A1 $\sum_{k=1}^{\infty} \underbrace{\frac{\ln k}{k}}_{c_k} x^k$ $\qquad R=1$ $\qquad$ a) $\left|\frac{c_{k+1}}{c_k}\right| \to 1$

nutost:

b) $x=-1$: m. ř. konv. neabs.

podrobně: a) $\frac{c_{k+1}}{c_k} = \frac{\ln(k+1)}{\ln k} \cdot \frac{k}{k+1} = \underbrace{\frac{\ln k + \overbrace{\ln\left(1+\frac{1}{k}\right)}^{\to 0}}{\ln k}} \cdot \frac{1}{1+\underbrace{\frac{1}{k}}_{\to 0}} \to 1$

$(\ln k \geq 1)$ $\qquad 1 + \underbrace{\frac{1}{\ln k}}_{\to 0} \cdot \underbrace{\ln\left(1+\frac{1}{k}\right)}_{\to 0}$

b) $\sum_{k=1} |c_k| \geq \sum_{k=3} \frac{1}{k} = +\infty$

$\sum_{k=1} (-1)^k c_k$ KONV. Leibniz: $c_k \to 0$ (známé limity)

? monotonie: $c_k = f(k)$

$a_1 = 0$

$a_2 = 0.3465$

$a_3 = 0.366$

$a_4 =$

$f'(x) = \left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x}\cdot x - 1\cdot \ln x}{x^2} < 0$ pro $x > e$; $\ln x > 1$

tj. $\{c_k\}$ klesá pro $k \geq 3$.

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teorie: $|x| < R \Rightarrow$ m. ř. konv. abs.

$|x| > R \Rightarrow$ m. ř. diverguje $\qquad \Rightarrow \quad R = |-1| = 1$.

? obecné $x \in \mathbb{C}$; $|x| = 1$.

$x = 1$: $\sum c_k$ div. (již víme)

$|x| = 1$; $x \neq 1$: $\left|\sum_{k=0}^{n} x^k\right| = \left|\frac{1-x^{n+1}}{1-x}\right| \leq \frac{1+|x^{n+1}|}{|1-x|} \leq \frac{2}{|1-x|}$

tj. má o.č.s.; tedy $\sum c_k x^k$ KONV. (Dirichlet)

(monotonie viz výše)

NEABS. (již víme).

A2| $\displaystyle\sum_{k=0}^{\infty} \underbrace{\frac{(k!)^2}{(2k)!}}_{c_k} x^k$; $\displaystyle\left|\frac{c_{k+1}}{c_k}\right| = \left(\frac{(k+1)!}{k!}\right)^2 \cdot \frac{(2k)!}{(2k+2)!} = \frac{(k+1)^2}{2(k+1)(2k+1)} \to \frac{1}{4}$

$R = 4.$

? konvergence pro $|x| = 4$:

$$|c_k x^k| = c_k 4^k =: a_k \; ; \quad \frac{a_{k+1}}{a_k} = \dots = \frac{2k+2}{2k+1} > 1$$

tj: $a_{k+1} \geq a_k \geq \dots \geq a_0 > 0$

$a_k \not\to 0 \Rightarrow$ řada div.

A3| $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k 5^k}(x-3)^k$ $R=5$; $|x-3|=5$, $x \neq 8$ KONV. NEABS.

$x = 8$: DIV.

A4| $\displaystyle\sum_{k=1}^{\infty} \underbrace{a^{k^2}}_{c_k} x^k$; $a>0$ $\quad \sqrt[k]{c_k} = a^k \to +\infty$; $a>1$

tj: $R = 0$

$\to 0$; $a<1$

tj: $R = +\infty$

$\to 1$, $a = 1$.

řada $\sum x^k$; $R = 1$;

pro $|x| = 1$: DIVERGUJE

A5| $\displaystyle\sum_{k=1}^{\infty} \underbrace{\frac{a^k+b^k}{k}}_{c_k} x^k$; BÚNO $a \geq b > 0$

$$\sqrt[k]{|c_k|} = \frac{1}{\sqrt[k]{k}} \sqrt[k]{1+\left(\frac{b}{a}\right)^k} \cdot a = P_1 \cdot P_2 \cdot a \to a, \text{ neboť}$$

$\cancel{P_1 = \exp\left(\frac{1}{k}\ln k\right) \to \exp(0) = 1}$

$\Rightarrow R = \frac{1}{a}$

$1 \leq P_2 \leq \sqrt[k]{1+1} = \exp\left(\frac{1}{k}\ln 2\right) \to 0$

A5 – dokončení $x = \frac{q}{a}$; $q \in \mathbb{C}$; $|q|=1$

$$\sum_{k=1}^{\infty} \frac{q^k}{k} \underbrace{\left(1+\left(\frac{b}{a}\right)^k\right)}_{\in [1,2]\text{; monotónní}} \quad \text{KONV.} \iff \sum_{k=1}^{\infty} \frac{q^k}{k} \text{ KONV.:}$$

ANO (neabs.) $q \neq 1$

NE $q=1$

A6 $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{k^2} (x-1)^k$; $\sqrt[k]{c_k} = \left(1+\frac{1}{k}\right)^k \to e$; $R = e^{-1}$

$x = 1+\frac{q}{e}$; $q \in \mathbb{C}$, $|q|=1$. $\rightsquigarrow \sum b_k q^k$

$$b_k = \left(1+\frac{1}{k}\right)^{k^2} e^{-k} = \exp\Big(\underbrace{k^2 \ln\left(1+\frac{1}{k}\right) - k}_{d_k}\Big)$$

$$d_k = f\left(\frac{1}{k}\right); \quad f(x) = \frac{\ln(1+x)-x}{x^2} = \frac{x - \frac{1}{2}x^2 + o(x^2) - x}{x^2} \to -\frac{1}{2}$$

$b_k \to e^{-1/2} \neq 0$; tj. $|x|=R$ – řada div.

A7 $\sum_k c_k x^k$; $c_k = (-1)^k \left(\frac{2^k (k!)^2}{(2k+1)!}\right)^\lambda$; $\lambda \in \mathbb{R}$:

$\lambda = 0$: $\sum (-1)^k x^k$; $R=1$; známo

$\lambda \neq 0$: $\left|\frac{c_{k+1}}{c_k}\right| = \left(\frac{2k^2}{(2k+3)(2k+2)}\right)^\lambda \to 2^{-\lambda}$; $R = 2^\lambda$

$x = 2^\lambda q$; kde $q \in \mathbb{C}$; $|q|=1$: $\sum_k b_k$; kde

$$b_k = (-1)^k \left(\frac{4^k (k!)^2}{(2k+1)!}\right)^\lambda; \quad \left|\frac{b_{k+1}}{b_k}\right| = \left(\frac{4k^2}{(2k+3)(2k+2)}\right)^\lambda \to 1$$

podílové krit.: $\emptyset$

A7 – dokončení

Raabe: $\left|\frac{b_k}{b_{k+1}}\right| = \left(1+\frac{3}{2k}\right)^{\lambda}\left(1+\frac{1}{k}\right)^{\lambda}$

$k\left(\left|\frac{b_k}{b_{k+1}}\right|-1\right) = f\left(\frac{1}{k}\right)$; $f(x) = \dfrac{(1+3/2\,x)^{\lambda}(1+x)^{\lambda}-1}{x}$

l'Hosp. „$\frac{0}{0}$": $\frac{3}{2}\lambda\left(1+\frac{3}{2}x\right)^{\lambda-1}(1+x)^{\lambda} + (1+3/2\,x)^{\lambda}\lambda(1+x) \to \frac{5}{2}\lambda$

$\lambda > \frac{2}{5}$: $\sum |b_k|$ KONV.

$< \frac{2}{5}$: DIV.

$\lambda \le 0$: $|b_{k+1}| \ge |b_k|$; tj. $b_k \to 0$; $\sum b_k$ div.

$\lambda \in (0, 2/5]$: $\sum b_k = \underbrace{\sum (-1)^k q^k}_{\text{o.č.s. vyjma } q=-1} \cdot d_k \to 0$; monotónně.

$0 < \frac{d_{k+1}}{d_k} = \left(\frac{1}{\left(1+\frac{3}{2k}\right)\left(1+\frac{1}{k}\right)}\right)^{\lambda} < 1$; tj. $d_{k+1} < d_k$

$d_{k+1} \le \left(1+\frac{1}{k}\right)^{-\lambda} d_k$

$d_{k+1} \le \prod_{l=1}^{k}\left(1+\frac{1}{l}\right)^{-\lambda} d_1$

$\ln d_{k+1} \le -\lambda \sum_{l=1}^{k} \underbrace{\ln\left(1+\frac{1}{k}\right)}_{\sim \frac{1}{k}} + \ln d_1 \to -\infty$

tedy $d_k \to 0$;

zjištěné závěry:

$R = 2^{\lambda}$; pro $|x| = 2^{\lambda}$

$\lambda > \frac{2}{5}$ KONV. ABS.

$\lambda > 0$ KONV. (vyjma $x = -2^{\lambda}$)

$\lambda \le 0$ DIV.

$\lambda < 2/5$ NEKONV. ABS.

B1/ $\sin^2 x = \frac{1}{2}\left(1-\cos 2x\right) = \frac{1}{2}\left(1 - \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k}}{(2k)!}\right)$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k-1}}{(2k)!} x^{2k} = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \dots, \quad x \in \mathbb{R}$$

B2/ $(1+x^2)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^{2k} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} + \dots, \quad |x|<1$

B3/ $\operatorname{arctg} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad ; \quad |x|<1$

$$\int_0^x \frac{\operatorname{arctg} t}{t}\, dt = \int_0^x \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{2k+1}\, dt = \sum_{k=0}^{\infty} \int_0^x (-1)^k \frac{t^{2k}}{2k+1}\, dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)^2} \quad ; \quad |x|<1$$

C1/ $f(x) = \sum_{k=1}^{\infty} k^2 x^{k-1} = \left(\sum_{k=1}^{\infty} k x^k\right)' = F'(x)$

$\frac{1}{x} F(x) = \sum_{k=1}^{\infty} k x^{k-1} = \left(\sum_{k=1}^{\infty} x^k\right)' = G'(x)$;

víme: $1+x+x^2+\dots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$; odsud $G(x) = \frac{x}{1-x}$

$$f(x) = F'(x) = \left(x G'(x)\right)' = \left(x\left(\frac{x}{1-x}\right)'\right)' = \frac{x+1}{(1-x)^3}$$

pro $\forall\, |x|<1 = R$: poloměr konvergence všech uvedených řad

C2/ $f(x) = \sum_{k=0}^{\infty} \frac{x^{4k+1}}{4k+1}$; $f'(x) = \sum_{k=0}^{\infty} x^{4k} = \frac{1}{1-x^4}$; $|x|<1$

$$\int \frac{1}{1-x^4}\, dx = \int \frac{1}{2}\cdot\frac{1}{1+x^2} + \frac{1}{4}\cdot\frac{1}{x+1} - \frac{1}{4}\cdot\frac{1}{x-1}\, dx$$

$= \frac{1}{4} \ln \frac{x+1}{|x-1|} + \frac{1}{2} \operatorname{arctg} x =: \tilde{f}(x)$; tj. $f(x) = \tilde{f}(x) + C$

$x=0 \Rightarrow C=0$.

*) C3| pozoruj!| $\binom{-\frac{1}{2}}{k} = \frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdots\left(-\frac{2k-1}{2}\right)}{1\cdot 2 \cdots k} = (-1)^k \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots(2k)}$

$= (-1)^k \frac{(2k-1)!!}{(2k)!!}$; odtud $\sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} x^k = (1-x)^{-\frac{1}{2}} - 1$

C4| $f(x) = \sum_{k=1}^{\infty} k(k+1)x^k = F'(x)$; $F(x) = \sum_{k=1}^{\infty} k\, x^{k+1}$

$\frac{1}{x^2} F(x) = \sum_{k=1}^{\infty} k\, x^{k-1} = G'(x)$; $G(x) = \sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$

$f(x) = F'(x) = \left(x^2 G'(x)\right)' = \frac{2x}{(1-x)^3}$

D1| víme: $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$; $|x|<1$

$\Rightarrow$ $-\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$

volme $x = \frac{1}{2}$: $\sum_{k=1}^{\infty} \frac{1}{k 2^k} = -\ln\frac{1}{2} = \ln 2$

D2| $f(x) = \sum_{k=1}^{\infty} \frac{k^2 x^k}{k!} = x\left(\sum_{k=1}^{\infty} \frac{k\, x^k}{k!}\right)' = x\, F'(x)$

$F(x) = \sum_{k=1}^{\infty} \frac{k x^k}{k!} = \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} = x \cdot \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = x e^x$

$f(x) = x\, F'(x) = x\left(x e^x\right)' = e^x (x^2 + x)$

$f(1) = 2e$

D3| $\operatorname{arctg} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$; $x \in (-1,1)$ – platí i pro $x=1$|

$\frac{\pi}{4} = \operatorname{arctg} 1 = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

D4) $f(x)=\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}x^k$; $f'(x)=\sum_{k=1}^{\infty}(-1)^k\frac{x^{k-1}}{k+1}=\frac{1}{x^2}F(x)$,

$$F(x)=\sum_{k=1}^{\infty}(-1)^k\frac{x^{k+1}}{k+1}=\ln(1+x)-x$$

$u=\ln(1+x)$
$v'=\frac{1}{x^2}$

$$f(x)=\underbrace{\int\frac{\ln(1+x)-x}{x^2}\,dx}_{I}+C\;;\quad I=I_1+I_2$$

p.p.

$$I_1=\int\frac{\ln(1+x)}{x^2}\,dx=-\frac{\ln(1+x)}{x}-\ln(1+x)+\ln|x|$$

$$=-\frac{\ln(1+x)}{x}-\ln(1+x)+C\qquad I_2=\int-\frac{dx}{x}\,dx=\ln|x|$$

$x\to 0+$: $f(0+)=0\Rightarrow C=1.$

E1) $y(x)=\sum_{k=0}^{\infty}c_kx^k$; $y(0)=c_0=1$
$y'(0)=c_1=0$

$$y''(x)=\sum_{k=2}^{\infty}k(k-1)c_kx^{k-2}$$

$$\sum_{k=2}^{\infty}k(k-1)c_kx^{k-2}-x\sum_{k=0}^{\infty}c_kx^k=0$$

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k-\sum_{k=1}^{\infty}c_{k-1}x^k=0$$

$k=0$: $2\cdot1\cdot c_2=0$

$k\geq1$: $(k+2)(k+1)c_{k+2}=c_{k-1}$

$\forall k\geq0$: $(k+3)(k+2)c_{k+3}=c_k$

$$\boxed{y(x)=\sum_{l=0}^{\infty}\underbrace{c_{3l}x^{3l}}_{a_l}\qquad\left|\frac{a_{l+1}}{a_l}\right|=\frac{|x|^3}{(3l+3)(3l+2)}\to0\;\Rightarrow R=+\infty.}$$

odkud $c_{3l+1}=c_{3l+2}=0$; $\forall l\geq0$

$$c_{3l}=\frac{1}{3l\cdot(3l-1)}\cdot\frac{1}{(3l-3)(3l-4)}\cdot\ldots\cdot\frac{1}{3\cdot2}\cdot\underbrace{c_0}_{=1}$$

E2/ poč. podm.: $c_0 = 0,\ c_1 = 1$

rovnice: $\underbrace{\sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}}_{\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2}} + \sum_{k=1}^{\infty} k c_k x^k - 2 \sum_{k=0}^{\infty} c_k x^k = 0$

x^0: $2c_2 - 2c_0 = 0$

x^k: $(k+2)(k+1) c_{k+2} + k c_k - 2 c_k = 0$

$$c_{k+2} = -c_k \frac{k-2}{(k+2)(k+1)}\ ; \quad \forall k \geq 1$$

$\Rightarrow$ odtud: $c_{2l} = 0$;

$$c_{2l+1} = (-1)^l \frac{(2l-1)(2l-3)\dots 3\cdot(-1)}{(2l+1)(2l)\dots 3\cdot 2}$$

? konvergence: $y(t) = \sum_{l=0}^{\infty} \underbrace{c_{2l+1} x^{2l+1}}_{b_l}$

$x = 0$: jistě konvergují

$x \neq 0$: $\left|\frac{b_{l+1}}{b_l}\right| = |x|^2 \cdot \left|\frac{c_{2l+3}}{c_{2l+1}}\right| = |x|^2 \cdot \frac{2l-1}{(2l+3)(2l+2)} \to 0$

$\sum b_l$ KONV. ABS. pro $\forall x \in \mathbb{C}$

$\Rightarrow$ poloměr konvergence řady $y(x)$ je $+\infty$.

E3/ $c_0 = 0,\ c_1 = 1$

$\underbrace{\sum_{k=2}^{\infty} \left(k(k-1) c_k x^{k-2} + k(k-1) c_k x^k \right)}_{\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k} - 2 \sum_{k=1}^{\infty} k c_k x^k + 2 \sum_{k=0}^{\infty} c_k x^k = 0$

x^0: $2\cdot 1 \cdot c_2 + 2c_0 = 0 \Rightarrow c_2 = 0$

x^1: $3\cdot 2\cdot c_3 - 2c_1 + 2c_1 = 0 \Rightarrow c_3 = 0$

E3 – dokončení

$$\overbrace{k(k-1)-2(k-1)}=(k-2)(k-1)$$

$k \geq 2$ obecně: $(k+2)(k+1)c_{k+2} + \overbrace{k(k-1)c_k - 2kc_k + 2c_k} = 0$

$$c_{k+2} = -\frac{(k-2)(k-1)}{(k+2)(k+1)}c_k \ .$$

$c_2 = 0 \Rightarrow c_4, c_6, \ldots = 0$

$c_3 = 0 \Rightarrow c_5, c_7 \ldots = 0 \ .$ $\Rightarrow \boxed{y(x) = x}$

$\Rightarrow c_1 = 1$ jediný nenulový koeficient