operatorname{rot}"] \arrow[d, "A_1"] & C^\infty(\Omega, \mathbb{R}^3) \arrow[r, "\operatorname{div}"] \arrow[d, "A_2"] & C^\infty(\Omega) \arrow[d, "A_3"] \\
\mathcal{E}^0(\Omega) \arrow[r, "d"] & \mathcal{E}^1(\Omega) \arrow[r, "d"] & \mathcal{E}^2(\Omega) \arrow[r, "d"] & \mathcal{E}^3(\Omega)
\end{tikzcd}$$

(5) $\Phi : (x, y, z) \mapsto (xy, x+y+z)$

$\omega = v^2\, du \wedge dv$

najděte $\Phi^* \omega$.

(6) $\Phi : (0, \infty) \times (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}^3$

sférické souřadnice

- najděte $\Phi^* (dx \wedge dy \wedge dz)$.

(7) $\Phi : t \longmapsto (\cos t, \sin t)$, $\mathbb{R} \to \mathbb{R}^2$

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

najděte $\Phi^* \omega$.

(8) Popište dvě (různé) orientace sféry S^2.

(9) $\varphi : U \to S$, $\psi : V \to S$ dvě mapy,

$S = \varphi(U) = \psi(V)$.

φ a ψ indukují stejnou orientaci

$\Longleftrightarrow$ $J(\psi^{-1} \circ \varphi)(u) = \det D(\psi^{-1} \circ \varphi)(u) > 0$, $u \in U$.

(10) Každá souvislá k-plocha má buď dvě různé orientace, nebo žádnou.

Př.: Möbiův pásek nemá žádnou orientaci (není orientovatelný).