A1. [**6b**] Vypočítejte limitu

$$\lim_{x\to 0+} \left(1+\sqrt[3]{\operatorname{tg} x}\right)^{\frac{1}{\sqrt{1+\sqrt{x}}-1}}$$

A2. [**5b**] Vypočítejte limitu

$$\lim_{x\to +\infty} \frac{x^2+\sin(x^2)+\ln(e^x\sqrt{x+1})}{1-\ln(e^{2x}+1)}$$

. . . .

B1. [**6b**] Vypočítejte limitu

$$\lim_{x\to -\infty} \left(1+\sqrt{x^2+1}-\sqrt{x^2+2}\right)^{\frac{1}{\exp(1/x)-1}}$$

B2. [**5b**] Vypočítejte limitu

$$\lim_{x\to +\infty} \frac{\ln(x+1)+\sqrt[3]{x}+\ln(1+e^{x/6})}{2+\cos\sqrt{x}+\sqrt{x}}$$

. . . .

A1) $\lim\limits_{x\to 0+} \left(1+\sqrt[3]{\operatorname{tg} x}\right)^{\frac{1}{\sqrt{1+\sqrt{x}}-1}}$

$$f(x) = \exp h(x); \quad h(x) = \frac{1}{\sqrt{1+\sqrt{x}}-1} \ln\left(1+\sqrt[3]{\operatorname{tg} x}\right).$$

úprava odmocniny: $\sqrt{1+\sqrt{x}}-1 = \dfrac{\sqrt{x}}{\sqrt{1+\sqrt{x}}+1}$

$$h(x) = \frac{\ln\left(1+\sqrt[3]{\operatorname{tg} x}\right)}{\sqrt[3]{\operatorname{tg} x}} \cdot \left(\sqrt{1+\sqrt{x}}+1\right) \cdot \frac{\sqrt[3]{\operatorname{tg} x}}{\sqrt{x}} = P_1 \cdot P_2 \cdot P_3$$

$$\left.\begin{aligned} &\frac{\ln(1+y)}{y} \to 0, \ y \to 0 \\ &\qquad\text{(zákl. limita)} \\ &\sqrt[3]{\operatorname{tg} x} \to 0; \ x \to 0+ \\ &\qquad \neq 0; \ x \in P_+(0,\delta) \\ &\qquad \text{(funkce } \sqrt[3]{y}, \operatorname{tg} x \text{ rostoucí,} \\ &\qquad\quad \text{spojité v okolí } 0\text{)} \end{aligned}\right\} \xRightarrow{\text{VoLSF(b)}} P_1 \to 1; \ x \to 0+.$$

$\sqrt{1+\sqrt{x}}+1 \to 2; \ x \to 0+$ dle VoLSF(a)

nebot' $\sqrt{y}$ je spojitá v $y_0 = 1$,

spojitá zprava v $y_0 = 0$.

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zbývá vyšetřit

$$P_3 = \frac{\sqrt[3]{\operatorname{tg} x}}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x}}{\sqrt{x}} = \sqrt[3]{\frac{\operatorname{tg} x}{x}} \cdot x^{-\frac{1}{6}};$$

$$\frac{\operatorname{tg} x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x} \to 1 \cdot 1 \ ; x \to 0+$$

(základní limita & spojitost cos v $y_0 = 0$)

$$x^{-\frac{1}{6}} = \exp\big(-\tfrac{1}{6}\underbrace{\ln x}_{\to -\infty}\big) \to +\infty \ ; x \to 0+$$

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celkem tedy: $P_3 \longrightarrow 1 \cdot (+\infty) = +\infty$

$h(t) \to +\infty$

$f(t) \to +\infty$

A2/ $$\lim_{x\to+\infty} \frac{x^2+\sin x^2+\ln\left(e^x\sqrt{x+1}\right)}{1-\ln\left(e^{2x}+1\right)}$$

úpravy logaritmů:

$$\ln\left(e^x\sqrt{x+1}\right) = \ln e^x + \ln\sqrt{x+1} = x + \frac{1}{2}\ln(x+1)$$

$$\ln\left(e^{2x}+1\right) = \ln e^{2x}\left(1+e^{-2x}\right) = 2x + \ln\left(1+e^{-2x}\right)$$

$$f(x) = \frac{x^2+\sin x^2 + x + \frac{1}{2}\ln(x+1)}{1-2x+\ln\left(1+e^{-2x}\right)}$$

$$= \frac{x^2\left(1+\frac{1}{x^2}\sin x^2+\frac{1}{x}+\frac{1}{2x^2}\ln(x+1)\right)}{x\left(\frac{1}{x}-2+\frac{1}{x}\ln\left(1+e^{-2x}\right)\right)} = x\cdot R(x).$$

ukážeme, že $R(x)\to -\frac{1}{2}$; tedy $f(x)\to-\infty$.

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dle VoAL stačí ukázat, že:

$\frac{1}{x^2}\sin x^2 \to 0$: dle Věty 2.8; $\frac{1}{x^2}\to 0$
$\sin x^2$ omezená fce

$$\frac{1}{2x^2}\ln(x+1) = \frac{x+1}{2x^2}\cdot\frac{\ln(x+1)}{x+1} = \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x^2}\right)\frac{\ln(x+1)}{x+1}$$

$$\to 0\cdot 0;$$

dle VoAL a základní limity

$$\frac{\ln y}{y}\to 0;\ y\to+\infty.$$

B1/ $\lim\limits_{x \to -\infty} \left(1+\sqrt{x^2+1}-\sqrt{x^2+2}\right)^{e^{1/x}-1}$

$$f(t) = e^{h(t)} \; ; \; h(t) = \frac{1}{e^{1/x}-1} \cdot \ln\left(1+\sqrt{x^2+1}-\sqrt{x^2+2}\right).$$

úprava odmocniny: $\sqrt{x^2+1}-\sqrt{x^2+2} = \dfrac{-1}{\sqrt{x^2+1}+\sqrt{x^2+2}} \underset{x\to-\infty}{\longrightarrow} 0,$

tedy píšeme:

$$h(t) = \frac{\ln\left(1+\sqrt{x^2+1}-\sqrt{x^2+2}\right)}{\sqrt{x^2+1}-\sqrt{x^2+2}} \cdot \frac{1/x}{e^{1/x}-1} \cdot \frac{-x}{\sqrt{x^2+1}+\sqrt{x^2+2}}$$

$$= P_1 \cdot P_2 \cdot P_3$$

$$\left.\begin{array}{l} \dfrac{\ln(1+y)}{y} \to 1 \; ; \; y \to 0 \\ \sqrt{x^2+1}-\sqrt{x^2+2} \to 0 \; ; x \to +\infty \\ \quad \text{a zjevně} \neq 0 \text{ pro } x \in P(-\infty) \end{array}\right\} \overset{\text{VoLSF(b)}}{\Longrightarrow} P_1 \to 1$$

$$\left.\begin{array}{l} \dfrac{y}{e^y-1} = \dfrac{1}{\frac{e^y-1}{y}} \to 1 \; ; \; y \to 0 \\ \qquad\qquad (\text{základní limita}) \\ \dfrac{1}{x} \to 0 \; ; \; x \to -\infty \\ \quad \neq 0 \; ; \; x \in P(-\infty) \end{array}\right\} \overset{\text{VoLSF(b)}}{\Longrightarrow} P_2 \to 1$$

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zbývá vyšetřit P_3.

víme, že $\sqrt{x^2+1} = \sqrt{x^2\left(1+\frac{1}{x^2}\right)} = |x|\cdot\sqrt{1+\frac{1}{x^2}} = -x\sqrt{1+\frac{1}{x^2}}$,

$x \in P(-\infty, \delta)$

$$P_3 = \frac{-x}{-x\left(\sqrt{1+\frac{1}{x^2}}+\sqrt{1+\frac{2}{x^2}}\right)} = \frac{1}{\sqrt{1+\frac{1}{x^2}}+\sqrt{1+\frac{2}{x^2}}} \to \frac{1}{2};$$

neboť $\frac{1}{x^2} \to 0$ dle VoAL

$\sqrt{y}$ spojitá v $y_0 = 1$.

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celkem tedy $h(x) = P_1 \cdot P_2 \cdot P_3 \to 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$

$$f(x) \to e^{1/2} = \sqrt{e}\,;\ x \to -\infty.$$

B2] $$\lim_{x\to+\infty} \frac{\ln(x+1) + \sqrt[3]{x} + \ln\left(1+e^{x/6}\right)}{2+\cos\sqrt{x}+\sqrt{x}} ;$$

úprava logaritmů :

$$\ln(x+1) = \ln x\left(1+\frac{1}{x}\right) = \ln x + \ln\left(1+\frac{1}{x}\right)$$

$$\ln\left(1+e^{x/6}\right) = \ln e^{x/6}\left(1+e^{-x/6}\right) = \frac{1}{6}x + \ln\left(1+e^{-x/6}\right)$$

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vytknutí vedoucího členu : nahoře x, dole $\sqrt{x}$:

$$f(x) = \sqrt{x}\cdot R(x) \to +\infty \cdot \frac{1}{6} = +\infty ;$$

neboť $$R(x) = \frac{\frac{1}{x}\ln x + \frac{1}{x}\ln\left(1+\frac{1}{x}\right) + \frac{\sqrt[3]{x}}{x} + \frac{1}{6} + \frac{1}{x}\ln\left(1+e^{-x/6}\right)}{\frac{2}{\sqrt{x}} + \frac{\cos\sqrt{x}}{\sqrt{x}} + 1} ;$$

stačí ukázat, že vše kromě $1/6$ a 1 jde do 0.

$\frac{1}{x}\ln x \to 0$; $x\to+\infty$ (základní limita)

$\frac{1}{x}\ln\left(1+\frac{1}{x}\right) \to 0\cdot\ln 1 = 0$; dle VoAL a spojitosti $\ln y$ v bodě $y_0 = 1$

$\frac{1}{x}\ln\left(1+e^{-x/6}\right) \to 0$ analogicky

$\frac{2}{\sqrt{x}} \to \frac{2}{+\infty} = 0$ dle VoAL

$\frac{\cos\sqrt{x}}{\sqrt{x}} \to 0$ dle V. 2.8 ; neboť $\frac{1}{\sqrt{x}} \to 0$, $\cos\sqrt{x}$ je omezená.