NMTP438, topic 3: random measures and point processes

1. Show that

a) $\mu \mapsto \mu(B)$ is a measurable mapping from $(\mathcal{M}, \mathfrak{M})$ to $([0,\infty], \mathcal{B}([0,\infty]))$ for every $B \in \mathcal{B}(E)$,
b) $\mu \mapsto \mu|_B$ is a measurable mapping from $(\mathcal{M}, \mathfrak{M})$ to $(\mathcal{M}, \mathfrak{M})$ for every $B \in \mathcal{B}(E)$.
c) $\mu \mapsto \int_E f(x)\,\mu(\mathrm{d}x)$ is a measurable mapping from $(\mathcal{M}, \mathfrak{M})$ to $([0,\infty], \mathcal{B}([0,\infty]))$ for every non-negative measurable function f on E.

2. Prove that Ψ is a random measure if and only if $\Psi(B)$ is a random variable for every $B \in \mathcal{B}$.

3. The Prokhorov distance for finite measures μ, ν is defined as

$$\varrho_P(\mu,\nu) = \inf\{\varepsilon > 0 : \mu(F) \leq \nu(F^\varepsilon) + \varepsilon, \nu(F) \leq \mu(F^\varepsilon) + \varepsilon) \text{ for every } F \in \mathcal{F}\},$$

where $F^\varepsilon = \{x \in E : \exists y \in F, d(x,y) < \varepsilon\}$ is an open ε-neighbourhood of a closed set F. Show that ϱ_P is a metric.

4. For $0 < a < b < c$ let us consider the sets $K_1 = \{0, a, a+b, a+b+c\}$ and $K_2 = \{0, a, a+c, a+b+c\}$. Let X_0 be a random variable with the uniform distribution on the interval $[0, a+b+c]$. We define simple point processes Φ_1 and Φ_2 on $\mathbb{R}$ such that $\operatorname{supp}\Phi_i = \{x \in \mathbb{R} : x = X_0 + y + z(a+b+c), y \in K_i, z \in \mathbb{Z}\}$, $i = 1, 2$. Show that $\mathbb{P}(\Phi_1(I) = 0) = \mathbb{P}(\Phi_2(I) = 0)$ for every interval $I \subseteq \mathbb{R}$ but the distributions of Φ_1 and Φ_2 are different.

5. Consider intependent random variables U_1 a U_2 with uniform distribution on the interval $[0, a]$, $a > 0$, and the point process Φ on $\mathbb{R}^2$ defined as

$$\Phi = \sum_{m,n \in \mathbb{Z}} \delta_{(U_1 + ma, U_2 + na)}.$$

Determine the intensity measure of this process.

6. Let Ψ be a random measure. Check that the following formulas hold for $B, B_1, B_2 \in \mathcal{B}$:

a) $\operatorname{var}\Psi(B) = M^{(2)}(B \times B) - \Lambda(B)^2$,
b) $\operatorname{cov}(\Psi(B_1), \Psi(B_2)) = M^{(2)}(B_1 \times B_2) - \Lambda(B_1)\Lambda(B_2)$.

7. Let Φ be a simple point process. Check that the following formulas hold for $B, B_1, B_2, B_3 \in \mathcal{B}$:

a) $M^{(2)}(B_1 \times B_2) = \Lambda(B_1 \cap B_2) + \alpha^{(2)}(B_1 \times B_2)$,
b) $M^{(3)}(B_1 \times B_2 \times B_3) = \Lambda(B_1 \cap B_2 \cap B_3) + \alpha^{(2)}((B_1 \cap B_2) \times B_3) + \alpha^{(2)}((B_1 \cap B_3) \times B_2) + \alpha^{(2)}((B_2 \cap B_3) \times B_1) + \alpha^{(3)}(B_1 \times B_2 \times B_3)$,
c) $\alpha^{(n)}(B \times \cdots \times B) = \mathbb{E}[\Phi(B)(\Phi(B) - 1) \cdots (\Phi(B) - n + 1)]$.