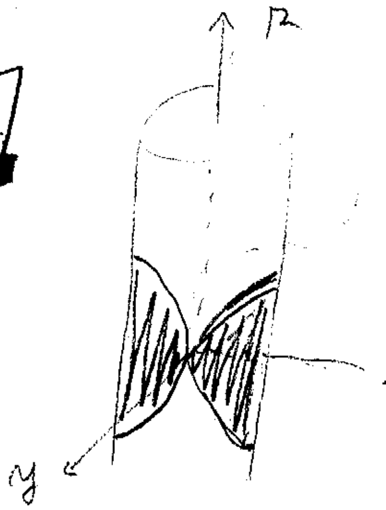
$P = \{x^2+y^2=1\} \cap \{y^2+z^2<1\}$

A1



1. rovnice = celý válec

$$\Psi \boxed{\begin{aligned} x &= \cos u \\ y &= \sin u \\ z &= v \end{aligned}} \qquad \begin{aligned} &u \in (0,2\pi) \\ \\ &v \in \mathbb{R} \end{aligned}$$

2. rce $\Rightarrow$ $v^2+\sin^2 u < 1$

$v^2 < \cos^2 u$

$|v| < |\cos u|$

$$\boxed{\begin{aligned} \Omega :\ & u \in (0,2\pi) \\ & |v| < |\cos u| \end{aligned}}$$

$$\frac{\partial \Psi}{\partial u} = (-\sin u, \cos u, 0)$$

$$\frac{\partial \Psi}{\partial v} = (0, 0, 1)$$

$$g^2 = \det \begin{pmatrix} (\Psi_u, \Psi_u), (\Psi_u, \Psi_v) \\ (\Psi_u, \Psi_v), (\Psi_v, \Psi_v) \end{pmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$P = \int_\Omega 1 \, du \, dv = \int_0^{2\pi} \left(\int_{-|\cos u|}^{|\cos u|} dv \right) du =$$

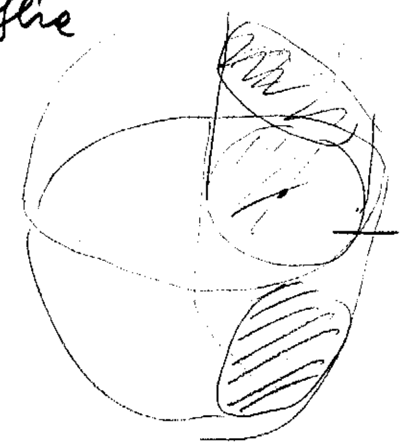
$$= 2 \int_0^{2\pi} |\cos u| \, du = \underline{\underline{8}}.$$

ověření: $2\pi \cdot 2 = 4\pi > 8$.

A2 $\{x^2+y^2<x\} \cap \{x^2+y^2+z^2=1\}$

vnitřek válce: $(x-\frac{1}{2})^2+y^2<(\frac{1}{2})^2$

sféra



Ω „podstava" – nízdole.

$z = \sqrt{1-(x^2+y^2)} = f(x,y)$

povrch: $\displaystyle\int_\Omega \sqrt{1+f_x^2+f_y^2}\,dxdy$.

$f_x = \dfrac{-2x}{\sqrt{1-(x^2+y^2)}}\cdot\dfrac{1}{2}$; $\quad g = \sqrt{1+\dfrac{x^2}{1-(x^2+y^2)}+\dfrac{y^2}{1-(x^2+y^2)}} =$

$f_y = \dfrac{-2y}{2\sqrt{1-(x^2+y^2)}}$; $\quad = \dfrac{1}{\sqrt{1-(x^2+y^2)}}$

$$\boxed{\int_\Omega \frac{dxdy}{\sqrt{1-(x^2+y^2)}}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^{\cos\varphi}\frac{r\cdot dr}{\sqrt{1-r^2}}\right)d\varphi = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\sqrt{1-r^2}\right]_0^{\cos\varphi} d\varphi$$

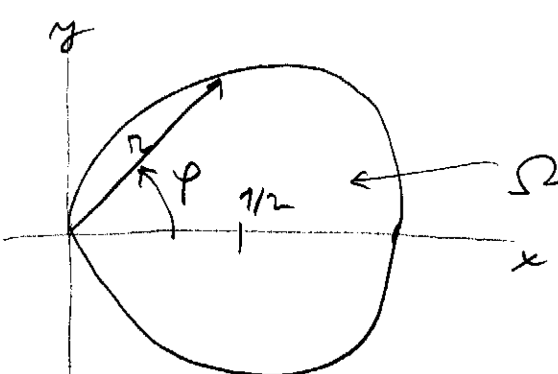
$$= +\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1-|\sin\varphi|\right)d\varphi$$

$$= \pi-2.$$

Polární:

$r^2 \le r\cdot\cos\varphi$

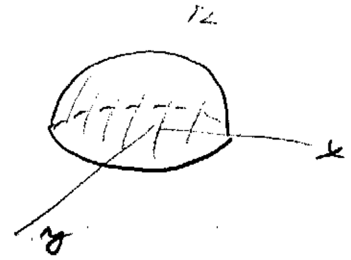
$r \le \cos\varphi \Rightarrow \varphi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$



Ω: $x^2+y^2<x$.

$\boxed{A3.}$ [redacted]

$z = 1 - (x^2+y^2)^{3/2} > 0$



$\Omega = \{x^2+y^2<1\}$

$f_x = -2x(x^2+y^2)^{1/2}\frac{3}{2} = -3x(x^2+y^2)^{1/2}$

$f_y = -2y(x^2+y^2)^{1/2}\cdot\frac{3}{2} = -3y(x^2+y^2)^{1/2}$

$g = \sqrt{1+9x^2(x^2+y^2)+9y^2(x^2+y^2)}$

$= \sqrt{1+(x^2+y^2)\cdot(9(x^2+y^2))} = \sqrt{1+9(x^2+y^2)^2}$

$\int_\Omega \sqrt{1+9(x^2+y^2)^2}\,dx\,dy = \int_0^{2\pi}\left(\int_0^1 r\sqrt{1+9r^4}\,dr\right)d\varphi = 2\pi\int_0^1 r\sqrt{1+9r^4}\,dr$

$= \boxed{\frac{2\pi}{2}\int_0^1\sqrt{1+9s^2}\,ds} = \pi/3\int_0^{\operatorname{argsinh}3}\cosh^2 t\,dt = \dots$

$r^2 = s$

$2r\,dr = ds$

$s = \frac{1}{3}\sinh t$

$ds = \frac{1}{3}\cosh t\,dt$

$\boxed{\frac{\pi}{6}\left(\operatorname{argsinh}3 + 3\sqrt{10}\right)}$

$t \in (0;\operatorname{argsinh}3)$

$\frac{1}{2}(e^t - e^{-t}) = \cancel{1}\ 3$

$e^t - e^{-t} = 6$

$z - \frac{1}{z} = 6 \quad / z$

$z^2 - 6z - 1 = 0$

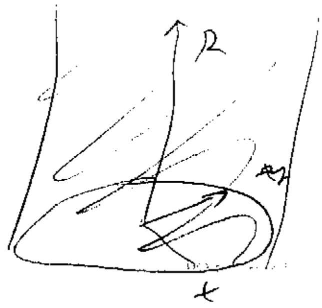
$D = 36 + 4 = 40$

$\cosh(1.81) = \sqrt{1+9} = \sqrt{10}$

$z_{1,2} = \frac{+6\pm 2\sqrt{10}}{2} = +3\pm\sqrt{10} = 1.81$

A4:

$$x^2+y^2 \le 1$$
$$x^2-y^2 = 2z$$
$$z \ge 0$$



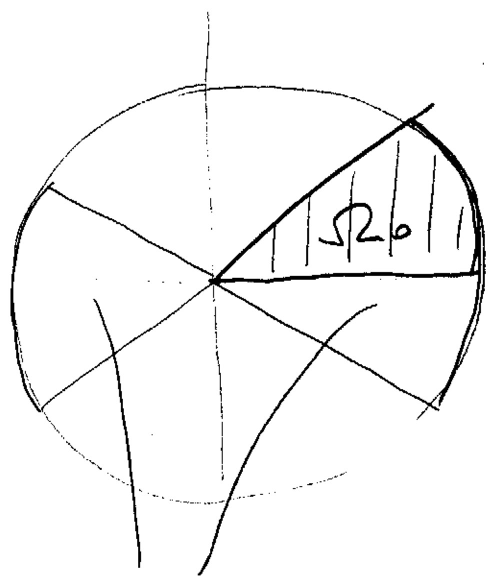
$$f = z = \frac{1}{2}\left(x^2-y^2\right) \ge 0\ ;\ |x| \ge |y|$$

$$\int_\Omega \sqrt{1+f_x^2+f_y^2}\,dx\,dy = \int_\Omega \sqrt{1+x^2+y^2} = 4\cdot\int_{\Omega_0}\dots$$

$$4\cdot\int_{\Omega_0}\sqrt{1+x^2+y^2}\,dx\,dy =$$

$$= \int_0^{\pi/4}\int_0^1 r\sqrt{1+r^2}\,dr\,d\varphi = \frac{\pi}{4}\cdot\left[\frac{1}{3}\left(1+r^2\right)^{3/2}\right]_0^1 =$$

$$= \frac{\pi}{12}\left(2^{3/2}-1\right).$$



polární souřadnice:

$x = r\cos\varphi \qquad \varphi\in(0,\frac{\pi}{4})$

$y = r\sin\varphi \qquad r\in(0,1)$

$$\Omega = \{x^2+y^2<1\}\cap\{|y|<|x|\}.$$

B1. $\varphi(t) = (3t, 3t^2, 2t^3)$

$$\varphi'(t) = (3, 6t, 6t^2) \quad ; \quad \|\varphi'(t)\|^2 = 9 + 36t^2 + 36t^4 = 9(1+4t^2+4t^4) = 9\cdot(1+2t^2)^2$$

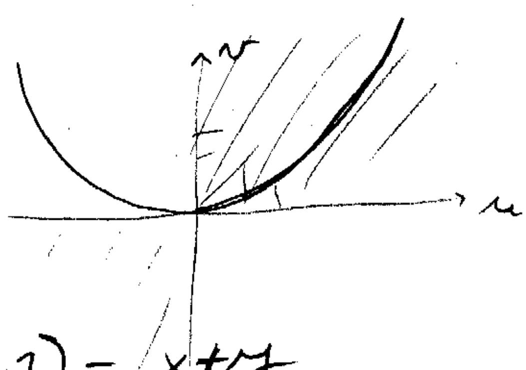
$$\int_0^1 \|\varphi'(t)\|\,dt = 3\int_0^1 \sqrt{1+4t^2+4t^4}$$

$$= 3\int_0^1 (1+2t^2)\,dt = 3\left(1+2\cdot\frac{1}{3}\right) = 3+2 = \underline{\underline{5}}$$

B2: $(x-y)^2 = \boxed{x+y}$ $\qquad [0,0,0] - \left[1,0,\frac{2\sqrt{2}}{3}\right]$

$$x^2 - y^2 = \frac{9}{8}z^2$$

$$\boxed{\begin{aligned} x-y &= u \\ x+y &= v \end{aligned}} \; : \; \boxed{\begin{aligned} u^2 &= v \\ uv &= \frac{9}{8}z^2 \end{aligned}}$$



$u = t \in [0,1] \qquad t \in [0,1] = x+y$

$$v = u^2 = t^2$$

$$z = \frac{2\sqrt{2}}{3}\cdot\sqrt{uv} = \frac{2\sqrt{2}}{3}\cdot t^{3/2}$$

$\longrightarrow$ zpět k původním:
$$x = \frac{1}{2}(u+v) = \frac{1}{2}(t+t^2)$$
$$y = \frac{1}{2}(v-u) = \frac{1}{2}(t^2-t)$$
$$z = \frac{2\sqrt{2}}{3}t^{3/2}$$

$$\varphi' = \left(\frac{1}{2}+t, t-\frac{1}{2}, \sqrt{2t}\right)$$

$$\|\varphi'\|^2 = 2\left(t+\frac{1}{2}\right)^2.$$

$$\varphi' = \left(\frac{1}{2}+t, t-\frac{1}{2}, \sqrt{2t}\right)$$
$$\|\varphi'\| = \left(t+\frac{1}{2}\right)^2 + \left(t-\frac{1}{2}\right)^2 + 2t = 2t^2 + \frac{1}{2} + 2t = 2\left(t^2+t+\frac{1}{4}\right) = 2\left(t+\frac{1}{2}\right)^2$$

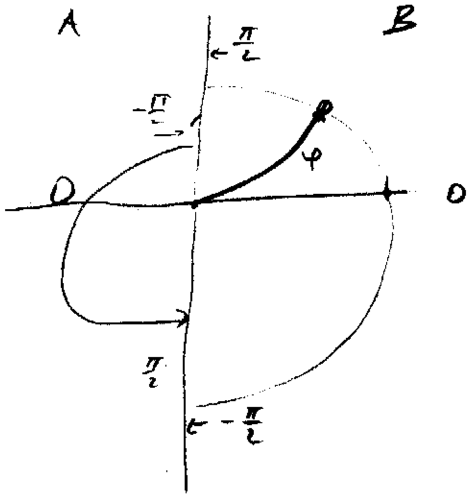
$$\text{délka} = \int_0^1 \sqrt{2}\left(t+\frac{1}{2}\right)dt = \sqrt{2}\cdot\left(\frac{1}{2}+\frac{1}{2}\right) = \boxed{\sqrt{2}}$$

B3: $x^2+y^2=z$

$\frac{y}{x}=\boxed{tg\, z}$

$z = \text{arctg}\left(\frac{y}{x}\right)$

$(0,0,0)\to\left[\sqrt{\frac{\pi}{8}},\sqrt{\frac{\pi}{8}},\frac{\pi}{4}\right]$



$x=r\cdot\cos\varphi$
$y=r\cdot\sin\varphi$
$z=\varphi$

$r^2=\varphi \qquad t=r\in\left(0,\sqrt{\frac{\pi}{4}}\right)$

$r=\sqrt{\varphi} \qquad \varphi=t^2=r^2$

$$\varphi:\ \boxed{\begin{aligned}x&=t\cdot\cos t^2\\ y&=t\cdot\sin t^2\\ z&=t^2\end{aligned}}$$

$$\varphi'=\left(\cos t^2+t\sin t^2(-2t),\ \sin t^2+t\cos t^2(+2t),\ 2t\right)$$

$$\|\varphi'\|^2=1+4t^4+4t^2=(1+2t^2)^2\ ; \quad L=\int_0^{\sqrt{\frac{\pi}{4}}}(1+2t^2)\,dt=\sqrt{\frac{\pi}{4}}+\frac{2}{3}\left(\frac{\pi}{4}\right)^{3/2}=\sqrt{\frac{\pi}{4}}\left(1+\frac{\pi}{6}\right)$$

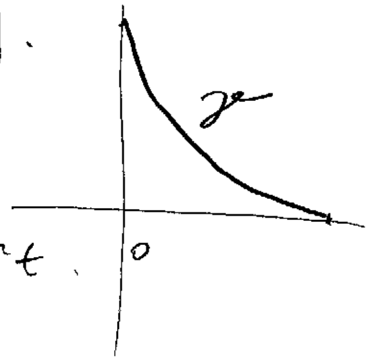
B4: $\varphi'(t)=\left(-e^{-t}\cos t-e^{-t}\sin t,\ -e^{-t}\sin t+e^{-t}\cos t,\ -e^{-t}\right)$

$\|\varphi'(t)\|^2=e^{-2t}+e^{-2t}+e^{-2t}=3e^{-2t}.$

$L(\gamma)=\int_0^\infty\sqrt{3}\,e^{-t}\,dt=\sqrt{3}$

C2: $\int_\gamma \left(|x|^{\frac{3}{4}}+|y|^{\frac{3}{4}}\right) ds$; $\gamma = \{ |x|^{\frac{2}{3}}+|y|^{\frac{2}{3}} = 1\}$.

$t \in [0, \frac{\pi}{2}]$.



$\varphi = (\cos^3 t, \sin^3 t)$; $\varphi' = 3(-\sin t \cos^2 t, \cos t \sin^2 t)$

$\|\varphi'\|^2 = 9(\sin^2 t \cos^4 t + \cos^2 t \sin^4 t) = 9 \sin^2 t \cos^2 t$.

$\|\varphi'\| = 3 \sin t \cos t$

$\frac{1}{4} I = 3\int_0^{\frac{\pi}{2}} \left(\cos^{\frac{9}{4}} t + \sin^{\frac{9}{4}} t\right) \sin t \cos t = \quad = \frac{4}{17}$

(symetrie sin, cos) $= 6 \cdot \int_0^{\frac{\pi}{2}} \sin^{\frac{13}{4}} t \cos t \, dt = 6 \cdot \int_0^1 y^{\frac{13}{4}} dy = 6 \cdot \frac{1}{\frac{13}{4}+1}$

$I = \frac{6 \cdot 4 \cdot 4}{17} = \frac{96}{17}$

$\boxed{C1}$ $\varphi'(t) = (-\sin t, \cos t, 1)$

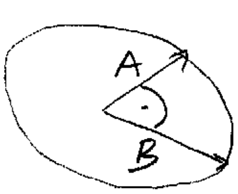
$\|\varphi'(t)\| = \sqrt{2}$.

$\int_0^{2\pi} (\cos^2 t + \sin^2 t + t^2)\sqrt{2}\, dt = \int_0^{2\pi} (1+t^2)\sqrt{2}\, dt \dots$

$\boxed{C3}$ γ ... je zřejmě kružnice o poloměru 1.

(jednotkové sféra $\cap$ rovina jdoucí skrz $\underset{\sim}{0}$)

parametrizace: $\varphi(t) = \underset{\sim}{A} \cos t + \underset{\sim}{B} \sin t$; $t \in (0, 2\pi)$.



$\|\varphi'\| = 1$ libovolné kolmé poloměry;

např.: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = A$

$\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = B$.

$I = \pi\left(\frac{1}{2} + \frac{1}{5}\right)$.

C4. průvodní = 0 (symetrie)

$$\int_S |x|+|y|+|z| \, dS \; ;$$

variante

$x = \cos u \cdot \cos v$
$y = \sin u \cdot \cos v$
$z = \sin v$

1. oktant.

$u \in (0, \frac{\pi}{2})$
$v \in (0, \frac{\pi}{2})$.

výsledek krát 8 (symetrie).

$$dS = \ldots \quad \varphi_u = (-\sin u \cos v, \cos u \cos v, 0)$$
$$\varphi_v = (-\cos u \sin v, -\sin u \sin v, \cos v)$$

$(\varphi_u, \varphi_u) = \cos^2 v$
$(\varphi_v, \varphi_v) = \sin^2 v + \cos^2 v$
$(\varphi_u, \varphi_v) = 0$

$$g^2 = \begin{vmatrix} \cos^2 v & 0 \\ 0 & 1 \end{vmatrix} = \cos^2 v$$

$$\boxed{g = |\cos v|}$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (\cos u \cdot \cos v + \sin u \cos v + \sin v) \cdot \cos v \; du \, dv$$

$$= \underbrace{2 \int_0^{\frac{\pi}{2}} \cos^2 v}_{\frac{\pi}{2}} + \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \underbrace{\sin v \cos v}_{\frac{1}{2} \sin 2v} = \frac{3\pi}{4}$$

$$\left[-\frac{1}{4} \cos 2v\right]_0^{\frac{\pi}{2}} = \frac{1}{2}$$

$\times 8 \longrightarrow$ Výsledek: $\boxed{6\pi}$