NMTP438, topic 2: random fields

1. Let $\{W^H(t) : t \in \mathbb{R}^d_+\}$ be a centered Gaussian random field with the covariances

$$\mathbb{E}\, W^H(t)W^H(s) = \frac{1}{2}(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}),\ t, s \in \mathbb{R}^d_+,$$

where $H \in (0,1)$. Such a random field is called the *Lévy's fractional Brownian random field.* Show that it is an intrinsically stationary random field and determine its variogram.

2. Consider a spherical model for the autocovariance function of a stationary isotropic random field:

$$C(\|h\|) = \sigma^2 \frac{|b(o,\varrho) \cap b(h,\varrho)|}{|b(o,\varrho)|},\ h \in \mathbb{R}^d.$$

This model is valid in the dimension d and all the lower dimensions. However, it is not valid in higher dimensions. Express this autocovariance function for $d = 1$ and check that it is a positive semidefinite function. Show that this function considered in $\mathbb{R}^2$ (using $\|h\|, h \in \mathbb{R}^2$, as its argument) is not positive semidefinite.

Hint: Consider the points $x_{ij} = (i\sqrt{2}\varrho, j\sqrt{2}\varrho)$, $i, j = 1, \ldots, 8$ and the coefficients $\alpha_{ij} = (-1)^{i+j}$.

3. Express the autocovariance function from the previous Exercise for $d = 2$ using elementary functions.

4. Determine the spectral density of a weakly stationary random field with the autocovariance function

$$C(h) = \exp\{-\|h\|^2\},\ h \in \mathbb{R}^d.$$

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