D. Pomocí vhodné substituce spočtěte plochu, ohraničenou křivkami:

1. $x + y = 1$, $x + y = 2$, $x = 3y$, $x = 4y$
2. $\sqrt{x} + \sqrt{y} = 1/3$, $\sqrt{x} + \sqrt{y} = 1/2$, $x = 2y$, $2x = y$
3. $xy = a^2$, $xy = 2a^2$, $x = y$, $y = 2x$

E. Pomocí polárních souřadnic spočtěte:

$$(1) \qquad \iint\limits_{R^2} \frac{dxdy}{(1 + x^2 + y^2)^p}$$

$$(2) \qquad \iint\limits_{M} \frac{dxdy}{\sqrt{1 - x^2 - y^2}}, \qquad M = \{x^2 + y^2 < 1\}$$

a plochu útvaru, ohraničeného křivkou:

3. $(x^2 + y^2)^2 = 8a^2xy$
4. $(x^3 + y^3)^2 = x^2 + y^2$

F. Pomocí (zobecněných) válcových souřadnic spočtěte objemy těles, ohraničených plochami:

1. $x^2 + y^2 = az$, $(x^2 + y^2)^2 = (x^2 - y^2)$, $z = 0$
2. $z = 6 - x^2 - y^2$, $z = \sqrt{x^2 + y^2}$
3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}$

G. Pomocí (zobecněných) sférických souřadnic spočtěte:

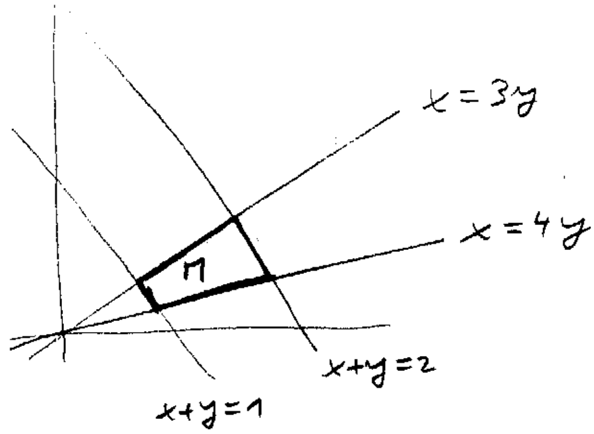
$$(1) \qquad \iiint\limits_{M} xyz\, dxdydz, \qquad M = \{1 < x^2 + y^2 + z^2 < 2,\ xyz > 0\}$$

$$(2) \qquad \iiint\limits_{M} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dxdydz, \qquad M = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1\}$$

a objemy těles, ohraničených plochami:

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$, $z = 0$
4. $x^2 + y^2 + z^2 = a^2$, $x^2 + y^2 + z^2 = b^2$, $x^2 + y^2 = cz^2$, $z = 0$
5. $(x^2 + y^2 + z^2)^3 = 3xyz$

(D1)



$\psi : (x,y) \longmapsto (u,v)$

$u = x+y$

$v = y/x$

$\varphi = \psi_{-1} : (u,v) \longmapsto (x,y)$

$x = \frac{u}{1+v}$

$y = \frac{uv}{1+v}$

$\varphi : \Omega \to \Pi$ je difeomorfismus

$\Omega = (1,2) \times (\frac{1}{4}, \frac{1}{3})$
(u, v)

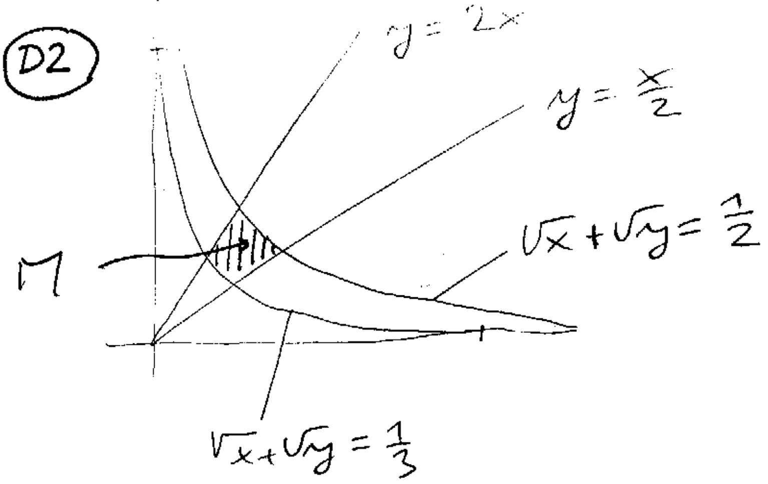
$$\lambda_2(\Pi) = \int_\Pi 1 \, dxdy = \int_\Omega |J\varphi| \, dudv =$$

$$D\varphi = \begin{pmatrix} \frac{1}{1+v}, & -\frac{u}{(1+v)^2} \\ \frac{v}{v+1}, & \frac{u}{(1+v)^2} \end{pmatrix} ; \quad J\varphi = \frac{u}{(1+v)^2}.$$

$$= \int_\Omega \frac{u}{(1+v)^2} \, dudv = \int_1^2 \left(\int_{1/4}^{1/3} \frac{u}{(1+v)^2} \, dv \right) du$$

$$= \int_1^2 \frac{u}{20} \, du = \frac{3}{40}.$$

(D2)



$\psi: (x,y) \longmapsto (u,v)$

$u = \sqrt{x}+\sqrt{y} \in \left(\frac{1}{3},\frac{1}{2}\right)$

$v = y/x \in \left(\frac{1}{2}, 2\right)$

$\varphi = \psi_{-1}: (u,v) \longmapsto (x,y)$

$x = \dfrac{u^2}{(1+\sqrt{v})^2}$

$y = \dfrac{u^2 v}{(1+\sqrt{v})^2}$

$\varphi: \Omega \to M$ je difeomorfismus;

$\lambda_2(M) = \iint_\Omega |J\varphi| \, du\, dv.$

$J\varphi$ lze vypočíst (a) přímo

(b) pomocí vzorečku: $D\psi_{-1} = [D\psi]^{-1} \circ \psi_{-1}$

$D\varphi = [D\psi]^{-1} \circ \varphi$

$J\varphi = \dfrac{1}{J\psi \circ \varphi}$

$D\psi = \begin{pmatrix} \frac{1}{2\sqrt{x}}, & \frac{1}{2\sqrt{y}} \\ -\frac{y}{x^2}, & \frac{1}{x} \end{pmatrix}$; $\quad J\psi = \dfrac{\sqrt{y}+\sqrt{x}}{2x^2}$;

$J\psi \circ \varphi = \dfrac{u}{2\left(\frac{u^2}{(1+\sqrt{v})^2}\right)^2} = \dfrac{(1+\sqrt{v})^4}{2u^3}$

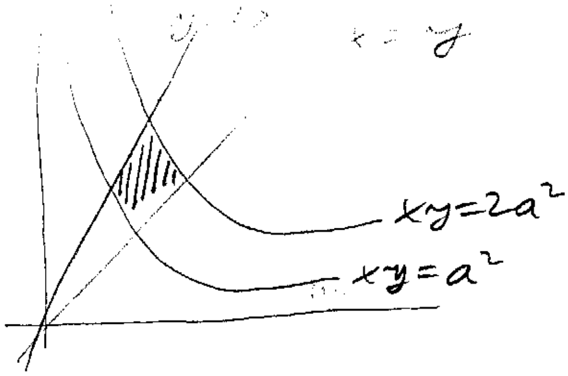
$\Rightarrow \boxed{J\varphi = \dfrac{2u^3}{(1+\sqrt{v})^4}}.$

$$\lambda_2(M) = \iint_\Omega \frac{2u^3}{(1+\sqrt{v})^4}\, du\, dv = \int_{1/2}^{2} \left(\int_{1/3}^{1/2} \frac{2u^3}{(1+\sqrt{v})^4}\, du \right) dv$$

$$= \frac{65}{2592} \int_{1/2}^{2} \frac{dv}{(1+\sqrt{v})^4} \;\Bigg|\; \begin{matrix}\text{subst:}\\ \sqrt{v} = t\end{matrix} \;\Bigg| = \frac{65}{2592} \cdot \left[\frac{2}{3(t+1)^3} - \frac{1}{(t+1)^2} \right]_{t=\frac{1}{\sqrt{2}}}^{t=\sqrt{2}}$$

$= 0.002130777\ldots$

D3



$\varphi : (x,y) \longmapsto (u,v)$

$u = xy \in (a^2, 2a^2)$

$v = y/x \in (1,2)$

$\varphi_{-1} = \varphi : (u,v) \longmapsto (x,y)$

$x = \sqrt{\frac{u}{v}} = u^{1/2} v^{-\frac{1}{2}}$

$y = \sqrt{uv} = u^{1/2} v^{1/2}$

$\Omega = \underset{u}{(a^2, 2a^2)} \times \underset{v}{(1,2)}$

$$\lambda_2(\sigma) = \iint_{\Omega} |J\varphi| \, du dv$$

$$\nabla\varphi = \begin{pmatrix} \frac{1}{2} u^{-\frac{1}{2}} v^{-\frac{1}{2}}, & -\frac{1}{2} u^{\frac{1}{2}} v^{-\frac{3}{2}} \\ \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}}, & \frac{1}{2} u^{\frac{1}{2}} v^{-\frac{1}{2}} \end{pmatrix} ; \quad J\varphi = \frac{1}{2v}$$

$$= \int_{a^2}^{2a^2} \left(\int_1^2 \frac{1}{2v} dv \right) du = \frac{a^2}{2} \ln 2 .$$

E1 $= \iint_{\Omega} \frac{r}{(1+r^2)^p}\, dr\, du = \int_0 \left(\int_0 \frac{r}{(1+r^2)^p}\, du \right) dr$

$= 2\pi \underbrace{\int_0^\infty \frac{r}{(1+r^2)^p}\, dr}_{I(p)} \dots$

(a) $p \leq 1$: $I(p) \geq I(1)$

$= \int_0^\infty \frac{r}{1+r^2}\, dr = \left[\frac{1}{2} \ln(1+r^2) \right]_0^\infty$

$= \infty - 0 = +\infty.$

(b) $p > 1$: $I(p) = \int_0^\infty r (1+r^2)^{-p}\, dr = \left[\frac{1}{2(1-p)} (1+r^2)^{1-p} \right]_0^\infty = \frac{1}{2(1-p)}.$

E2 $\iint_{\substack{r\in(0,1)\\ u\in(0,2\pi)}} \frac{r}{\sqrt{1-r^2}}\, dr\, du = 2\pi \cdot \int_0^1 \frac{r}{\sqrt{1-r^2}}\, dr = 2\pi \left[-(1-r^2)^{\frac{1}{2}} \right]_0^1 = 2\pi.$

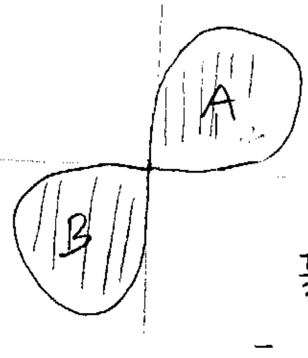
E3 $(x^2+y^2)^2 = 8a^2 xy$ $\qquad \varphi$: $x = r\cos u$, $y = r \sin u$

$(r^2)^2 = 8a^2 r^2 \cos u \sin u$

$r^2 = 4a^2 \sin 2u > 0$ $\qquad A = \varphi(\Omega)$

$u \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ $\qquad \Omega: u \in (0, \frac{\pi}{2})$

$r \in (0, 2a\sqrt{\sin 2u}).$



$\lambda_2(A) = \iint_{\Omega} r\, dr\, du$

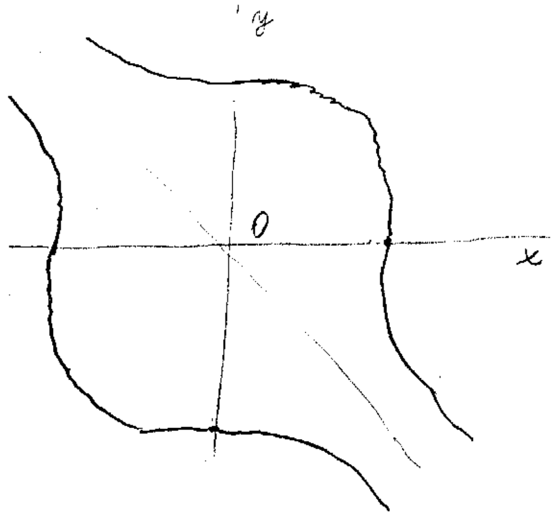
$= \int_0^{\frac{\pi}{2}} \left(\int_0^{2a\sqrt{\sin 2u}} r\, dr \right) du = \frac{1}{2} \int_0^{\frac{\pi}{2}} 4a^2 \sin 2u\, du = \boxed{2a^2}$

(E4) $(x^3+y^3)^2 = x^2+y^2$

$$r^6(\cos^3 u + \sin^3 u)^2 = r^2$$

$$r^4 = \frac{1}{(\cos^3 u + \sin^3 u)^2}$$

$$r = \frac{1}{|\cos^3 u + \sin^3 u|^{1/2}} =$$



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uvažujme navíc: $x, y \geq 0$ $\left(\iff u \in (0, \frac{\pi}{2})\right)$

$$r \in \left(0, \frac{1}{(\cos^3 u + \sin^3 u)^{1/2}}\right)$$

$$\lambda_2(M) = \iint_\Omega r \, dr\, du$$

$$= \int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{1}{(\cos^3 u + \sin^3 u)^{1/2}}} r \, dr \right) du = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{du}{\cos^3 u + \sin^3 u}$$

$$\cos^3 u + \sin^3 u = (\cos u + \sin u)(\cos^2 u - \cos u \sin u + \sin^2 u)$$
$$= (\cos u + \sin u)(1 - \cos u \sin u)$$

---

subst: $t = \operatorname{tg} \frac{x}{2} \in (0,1)$

$$\cos u = \frac{1-t^2}{1+t^2} \qquad du = \frac{2}{1+t^2}\,dt$$

$$\sin u = \frac{2t}{1+t^2}$$

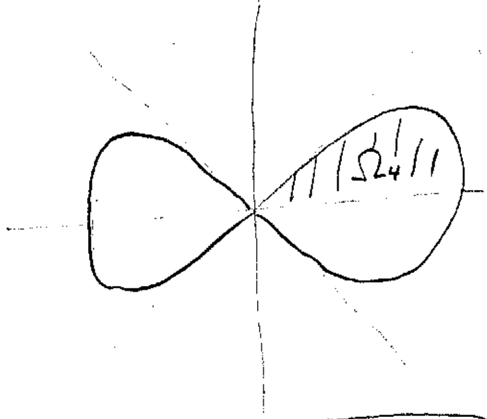
F1 $M = \{ 0 < z < \frac{1}{a}(x^2+y^2);\ (x,y)\in\Omega \}$

„podstava" Ω : $(x^2+y^2)^2 = x^2-y^2$

polární souřadnice $\quad (r^2)^2 = r^2(\cos^2 u - \sin^2 u)$

$r^2 = \cos 2u \geq 0$: $u \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$

$r \in \left(0, \sqrt{\cos 2u}\right)$



$$\frac{1}{4}\lambda_3(M) = \iint_{\Omega_4} \frac{1}{a}(x^2+y^2)\,dx\,dy$$

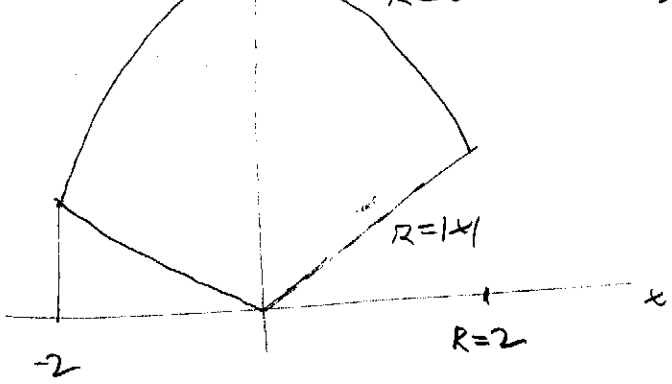
$$= \iint_{O_4} \frac{1}{a} r^2 \cdot r\,dr\,du$$

O_4: $u \in \left(0, \frac{\pi}{4}\right)$
$r \in \left(0, \sqrt{\cos 2u}\right)$

$$= \frac{1}{a}\int_0^{\frac{\pi}{4}}\left(\int_0^{\sqrt{\cos 2u}} r^3\,dr\right)du = \frac{1}{4a}\int_0^{\frac{\pi}{4}}\cos^2 2u\,du = \frac{1}{4a}\cdot\frac{\pi}{8}.$$

F2 $\sqrt{x^2+y^2} < z < 6-x^2-y^2$: $\lambda_3(M) = \iint_{\Omega}\left(6-x^2-y^2-\sqrt{x^2+y^2}\right)dx\,dy$

řez z boku:



(polární subst.) $\quad = \int_0^{2\pi}\left(\int_0^2 (6-r^2-r)\,r\,dr\right)du$

$$= 2\pi\cdot\frac{16}{3}.$$

podstava: $\{x^2+y^2 < 4\} = \Omega$

(F3) zobecněné válcové souřadnice

$\varphi: (r,u,v) \longmapsto (x,y,z)$

$x = a \cdot r \cos u$

$y = b \cdot r \sin u$

$z = c \cdot v$

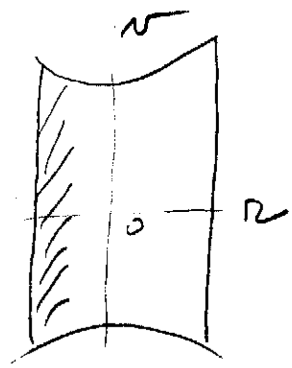
$+$

$$D\varphi = \begin{pmatrix} a\cos u, & -br\sin u, & 0 \\ a\sin u, & br\cos u, & 0 \\ 0, & 0, & c \end{pmatrix} \quad ; \quad J\varphi = abcr$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \quad : \quad r^2 - v^2 = -1$$

$$r^2 + 1 = v^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad : \quad r^2 = 1 .$$



„podstava"

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sedy: $M = \varphi(\Omega)$; Ω: $u \in (0, 2\pi)$

$r \in (0,1)$

$v \in \left(-\sqrt{1+r^2}, \sqrt{1+r^2}\right)$

$$\lambda_3(M) = \int_\Omega |J\varphi| \, dr\, du\, dv$$

$$= abc \int_0^{2\pi} \left(\int_0^1 \left(\int_{-\sqrt{1+r^2}}^{\sqrt{1+r^2}} r \, dv \right) dr \right) du$$

$$= 2abc \cdot 2\pi \cdot \int_0^1 r\sqrt{1+r^2} \, dr = 2abc \cdot 2\pi \cdot \frac{1}{3}\left(2\sqrt{2} - 1\right)$$

(F4) substituce

$$x = a x', \quad y = b y', \quad r = c r'$$

$(x', y', r') \in M'$:

$$(x')^2 + (y')^2 = r'$$

$$(x')^2 + (y')^2 = x' + y'$$

$$\lambda_3(M) = abc\, \lambda_3(M') \; ;$$ ⇝ stačí spočítat $\lambda_3(M')$

' dále vynecháváme...

---

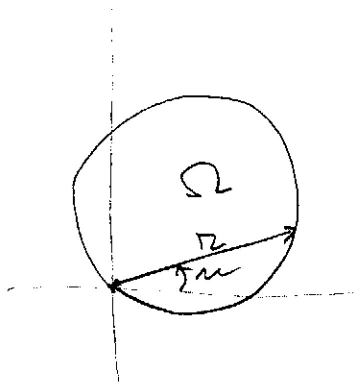
podstava:
(polární souřadnice)

$$x^2 + y^2 = x + y$$

$$r^2 = r(\cos u + \sin u)$$

$$r = \sqrt{2} \sin\left(u + \frac{\pi}{4}\right) > 0$$

$$u \in \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right).$$



$$M = \{ \underline{0} < r < x^2 + y^2 ;\ (x,y) \in \Omega \}$$

tato podmínka v zadání chybí

$$\lambda_3(M) = \iint_\Omega (x^2 + y^2)\, dx\, dy = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\int_0^{\sqrt{2}\sin\left(u + \frac{\pi}{4}\right)} r^3\, dr \right) du$$

$$= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} 4 \sin^4\left(u + \frac{\pi}{4}\right) du = \int_0^\pi \sin^4 u\, du = \frac{3\pi}{8}.$$

$$\sin^4 u = (\sin^2 u)^2 = \left(\frac{1}{2}(1 - \cos 2u) \right)^2$$

$$= \frac{1}{8}\left(\cos 4u - 4\cos 2u + 3 \right)$$

(61)

$x = r \sin u \cos v$

$y = r \sin u \sin v$

$z = r \cos u$

$\Omega \left(\begin{array}{l} r \in (1, \sqrt{2}) \\ u \in (0, \frac{\pi}{2}) \\ v \in (0, \frac{\pi}{2}) \end{array} \right.$ jen 1. oktant (výsledek 4x).

$J = r^2 \sin u$

$$\int_{\Pi} xyz \, dx\,dy\,dz = \int_{\Omega} r \sin u \cos v \; r \sin u \sin v \; r \cos v \cdot r^2 \sin u \; dr\,du\,dv$$

$$= \int_1^{\sqrt{2}} r^5 \, dr \cdot \int_0^{\frac{\pi}{2}} \sin^3 u \, du \cdot \int_0^{\frac{\pi}{2}} \cos^2 v \sin v \, dv$$

$$= \frac{1}{6}(8-1) \cdot \frac{2}{3} \cdot \frac{1}{3}.$$

(62) zobecněné sférické souř. (elipsoid)

$x = a \cdot r \sin u \cos v$, $\quad r \in (0, 1)$

$y = b \cdot r \sin u \sin v$, $\quad u \in (0, \pi)$

$z = c \cdot r \cos u$, $\quad v \in (0, 2\pi)$ $\quad \Big) \; \Omega$

$$\int_{\Pi} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx\,dy\,dz = \int_{\Omega} r^2 \cdot \underbrace{abc\, r^2 \sin u}_{\text{jakobián}} \, dr\,du\,dv$$

$$= 2\pi \cdot \int_0^1 r^4 \, dr \cdot \int_0^{\pi} \sin u \, du \cdot abc$$

$$= \underline{2\pi \cdot \frac{1}{5} \cdot 2 \cdot abc}.$$

(G3) zobecn. sfér. souřn. (viz G2)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad : \quad r^2 = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \quad : \quad \sin^2 u = \cos^2 u$$

$$|\sin u| = |\cos u|$$

$$u = \frac{\pi}{4}, \frac{3\pi}{4}$$

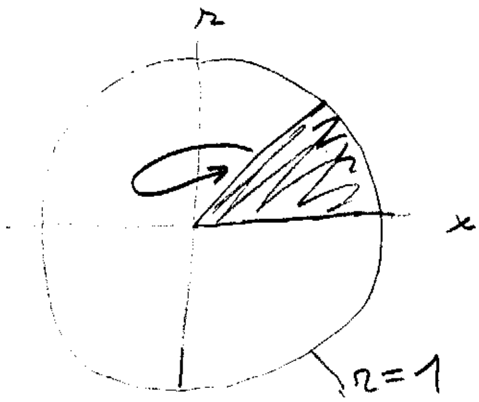
$$z = 0 : \quad u = \frac{\pi}{2}.$$

Ω :

$r \in (0,1)$

$u \in (\frac{\pi}{4}, \frac{\pi}{2})$

$v \in (0, 2\pi)$



$$\lambda_3(M) = \int_\Omega abc\, r^2 \sin u \, dr\, du\, dv$$

$$= \int_0^{2\pi} dv \cdot \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin u \, du \cdot \int_0^1 r^2 dr \cdot abc$$

$$= 2\pi \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{3} \cdot abc$$

(G4) sférické souřadnice: $x = r \sin u \cos v$

$y = r \sin u \sin v$; $J = r^2 \sin u$

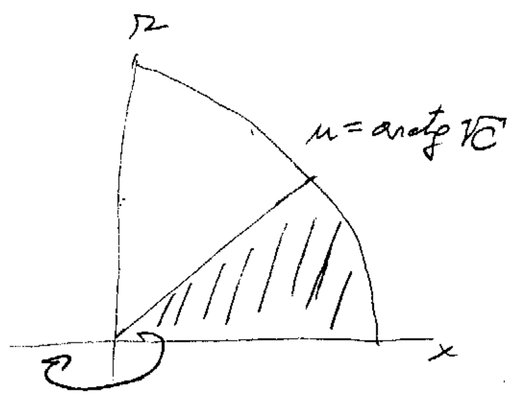
$z = r \cos u$

1. & 2. rce $\rightsquigarrow$ $r \in (a, b)$

3. rce $\longrightarrow$ $z \geq 0$: $u \in (0, \frac{\pi}{2})$

2. rce $\rightsquigarrow$ $\sin^2 u = C \cos^2 u$

$\operatorname{tg}^2 u = C$



$$\Omega : \begin{pmatrix} r \in (a, b) \\ u \in (\operatorname{arctg} \sqrt{C}, \frac{\pi}{2}) \\ v \in (0, 2\pi) \end{pmatrix}$$

$$\lambda_3(\Gamma) = \int_\Omega r^2 \sin u = \int_a^b r^2 \, dr \cdot \int_{\operatorname{arctg}\sqrt{C}}^{\frac{\pi}{2}} \sin u \, du \cdot \int_0^{2\pi} dv = \frac{1}{3}(b^3 - a^3) \cdot \underbrace{\cos(\operatorname{arctg}\sqrt{C})}_{\frac{1}{\sqrt{1+C}}} \cdot 2\pi$$

(G5) sférické souřadnice $(x^2 + y^2 + z^2)^3 = 3xyz$

$$r^6 = 3r^3 \sin^2 u \cos u \cos v \sin v$$

$$r \in (0, \sqrt[3]{3 \sin^2 u \dots})$$

jen 1. oktant (výsledek $\times 4$) : $u, v \in (0, \frac{\pi}{2})$:

$$\frac{1}{4} \lambda_3(\Gamma) = \int_{u, v \in (0, \frac{\pi}{2})} \left(\int_0^{\sqrt[3]{\dots}} r^2 \sin u \, dr \right) du \, dv =$$

$$= \frac{1}{3} 3 \int_0^{\frac{\pi}{2}} \sin^3 u \cdot \cos u \, du \cdot \int_0^{\frac{\pi}{2}} \cos v \sin v \, dv = \underline{\underline{\frac{1}{4} \cdot \frac{1}{2}}} .$$