5. cvičení

(1) Řešte vlnovou rovnici

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad x\in(0,\tfrac{1}{2}) \qquad t\in\mathbb{R}^+$$

$$u(0,x) = 0$$

$$\frac{\partial u}{\partial t}(0,x) = \delta_b \qquad b\in(0,\tfrac{1}{2})$$

a) $u(t,0) = u(t,\tfrac{1}{2}) = 0$ b) $\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,\tfrac{1}{2}) = 0$

Řešení

a) Je třeba provést liché prodloužení poč. podm.

$$\frac{\partial u^s}{\partial t}(0,x) = g^L(x) = \delta_b - \delta_{-b}$$

Podle předch. lemma a 3 FT máme

$$u(t,x) \overset{FT}{=} \sum_{n\in\mathbb{Z}} \mathcal{F}(g^L)(n)\, \mathcal{F}(\mathcal{E})(t,n)\, e^{2\pi i n x}$$

$$\mathcal{F}(\mathcal{E})(t,n) = \begin{cases} \frac{\sin(2\pi a n t)}{2\pi a n} & n\neq 0 \\ t & n=0 \end{cases}$$

$$\langle \mathcal{F}(\delta_b), \varphi\rangle = \langle \delta_b, \mathcal{F}(\varphi)\rangle = \langle \delta_b, \int_{\mathbb{R}} e^{-2\pi i s x}\varphi(x)\,dx\rangle = \langle e^{-2\pi i b x}, \varphi\rangle$$

$$\mathcal{F}(g^L)(n) = (\mathcal{F}(\delta_b) - \mathcal{F}(\delta_{-b}))(n) = (e^{-2\pi i b x} - e^{2\pi i b x})(n) = (-2i\sin(2\pi b x))(n) = -2i\sin(2\pi b n)$$

$$u(t,x) = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} -2i\sin(2\pi b n)\frac{\sin(2\pi a n t)}{2\pi a n} e^{2\pi i n x} + 0 = \boxed{-2i\sum_{n=1}^{\infty}\frac{\sin(2\pi b n)\sin(2\pi a n t)\sin(2\pi n x)}{\pi a n}}$$

b) změna g nula/nadav

$$g^s(x) = \delta_b + \delta_{-b}$$

$$\mathcal{F}(g^s)(m) = (\mathcal{F}(\delta_b) + \mathcal{F}(\delta_{-b}))(m) = (e^{-2\pi i b x} + e^{2\pi i b x})(m) = 2\cos(2\pi b m)$$

Tuy

$$u(t,x) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} 2\cos(2\pi b m)\, \frac{\sin(2\pi m t)}{2\pi m}\, e^{2\pi i m x} + 2\cdot 1\cdot t\cdot 1$$

$$= \sum_{m=1}^{\infty} \frac{2\cos(2\pi b m)\sin(2\pi m t)\cos(2\pi m x)}{\pi m} + 2t$$

(2) $\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \qquad x \in B_{\frac{1}{2}}(0) \subset \mathbb{R}^3 \qquad t \in \mathbb{R}^+$

$$u(0,x) = |x|$$

$$\frac{\partial u}{\partial t}(0,x) = 1$$

$$u(t,x) = 0 \qquad |x| = \frac{1}{2}$$

Hledejme řešení $u(t,x) = v(t,|x|) = v(t,r)$

$$\frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial r^2} - \frac{2}{r}\frac{\partial v}{\partial r} = 0 \qquad w(t,r) = r\,v(t,r)$$

$$\frac{\partial^2 w}{\partial r^2} = r\frac{\partial^2 v}{\partial r^2} + 2\frac{\partial v}{\partial r}$$

$$\frac{1}{a^2}\frac{\partial^2 w_1}{\partial t^2} - \frac{\partial^2 w_1}{\partial r^2} = 0$$

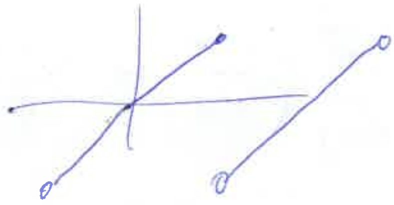
$$w_1(0,x) = 0$$

$$\frac{\partial w_1(0,x)}{\partial t} = r$$

$w_1(t,0) = w_1(t,\frac{1}{2}) = 0$ (nebot' už v nuli bude platit)

Poč. podmínky: liché

$u_0^{l}(r) = r$ na $[-\frac{1}{2}, \frac{1}{2}]$



$$u_1(t,x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathcal{F}(u_0^{l})(n)\, \frac{\sin(2\pi n t)}{2\pi n}\, e^{2\pi i n x} + \underset{\substack{\| \\ 0}}{\mathcal{F}(u_0^{l})(0)}\, t \cdot 1$$

$$\mathcal{F}(u_0^{l})(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n x}\, x\, dx \quad -2i \int_0^{\frac{1}{2}} \sin(2\pi n x)\, x\, dx = -2i \left\{ \left[-\frac{\cos(2\pi n x)}{2\pi n} x \right]_0^{\frac{1}{2}} + \underbrace{\int_0^{\frac{1}{2}} \frac{\cos(2\pi n x)}{2\pi n}\, dx}_{=0} \right\}$$

$$= i\, \frac{\cos(\pi n)}{2\pi n} = i\, \boxed{\frac{(-1)^n}{2\pi n}}$$

$$\boxed{u_1(t,x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n}\, \frac{\sin(2\pi n t)\, \sin(2\pi n x)}{2\pi n}}$$

$$\boxed{u_1(t,x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n}\, \frac{\sin(2\pi n t)}{2\pi n}\, \frac{\sin(2\pi n |x|)}{|x|}}$$

Derivování je než

$$w_2(t,r) = \frac{\partial}{\partial t}\left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathcal{F}(u_1^{l})(n)\, \frac{\sin(2\pi n t)}{2\pi n}\, e^{2\pi i n r} + \overbrace{\underbrace{\mathcal{F}(u_1^{l})(0)}_{=0} \cdot t \cdot 1}^{=0} \right)$$

$$w_1^{l}(r) = \begin{cases} r^2 & r > 0 \\ -r^2 & r < 0 \end{cases}$$

$$\mathcal{F}(w_1^{l})(r) = -2i \int_0^{\frac{1}{2}} \sin(2\pi n x)\, x^2\, dx = -2i \left[\frac{-\cos(2\pi n x) \cdot x^2}{2\pi n} \right]_0^{\frac{1}{2}} + 2i \int_0^{\frac{1}{2}} \frac{\cos(2\pi n x)}{2\pi n} \cdot 2x\, dx$$

$$= \frac{i(-1)^n}{4\pi n} - 4i \underbrace{\left[\frac{\sin 2\pi n x}{(2\pi n)^2} \cdot 2x \right]_0^{\frac{1}{2}}}_{=0} + i \int_0^{\frac{1}{2}} \frac{\sin(2\pi n x)}{(\pi n)^2}\, dx = \frac{i(-1)^n}{4\pi n} + i\, \frac{[-\cos 2\pi n x]_0^{\frac{1}{2}}}{2(\pi n)^3}$$

$$= i \left(\frac{(-1)^n}{4\pi n} + \frac{1-(-1)^n}{2(\pi n)^3} \right)$$

Tedy $w_2(t,r) = \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{2\pi n} + \frac{(-1)^n-1}{(\pi n)^3}\right)\frac{\sin(2\pi n t)}{2\pi n}\sin(2\pi n r)\right)$

$$= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{2\pi n} + \frac{(-1)^n-1}{(\pi n)^3}\right)\cos(2\pi n t)\sin(2\pi n r)$$

$$\underline{u_2(t,x) = \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{2\pi n} + \frac{(-1)^n-1}{(\pi n)^3}\right)\cos(2\pi n t)\frac{\sin(2\pi n|x|)}{|x|}}$$

$$\underline{u(t,x) = u_1(t,x) + u_2(t,x)}.$$

3. $\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$ $\quad x\in\mathbb{R}^3 \quad t\in\mathbb{R}^+$

$u(0,x) = e^{-\alpha|x|^2}$ $\quad \alpha > 0$

$\frac{\partial u}{\partial t}(0,x) = 0$

~~$u(t,x) = \frac{\partial}{\partial t}\left(u_0 * e\right)(t,x) = \frac{\partial}{\partial t}\Big($~~

Převede se opět na úlohu na polopřímce:

$\frac{1}{a^2}\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial r^2} = 0$ $\quad r\in(0,\infty)$

$w(0,r) = r e^{-\alpha r^2}$ $\to$ liché prodloužení na $\mathbb{R}$

$\frac{\partial w}{\partial t}(0,r) = 0$

Tedy

$w(t,r) = \frac{1}{2}\left((r+at)e^{-\alpha(r+at)^2} + (r-at)e^{-\alpha(r-at)^2}\right)$

Tedy $u(t,x) = \frac{1}{2|x|}\left((|x|+at)e^{-\alpha(|x|+at)^2} + (|x|-at)e^{-\alpha(|x|-at)^2}\right)$

Všimněme si, že $\lim_{|x|\to 0} u(t,x) = e^{-\alpha a^2 t^2}(1-2\alpha a^2 t^2)$ (je konečné) – libovolně

Fundamentální řešení jiných rovnic

(1) $\Delta^2 u = \Delta(\Delta u) = \delta$ v $\mathbb{R}^N$ $N=2,3$

Hledejme řešení $-\Delta u = \mathcal{E}$ (radiálně symetrické)

$$\boxed{v'' + \frac{N-1}{r} v' = -\mathcal{E}}$$

a) $N=2$

$$r^2 v'' + r v' = \frac{1}{2\pi} r^2 \ln r$$

Pro homogenní rovnici: $v_H(r) = C_1 \ln r + C_2$ — ale $C_1 \equiv 0$ ($\Delta \ln r = \delta$, ne 0!)

partikulární řešení $v_p(r) = r^2(A \ln r + B)$

$$r^2 v_p'' + r v_p' = 2r^2(A \ln r + B) + 4Ar^2 - Ar^2 + 2r^2(A\ln r + B) + Ar^2$$

$$= 4Ar^2 \ln r + 4r^2(A+B)$$

$$v_p(r) = \frac{1}{8\pi} r^2(\ln r - 1)$$

Protože všechny polynomy nulu řeší $\Delta^2 P = 0$ a speciálně $\Delta^2 |x|^2 = 0$,

ji $\boxed{u_2(x) = \frac{1}{8\pi} |x|^2 \ln|x| + P(x)}$ P .. polynom v x_1 a x_2 splňující $\Delta^2 P(x) = 0$.

b) $N=3$

$$r^2 v'' + 2r v' = -\frac{r}{4\pi}$$

$$(rv)'' = -\frac{1}{4\pi}$$

$$v(r) = \frac{-r}{8\pi} + C + \frac{D}{r}$$ $D=0$ .. 3D

$\boxed{u(x) = \frac{-|x|}{8\pi} + P(x)}$ $P(x)$: polynom $\Delta^2 P(x) = 0$ v x_1, x_2, x_3.

(36)

② $(-\Delta - k^2)u = \delta \qquad x \in \mathbb{R}^3 \quad k \in \mathbb{R}^+$

Dodat do zápisu větu o Δ

Hledejme opět radiálně symetrické řešení. Řešíme

$$-(v''(r) + \frac{2}{r} v'(r) + k^2 v) = 0 \qquad \underline{rw = rv}$$

$$\underbrace{r v''(r) + 2v'(r)}_{w''(r)} + \underbrace{k^2 r v}_{+k^2 w} = 0$$

Obecné řešení má tvar $w(r) = A\cos(kr) + B\sin(kx)$

Proto

$$\underline{v(r) = \frac{A}{r}\cos(kr) + \frac{B}{r}\sin(kr)}$$

Ukážeme, že pokud $v = v(r)$ v $\mathbb{R}^N$ je hladká funkce mimo počátek, $\Delta v = 0$ mimo počátek,

pak $\Delta T_v = T_{\Delta v} + A\varkappa_N \delta$, $\lim_{r\to 0} v(r) r^{N-1} = 0$, $\int_{\mathbb{R}^N} r^{N-2}|v(r)|\,dr < \infty$

kde $A = \lim_{r\to 0+} r^{N-1} v'(r)$.

Počítejme pro $\varphi = \varphi(r)$ (obecně analogicky)

$$\langle \Delta T_v, \varphi\rangle = \langle T_v, \Delta\varphi\rangle = \langle T_v, \frac{d^2\varphi}{dr^2} + (N-1)\frac{d\varphi}{dr}\frac{1}{r}\rangle$$

$$= \varkappa_N \int_0^\infty \left(v\varphi''(r) + (N-1)v\frac{\varphi'(r)}{r}\right) r^{N-1}\,dr$$

$$= -\varkappa_N \int_0^\infty (v'\varphi'(r) r^{N-1})\,dr + \varkappa_N \int_0^\infty \underbrace{(v\varphi' r^{N-2}(N-1) - v\varphi'(N-1)r^{N-2})}_{=0}\,dr$$

$$= -\varkappa_N \lim_{\varepsilon\to 0}\left(\int_\varepsilon^\infty \left(-v''\varphi - (N-1)\varphi\frac{v'}{r}\right) r^{N-1}\,dr - v'(\varepsilon)\varepsilon^{N-1}\varphi(\varepsilon)\right)$$

$\Delta T_v = T_{\Delta v} + A\varkappa_N\delta$ pro $T_{\Delta v} \in \dots$

(pozn. $v'' + \frac{N-1}{r}v' \in C^1_{loc}(\mathbb{R}^N)$)

$$= + \underbrace{\int_{\mathbb{R}^3} \Delta v\,\varphi\,dx}_{0} + \lim_{r\to 0+}(v'(r) r^{N-1}\varkappa_N)\,\varphi(0).$$

Tedy $\Delta T_v = A\varkappa_N\delta$, kde $A = \lim_{r\to 0} r^{N-1} \dots$

Teď vrátíme k našemu příkladu 7

$v(r) = \frac{A}{r}\cos(kr) + \frac{B}{r}\sin(kr)$ splňuje naši původní rovnici ($u(x) = v(|x|)$)

$(-\Delta - k^2)u = 0$

Hledáme teď takové koeficienty, aby

$\lim_{r\to 0+} r^2 \underbrace{\frac{d}{dr}\left(A\frac{\cos kr}{r} + B\frac{\sin kr}{r}\right)}$ jde úloha splněna, pro nalezení

$-A\frac{k\sin(kr)}{r} - A\frac{\cos kr}{r^2} + \frac{B\cos(kr)\cdot k}{r} - B\frac{\sin(kr)}{r^2}$

$\boxed{\lim_{r\to 0+} r^2 \left(\quad\right) = -A}$

Protože $\varkappa_3 = 4\pi$, dostáváme, že $A = +\frac{1}{4\pi}$, B libovolné
$((-\Delta - k^2)u)$

Tedy

$\underline{v(r) = \frac{1}{4\pi r}\cos(kr) + \frac{B}{r}\sin kr}$ $B\in\mathbb{R}$

$\underline{u(x) = \frac{1}{4\pi|x|}\cos(k|x|) + \frac{B}{|x|}\sin(k|x|)}$ $B\in\mathbb{R}$

Protože $\Delta\left(\frac{1}{4\pi|x|}\right) = 0$ mimo počátek (bodově) je pak na nulovou rovnici všech členů splněna!

③ $(-\Delta + k^2)u = \delta$ $x\in\mathbb{R}^3$ $k\in\mathbb{R}^+$

Zde jsme došli metodou založenou na Fourierově transformaci

$(-\Delta + k^2)u = \delta$

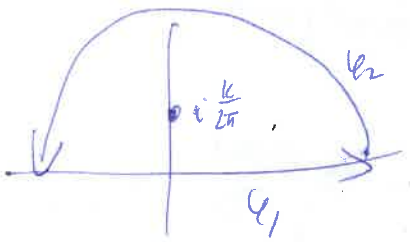
$\mathcal{F}(u)\left(|\xi|^2 \cdot 4\pi^2 + k^2\right) = 1$

$\underline{\mathcal{F}(u)(\xi) = \frac{1}{|\xi|^2 4\pi^2 + k^2}}$

Výpočet zpětné FT lze pomocí vzorce FT radiálně symetrické fce + reziduová věta

$$u(x) = \frac{2}{|x|}\int_0^\infty \frac{\rho \sin(2\pi\rho|x|)}{k^2+4\pi^2\rho^2}\,d\rho = \frac{1}{|x|}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{\rho\, e^{2\pi i\rho|x|}}{k^2+4\pi^2\rho^2}\,d\rho$$

v hlavním smyslu



$$\Rightarrow \quad u(x) = \frac{1}{|x|}\,\mathrm{Im}\left(2\pi i\,\mathrm{Res}_{i\frac{k}{2\pi}}\frac{z\,e^{2\pi i z|x|}}{k^2+4\pi^2 z^2}\right)$$

$$= \frac{2\pi}{|x|}\,\mathrm{Im}\left(\frac{i}{8\pi^2}\,e^{-k|x|}\right) =$$

$$= \boxed{\frac{1}{4\pi|x|}\,e^{-k|x|}}$$

(4) $(\Delta^2 - k^2\Delta + k^4)u = \delta \qquad x\in\mathbb{R}^3,\ k\in\mathbb{R}^+$

Opět pomocí FT

$$\mathcal{F}(u)(\xi)\left(|\xi|^4(2\pi)^4 + k^2(2\pi)^2|\xi|^2 + k^4\right) = 1$$

Tedy
$$u(x) = \mathcal{F}^{-1}\left(\frac{1}{16\pi^4|\xi|^4 + 4\pi^2k^2|\xi|^2 + k^4}\right)(x)$$

$$= \frac{2}{r}\int_0^\infty \frac{\rho\sin(2\pi r\rho)}{16\pi^4\rho^4 + k^2\cdot 4\pi^2\rho^2 + k^4}\,d\rho$$

$$= \frac{1}{r}\left(\mathrm{Im}\int_{-\infty}^{\infty}\frac{\rho\, e^{2\pi r\rho i}}{16\pi^4\rho^4 + k^2 4\pi^2\rho^2 + k^4}\,d\rho\right)$$

$$\boxed{u(x) = \frac{1}{r}\, \mathrm{Im}\left(2\pi i \sum_{\mathrm{Im}\, z_j > 0} \mathrm{Res}_{z_j} \frac{z\, e^{i 2\pi r z}}{16\pi^4 z^4 + k^2 4\pi^2 z^2 + k^4} \right)}$$

Lze spočítat, že kořeny jmenovatele jsou

$$z_1 = \frac{k}{2\pi} e^{i\frac{\pi}{3}} = \frac{1}{2}(1 + i\sqrt{3})\frac{k}{2\pi} \qquad z_2 = \frac{1}{2}(-1 + i\sqrt{3})\frac{k}{2\pi}$$

$$z_3 = \bar{z}_1, \qquad z_4 = \bar{z}_2$$

Proto $(\mathrm{Res}_{z_1} + \mathrm{Res}_{z_2})(\,) = \dfrac{e^{-kr\frac{\sqrt{3}}{2}}}{\sqrt{3}} \dfrac{1}{(2\pi k)^2} \sin\left(\dfrac{kr}{2}\right)$

$$\boxed{u(x) = \frac{e^{-k|x|\frac{\sqrt{3}}{3}} \sin\frac{k|x|}{2}}{2\pi k^2 |x| \sqrt{3}}}$$