(1) Wyznaczyć wszystkie ekstremale funkcjonału

6b

$$\Phi(y) = \int_{e}^{e^2} \left(y + \frac{(y')^2}{2} \frac{1}{\ln^2 x} \right) dx$$

na zbiorze $V = \{ u \in C^1([e,e^2]) \quad u(e) = 2e + \frac{e^2}{4} \quad u(e^2) = 2e^2 + \frac{5}{4}e^4 + e \}$

Rozw.

$$\frac{\partial}{\partial y} L - \frac{d}{dx} \frac{\partial}{\partial y'} (L) = 0 \qquad 0,5b$$

$$1 - \frac{d}{dx}\left(y' \frac{1}{\ln^2 x} \right) = 0$$

$$\left(y' \frac{1}{\ln^2 x} \right)' = 1$$

$$y' \frac{1}{\ln^2 x} = x + C$$

$$y' = x \ln^2 x + C \ln^2 x \qquad 1b$$

Integrujemy przez części

$$y = \int (x \ln^2 x + C \ln^2 x) dx = \frac{x^2}{2} \ln^2 x + Cx \ln^2 x - \int \frac{x^2}{2} \cdot 2 \ln x \frac{1}{x} dx - \int x \cdot 2C \frac{\ln x}{x} dx$$

$$= \frac{x^2}{2} \ln^2 x + Cx \ln^2 x - \int x \ln x \, dx - 2C \int \ln x \, dx$$

$$= \frac{x^2}{2} \ln^2 x + Cx \ln^2 x - \frac{x^2}{2} \ln x + \int \frac{x}{2} dx - 2Cx \ln x + 2Cx$$

$$= \frac{x^2}{2} \ln^2 x + Cx \ln^2 x - \frac{x^2}{2} \ln x + \frac{x^2}{4} - 2Cx \ln x + 2Cx + D \qquad 2b$$

$$u(e) = 2e + \frac{e^2}{4} \Rightarrow \cancel{\frac{e^2}{2}} + Ce - \cancel{\frac{e^2}{2}} + \cancel{\frac{e^2}{4}} - \cancel{2Ce} + 2Ce + D = 2e + \cancel{\frac{e^2}{4}} \qquad 1b$$

$$u(e^2) = 2e^2 + \frac{5e^4}{4} + e \Rightarrow \frac{e^4}{2} \cdot 4 + \cancel{4Ce^2} - e^4 + \frac{e^4}{4} - \cancel{4Ce^2} + 2Ce^2 + D = \frac{5}{4}e^4 + 2e^2 + e$$

$$Ce + D = 2e$$

$$2Ce^2 + D = 2e^2 + e$$

$$C(2e^2 - e) = 2e^2 - e = \cancel{e(2e-1)} \Rightarrow \boxed{C = 1}, \Rightarrow D = e \qquad 1b$$

$$\boxed{y(x) = \frac{x^2}{2} \ln^2 x + x \ln^2 x - \frac{x^2}{2} \ln x + \frac{x^2}{4} - 2x \ln x + 2x + e}, \quad x \in [e, e^2] \qquad 0,3b$$

(2) Vyšetřete obor konvergence (stejnoměrné konv., bodové konv., bodovou konvergenci) posloupnosti funkcí

1,5

$$f_n(x) = (1+x^{2n})^{\frac{2}{n}}$$

Řešení

Tvrzení pro $0 \le x \le 1$ je $(1+x^{2n})^{\frac{2}{n}} \le 2^{\frac{2}{n}} \to 1$

$(1+x^{2n})^{\frac{2}{n}} \ge 1^{\frac{2}{n}} \to 1$ 0,375

Tedy $f_n(x) \to 1$ pro $x \in [0,1]$

Pro $x>1$ je $(1+x^{2n})^{\frac{2}{n}} \to x^4$ 0,375

$\Big((1+x^{2n})^{\frac{2}{n}} = x^4\left(\frac{1}{x^{2n}}+1\right)^{\frac{2}{n}} \le x^4 2^{\frac{2}{n}}$, $\ge (x^{2n})^{\frac{2}{n}} = x^4 \Big)$

Studujme stejnoměrnou konvergenci

a) $x \le 1$: $(1+x^{2n})^{\frac{2}{n}} - 1 \le 2^{\frac{2}{n}} - 1 \xrightarrow[n\to\infty]{} 0$

Tedy $(1+x^{2n})^{\frac{2}{n}} \rightrightarrows 1$ na $[0,1]$, neboť $\sup_{[0,1]} f_n(x) - 1 = f_n(1) - 1$ 1,5

b) $x > 1$

$0 \le (1+x^{2n})^{\frac{2}{n}} - x^4$

$\Rightarrow \left((1+x^{2n})^{\frac{2}{n}} - x^4\right)' = \frac{2}{n}\cdot(1+x^{2n})^{\frac{2}{n}-1}\cdot 2n x^{2n-1} - 4x^3$ 0,66

Tvrdím, že $PS < 0$

$4\cdot x^{2n-1}\cdot(1+x^{2n})^{\frac{2}{n}-1} < 4x^3$

$(1+x^{2n})(1+x^{2n})^{\frac{2}{n}-1} < x^{2n}(1+x^{2n})^{\frac{2}{n}-1} < x^4$

$(1+x^{2n}) < x^{4\cdot\frac{n}{2}}$ O.K. 1,5

Tedy f_n je klesající, proto $\sup_{(1,\infty)} (1+x^{2n})^{\frac{2}{n}} - x^4 = 2^{\frac{2}{n}} - 1 \xrightarrow[n\to\infty]{} 0$. 0,33

Závěr:

$(1+x^{2n})^{\frac{2}{n}} \underset{[0,\infty)}{\rightrightarrows} f(x)$, kde $f(x) = \begin{cases} 1 & x \le 1 \\ x^4 & x > 1 \end{cases}$ 0,5

③ Spodle [illegible]

(103) $I = \int_0^{\infty} \frac{1-\cos x}{x^2}$

Návod Nejprve dokažte, že integrál konverguje. Za použití Lebesgueovy věty definujte

$$\varphi(a) := \int_0^{\infty} e^{-ax} \frac{1-\cos x}{x^2}\,dx$$

a spočtěte $\varphi''(a)$. Potom integrováním a s tím, že $\lim_{a\to+\infty} \varphi'(a) = \lim_{a\to+\infty} \varphi(a) = 0$, nalezněte $\varphi(a)$ pro $a>0$. Nakonec spočtěte $\lim_{a\to 0^+} \varphi(a) = I$. Ověřte předpoklady všech vět, které použijete (důvod, možná [illegible] [illegible])

Řešení

- Zřejmě $\frac{1-\cos x}{x^2} \sim \frac{1}{2}$ pro $x \to 0^+$, $\left|\frac{1-\cos x}{x^2}\right| \leq \frac{C}{x^2}$ pro $x \to +\infty$ $\Rightarrow$ integrál konverguje 1b

- $\varphi(a) = \int_0^{\infty} e^{-ax} \frac{1-\cos x}{x^2}\,dx$

$\varphi'(a) = -\int_0^{\infty} e^{-ax} \frac{1-\cos x}{x}\,dx$ 0,5b

- Zřejmě $\varphi(a)$ je spojitá na $[0,\infty)$ (majoranta $\frac{1-\cos x}{x^2}$); obdobně vlastně 1,5+
- zřejmě $\lim_{a\to+\infty} \varphi(a) = 0$ (Lebesgueova dom. konv. $e^{-ax}\frac{1-\cos x}{x^2} \xrightarrow{a\to+\infty} 0$, majoranta $\frac{1-\cos x}{x^2}$)
- (majoranta je $e^{-a_0 x}\frac{1-\cos x}{x}$ pro $0<a_0\leq a$) obdobně platí
- $\lim_{a\to+\infty} \varphi'(a) = 0$ (obdobně jako u φ, majoranta je $e^{-a_0 x}\frac{1-\cos x}{x}$)

$\varphi''(a) = \int_0^{\infty} e^{-ax}(1-\cos x)\,dx$

- (majoranta je $e^{-a_0 x}(1-\cos x)$ pro $0<a_0\leq a$) 2b

$$\varphi''(a) = \int_0^{\infty} e^{-ax}\,dx - \mathrm{Re}\int_0^{\infty} e^{x(-a+i)}\,dx = \frac{1}{a} - \mathrm{Re}\left[\frac{1}{-a+i} e^{x(-a+i)}\right]_0^{\infty}$$

$$= \frac{1}{a} + \mathrm{Re}\left[\frac{1}{-a+i}\right] = \frac{1}{a} + \mathrm{Re}\,\frac{-a-i}{a^2+1} = \frac{1}{a} - \frac{a}{a^2+1}$$ 1b

$$\varphi'(a) = \ln a - \frac{1}{2}\ln(a^2+1) + C = \ln\frac{a}{\sqrt{a^2+1}} + C$$

$$0 = \lim_{a\to+\infty} \varphi'(a) = C \Rightarrow C = 0$$

$$\varphi'(a) = \ln a - \frac{1}{2}\ln(a^2+1), \quad a>0$$ 1b

$$\varphi(a) = \int \left(\ln a - \tfrac{1}{2}\ln(a^2+1)\right) da = \int 1 \cdot \ln a \, da - \int 1 \cdot \tfrac{1}{2}\ln(a^2+1)\, da$$

$$= a\ln a - a - \tfrac{1}{2}a\ln(a^2+1) + \int \frac{a^2}{a^2+1}\, da$$

$$= a\ln a - \cancel{a} - \tfrac{1}{2}a\ln(a^2+1) + \cancel{a} - \operatorname{arctg}(a) + C$$

1b

Zřejmě $\lim_{a\to 0^+} \varphi(a) = C$ a potřebujeme určit C. To určíme z toho, že $\lim_{a\to+\infty} \varphi(a) = 0$

$$0 = \lim_{a\to+\infty} \left(a\ln a - \tfrac{1}{2}a\ln(a^2+1) - \operatorname{arctg} a + C\right) =$$

$$\lim_{a\to+\infty} a \ln\left(\frac{a}{\sqrt{a^2+1}}\right) = \lim_{a\to\infty} a \ln\left(1 - \frac{1}{2a^2}\right) = \lim_{a\to+\infty} -\frac{1}{2a} = 0$$

$$\frac{a}{\sqrt{a^2+1}} = \frac{a}{a\left(1+\frac{1}{a^2}\right)^{1/2}} \approx 1 - \frac{1}{2a^2}$$

1b

$$\lim_{a\to+\infty} \operatorname{arctg} a = \frac{\pi}{2}$$

Tedy $0 = \lim_{a\to\infty} \varphi(a) = C - \frac{\pi}{2} \Rightarrow C = \frac{\pi}{2}$

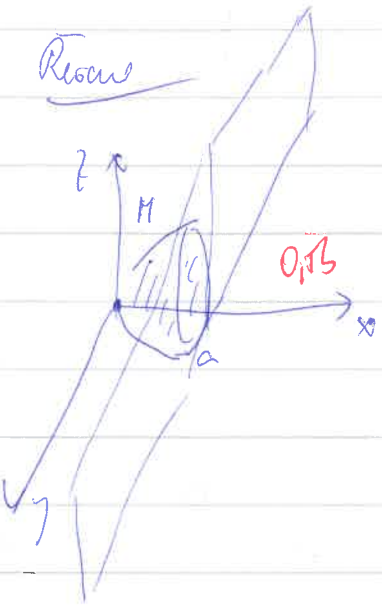
$$\varphi(a) = a\ln a - \tfrac{1}{2}a\ln(a^2+1) - \operatorname{arctg} a + \frac{\pi}{2}$$

Proto $I = \int_0^\infty \frac{1-\cos x}{x^2}\, dx = \lim_{a\to 0^+} \varphi(a) = \frac{\pi}{2}$

1b

(4) Spočítejte tok pole $\vec{v} = (x^2, 4y, 2z)$ ven z tělesa
(6b) omezeného paraboloidem $\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2x}{a}$ $(a, b, c > 0)$ a rovinou
$x = a$.

Řešení



0,5b

a) Přes Gaussovu větu

$$\int_\Omega \operatorname{div} \vec{v} \, dx = \int_{\partial\Omega} \vec{v} \cdot \vec{n} \, dS$$ 1b

$$\operatorname{div} \vec{v} = 2x + 4$$ 0,5b

$y = b r \cos\varphi$ $\quad \varphi \in (0, 2\pi)$
$z = c r \sin\varphi$ $\quad r \in (0, \sqrt{2})$
$0 \leq x \leq \frac{a}{2} r^2$
zob. Válcové souřadnice $\quad J = bcr$ 1b

$$\int_{\Omega} \operatorname{div} \vec{v} \, dx$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{\frac{a}{2}r^2}^{a} bcr \, (2x+4) \, dx \, dr \, d\varphi$$ 0,5b

$$= 2\pi bc \int_0^{\sqrt{2}} r \left(\int_{\frac{a}{2}r^2}^{a} (2x+4) \, dx \right) dr = 2\pi bc \int_0^{\sqrt{2}} r \, [x^2 + 4x]_{\frac{a}{2}r^2}^{a} \, dr$$ 1b

$$\cancel{= 2\pi bc \int_0^{\sqrt{2}} r \left(\frac{a^2}{4} r^4 + 2ar^2 \right) dr = \frac{\pi}{2} a^2 bc \left(a \frac{1}{6} \cdot 8 + 2 \cdot 4 \right)}$$ 1b

$$\cancel{= \frac{\pi}{2} abc \left(\frac{4}{3} a + 8 \right) = 2\pi abc \left(\frac{a}{3} + 2 \right)}$$ 0,5b

$$= 2\pi bc \int_0^{\sqrt{2}} r \left(a^2 + 4a - \frac{a^2}{4} r^4 - 2ar^2 \right) dr = 2\pi a^2 bc \left[\frac{r^2}{2} - \frac{r^6}{24} \right]_0^{\sqrt{2}}$$

$$+ 2\pi abc \left[4 \frac{r^2}{2} - 2 \frac{r^4}{4} \right]_0^{\sqrt{2}} = 2\pi a^2 bc \left(1 - \frac{1}{3} \right) + 2\pi abc (4 - 2)$$ 1b

$$= \underline{4\pi abc \left(\frac{a}{3} + 1 \right)}$$ 0,5b

Jiný up[illegible]

Tok podstavou:

$\vec{n} = (1,0,0)$ $y = cr\cos\varphi$ $z = rb\sin\varphi$ $J = bcr$ 1b

$\vec{v} = (x^2, 2y, 2z)$

$x = a$ $T_1 = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} a^2bc \cdot r\, dr = 2\pi a^2bc$

tok pláštěm

$x = \frac{a}{2}r^2$

$y = br\cos\varphi$

$z = cr\sin\varphi$

$\vec{n}$ ... [illegible]

1b 1b

$$\begin{pmatrix} ar & 0 & \vec{e}_1 \\ b\cos\varphi & -br\sin\varphi & \vec{e}_2 \\ c\sin\varphi & cr\cos\varphi & \vec{e}_3 \end{pmatrix} = \vec{e}_1 \cdot (bcr) + \vec{e}_2 \cdot (-acr^2\cos\varphi) + \vec{e}_3 \cdot (-abr^2\sin\varphi)$$

$\Rightarrow \vec{n} = (-bcr, acr^2\cos\varphi, abr^2\sin\varphi)$

$\vec{v} = (\frac{a^2}{2^2} \cdot r^4, 2br\cos\varphi, 2cr\sin\varphi)$ 0,5b

$T_2 = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} \left(-\frac{a^2bc}{4} r^5 + 2abcr^3\right) dr$ 0,5b

$= 2\pi \cdot \left(-\frac{a^2bc}{24} \cdot 8 + \frac{2abc}{4} \cdot 4\right) = 4abc\pi - \frac{2}{3}\pi a^2bc$ 0,5b

$\underline{T = T_1 + T_2 = 4\pi abc + \frac{4}{3}\pi a^2bc}$ 0,5b