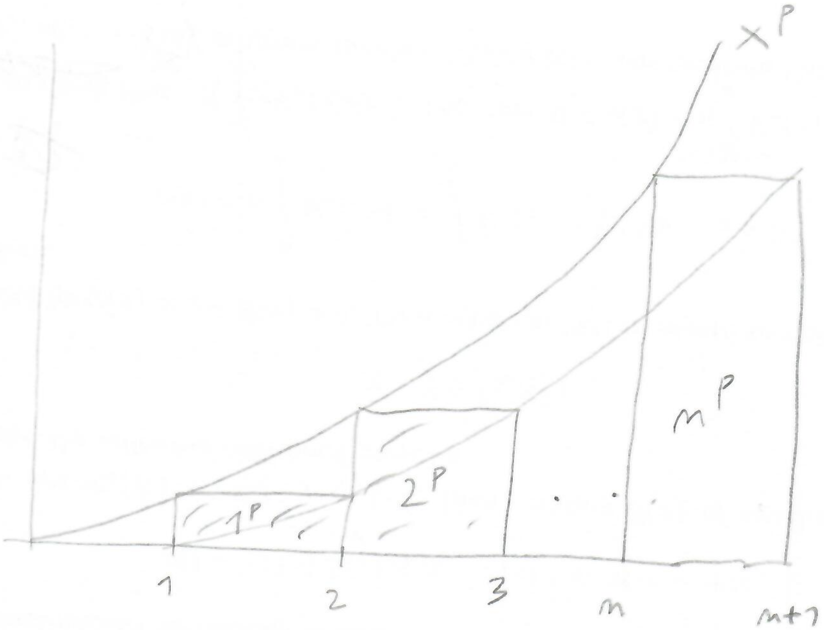
(a) $\lim\limits_{n\to\infty} \dfrac{1^p+2^p+\dots+n^p}{n^{p+1}} = ?\ ;\ (p>0).$

heuristicky: $s_n = 1^p+2^p+\dots+n^p \approx \int\limits_0^n x^p\,dx = \dfrac{n^{p+1}}{p+1}$

$\Rightarrow\ a_n \to \dfrac{1}{p+1}$

rigorózně:



$$s_n \le \int\limits_1^{n+1} x^p\,dx = \left[\frac{x^{p+1}}{p+1}\right]_1^{n+1} = \frac{1}{p+1}\left((n+1)^{p+1}-1\right)$$

$$s_n \ge \int\limits_0^n x^p\,dx = \frac{1}{p+1}\cdot n^{p+1}.$$

$$\frac{1}{p+1} \le a_n \le \frac{1}{p+1}\cdot\left(\left(1+\frac{1}{n}\right)^{p+1} - \left(\frac{1}{n}\right)^{p+1}\right) \to \frac{1}{p+1}$$

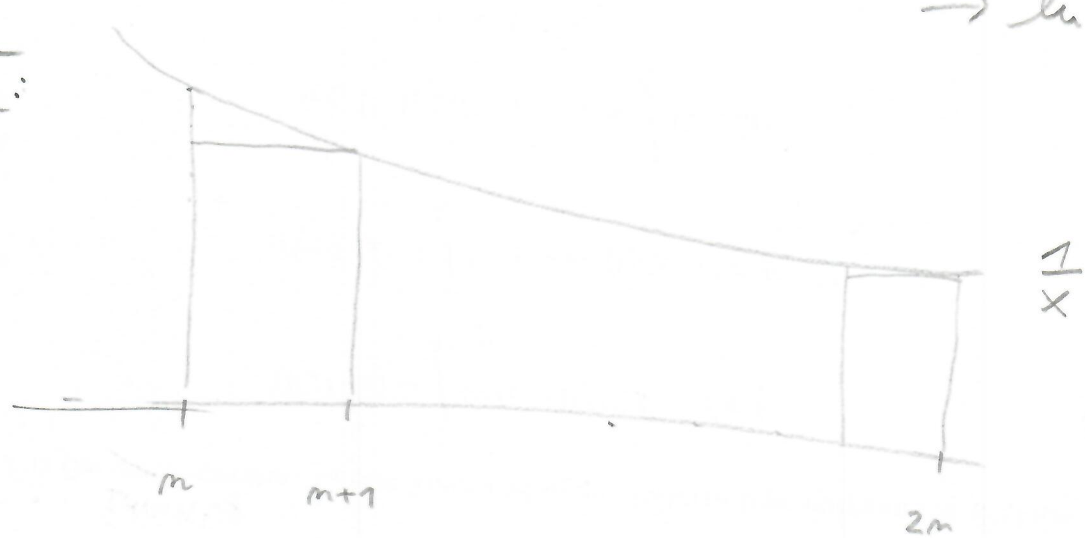
věta „o dvou policajtech": $a_n \to \dfrac{1}{p+1}$

(f) $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} = \sum_{k=1}^{n} \frac{1}{n+k}$

heuristicky: $a_n \approx \int_1^n \frac{dx}{n+x} = \left[\ln(n+x)\right]_1^n = \ln\left(\frac{2n}{n+1}\right)$

$\to \ln 2.$

rigorózně:



$a_n \le \int_n^{2n} \frac{1}{x}\,dx = \left[\ln x\right]_n^{2n} = \ln 2$

$a_n \ge \int_{n+1}^{2n+1} \frac{1}{x}\,dx = \left[\ln x\right]_{n+1}^{2n+1} = \ln \frac{2n+1}{n+1} \to \ln 2$

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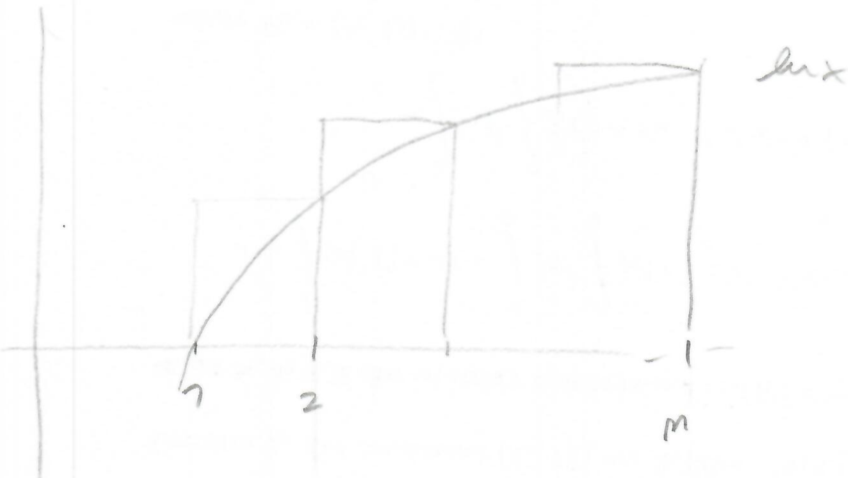
věta „o dvou policajtech": $a_n \to \ln 2$

(c) $\lim\limits_{n\to\infty} \frac{1}{n} \ln(n!)$

$$\ln(n!) = \ln n\cdot(n-1)\cdots 2\cdot 1 = \ln n + \ln(n-1) + \dots + \ln 2 + \ln 1$$

$$= \sum_{k=1}^{n} \ln k = \sum_{k=2}^{n} \ln k \geq \int_1^n \ln x \, dx = \Big[(x-1)\ln x\Big]_1^n$$



$$= (n-1)\ln n;$$

$$\Rightarrow \quad a_n \geq \frac{n-1}{n}\cdot \ln n \to +\infty; \quad a_n \to +\infty.$$