Mathematical theory of compressible viscous fluids

Eduard Feireisl and Milan Pokorný

Another advantage of a mathematical statement is that it is so definite that it might be definitely wrong. . . Some verbal statements have not this merit.

F.L.Richardson (1881-1953)

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Chapter 1

Introduction

Despite the concerted effort of generations of excellent mathematicians, the fundamental problems in partial differential equations related to continuum fluid mechanics remain largely open. Solvability of the Navier–Stokes system describing the motion of an incompressible viscous fluid is one in the sample of millenium problems proposed by Clay Institute, see [7]. In contrast with these apparent theoretical difficulties, the Navier–Stokes system became a well established model serving as a reliable basis of investigation in continuum fluid mechanics, including the problems involving turbulence phenomena. An alternative approach to problems in fluid mechanics is based on the concept of weak solutions. As a matter of fact, the balance laws, expressed in classical fluid mechanics in the form of partial differential equations, have their origin in integral identities that seem to be much closer to the modern weak formulation of these problems. Leray [13] constructed the weak solutions to the incompressible Navier–Stokes system as early as in 1930, and his "turbulent solutions" are still the only ones available for investigating large data and/or problems on large time intervals. Recently, the real breakthrough is the work of Lions [14] who generalized Leray's theory to the case of barotropic compressible viscous fluids (see also Vaigant and Kazhikhov [19]). The quantities playing a crucial role in the description of density oscillations as the effective viscous flux were identified and used in combination with a renormalized version of the equation of continuity to obtain first large data/large time existence results in the framework of compressible viscous fluids.

The main goal of this lecture series is to present the mathematical theory of *compressible barotropic fluids* in the framework of Lions [14], together with the extensions developed in [8]. After an introductory part we first focus on the crucial question of *stability* of a family of weak solutions that is the core of the abstract theory, with implications to numerical analysis and the associated real world applications. For the sake of clarity of presentation, we discuss first the case, where the pressure term has sufficient growth for large value of the density yielding sufficiently strong energy bounds. We also start with the simplest geometry of the physical space, here represented by a cube, on the boundary of which the fluid satisfies the slip boundary conditions. As is well-known, such a situation may be reduced to studying the purely spatially periodic case, where the additional difficulties connected with the presence of boundary conditions is entirely eliminated. Next part of this lecture series will be devoted to the detailed existence proof with (nowadays) optimal restriction on the pressure function. We will also consider the case of homogeneous Dirichlet boundary conditions.

Chapter 2

Mathematical model

As the main goal of this lecture series is the *mathematical theory*, we avoid a detailed derivation of the mathematical model of a compressible viscous fluid. Remaining on the platform of *continuum fluid mechanics*, we suppose that the motion of a compressible barotropic fluid is described by means of two basic *fields*:

the mass density .... $\varrho = \varrho(t, x)$,
the velocity field .... $\mathbf{u} = \mathbf{u}(t, x)$,

functions of the time $t \in \mathbb{R}$ and the spatial position $x \in \mathbb{R}^3$.

2.1 Mass conservation

Let us recall the classical argument leading to the mathematical formulation of the physical principle of *mass conservation*, see e.g. Chorin and Marsden [3]. Consider a volume $B \subset \mathbb{R}^3$ containing a fluid of density ϱ. The change of the total mass of the fluid contained in B during a time interval $[t_1, t_2]$, $t_1 < t_2$ is given as

$$\int_B \varrho(t_2, x)\ \mathrm{d}x - \int_B \varrho(t_1, x)\ \mathrm{d}x.$$

One of the basic laws of physics incorporated in continuum mechanics as the *principle of mass conservation* asserts that mass is neither created nor destroyed. Accordingly, the change of the fluid mass in B is only because of the mass *flux* through the boundary ∂B, here represented by $\varrho\mathbf{u} \cdot \mathbf{n}$, where

$\mathbf{n}$ denotes the *outer* normal vector to ∂B:

$$\int_B \varrho(t_2, x)\ \mathrm{d}x - \int_B \varrho(t_1, x)\ \mathrm{d}x = -\int_{t_1}^{t_2} \int_{\partial B} \varrho(t, x)\mathbf{u}(t, x) \cdot \mathbf{n}(x)\ \mathrm{dS}_x\ \mathrm{d}t. \tag{2.1}$$

One should remember formula (2.1) since it contains *all* relevant piece of information provided by *physics*. The following discussion is based on *mathematical* arguments based on the (unjustified) hypotheses of *smoothness* of all fields in question. To begin, apply Gauss–Green theorem to rewrite (2.1) in the form:

$$\int_B \varrho(t_2, x)\ \mathrm{d}x - \int_B \varrho(t_1, x)\ \mathrm{d}x = -\int_{t_1}^{t_2} \int_B \mathrm{div}_x \Big(\varrho(t, x)\mathbf{u}(t, x) \Big)\ \mathrm{d}x\ \mathrm{d}t.$$

Furthermore, fixing $t_1 = t$ and performing the limit $t_2 \to t$ we may use the mean value theorem to obtain

$$\int_B \partial_t \varrho(t, x)\ \mathrm{d}x = \lim_{t_2 \to t} \frac{1}{t_2 - t} \Big(\int_B \varrho(t_2, x)\ \mathrm{d}x - \int_B \varrho(t, x)\ \mathrm{d}x \Big) \tag{2.2}$$

$$= -\lim_{t_2 \to t} \frac{1}{t_2 - t} \int_t^{t_2} \int_B \mathrm{div}_x \Big(\varrho(t, x)\mathbf{u}(t, x) \Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$= -\int_B \mathrm{div}_x \Big(\varrho(t, x)\mathbf{u}(t, x) \Big)\ \mathrm{d}x.$$

Finally, as relation (2.2) should hold for *any* volume element B, we may infer that

$$\partial_t \varrho(t, x) + \mathrm{div}_x \Big(\varrho(t, x)\mathbf{u}(t, x) \Big) = 0. \tag{2.3}$$

Relation (2.3) is a first order partial differential equation called *equation of continuity.*

2.2 Balance of momentum

Using arguments similar to the preceding part, we derive *balance of momentum* in the form

$$\partial_t \Big(\varrho(t, x)\mathbf{u}(t, x) \Big) + \mathrm{div}_x \Big(\varrho(t, x)\mathbf{u}(t, x) \otimes \mathbf{u}(t, x) \Big) = \mathrm{div}_x \mathbb{T}(t, x) + \varrho(t, x)\mathbf{f}(t, x), \tag{2.4}$$

or, equivalently (cf. (2.3)),

$$\varrho(t,x)\Big[\partial_t\mathbf{u}(t,x)+\mathbf{u}(t,x)\cdot\nabla_x\mathbf{u}(t,x)\Big]=\mathrm{div}_x\mathbb{T}(t,x)+\varrho(t,x)\mathbf{f}(t,x),$$

where the tensor $\mathbb{T}$ is the Cauchy stress and $\mathbf{f}$ denotes the (specific) external force acting on the fluid.

We adopt the standard *mathematical* definition of fluids in the form of *Stokes' law*

$$\mathbb{T}=\mathbb{S}-p\mathbb{I},$$

where $\mathbb{S}$ is the viscous stress and p is a scalar function termed pressure. In addition, we suppose that the viscous stress is a *linear* function of the velocity gradient, specifically $\mathbb{S}$ obeys *Newton's rheological law*

$$\mathbb{S}=\mathbb{S}(\nabla_x\mathbf{u})=\mu\left(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right)+\eta\mathrm{div}_x\mathbf{u}\mathbb{I}, \tag{2.5}$$

with the shear viscosity coefficient μ and the bulk viscosity coefficient η, here assumed constant, $\mu>0$, $\eta\geq 0$.

In order to close the system, we suppose the fluid is *barotropic*, meaning the pressure p is an explicitly given function of the density $p=p(\varrho)$. Accordingly, for $\lambda=\eta-\frac{2}{3}\mu$

$$\mathrm{div}_x\mathbb{T}=\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla_x\mathrm{div}_x\mathbf{u}-\nabla_x p(\varrho),\ \ \mu>0,\ \ \lambda\geq-\frac{2}{3}\mu, \tag{2.6}$$

and equations (2.3), (2.4) can be written in a concise form as

NAVIER–STOKES SYSTEM

$$\partial_t\varrho+\mathrm{div}_x(\varrho\mathbf{u})=0, \tag{2.7}$$

$$\partial_t(\varrho\mathbf{u})+\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_x p(\varrho)=\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla_x\mathrm{div}_x\mathbf{u}+\varrho\mathbf{f}. \tag{2.8}$$

The system of equations (2.7), (2.8) should be compared with a "more famous" incompressible Navier–Stokes system, where the density is constant, say $\varrho\equiv 1$, while (2.7), (2.8) "reduces" to

$$\mathrm{div}_x\mathbf{u}=0, \tag{2.9}$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mu \Delta \mathbf{u} + \mathbf{f}. \tag{2.10}$$

Unlike in (2.8), the pressure p in (2.10) is an unknown function determined (implicitly) by the fluid motion! The pressure in the *incompressible* Navier–Stokes system has non-local character and may depend on the far field behavior of the fluid system.

2.3 Spatial domain and boundary conditions

In the real world applications, the fluid is confined to a bounded spatial domain $\Omega \subset \mathbb{R}^3$. The presence of the physical boundary $\partial\Omega$ and the associated problem of fluid-structure interaction represent a source of substantial difficulties in the mathematical analysis of fluids in motion. In order to avoid technicalities, we suppose in Part II of the lecture series that the motion is *space-periodic*, specifically,

$$\varrho(t,x) = \varrho(t, x + \mathbf{a}^i), \ \mathbf{u}(t,x) = \mathbf{u}(t, x + \mathbf{a}^i) \text{ for all } t, x, \tag{2.11}$$

where the period vectors $\mathbf{a}^1 = (a_1, 0, 0)$, $\mathbf{a}^2 = (0, a_2, 0)$, $\mathbf{a}^3 = (0, 0, a_3)$ are given. Equivalently, we may assume that Ω is a flat *torus*,

$$\Omega = [0, a_1]|_{\{0,a_1\}} \times [0, a_2]|_{\{0,a_2\}} \times [0, a_3]|_{\{0,a_3\}}.$$

The space-periodic boundary conditions have a nice physical interpretation in fluid mechanics, see Ebin [5]. Indeed, if we restrict ourselves to the classes of functions defined on the torus Ω and satisfying the extra geometric restrictions:

$$\varrho(t,x) = \varrho(t,-x), \ u_i(t,\cdot,x_i,\cdot) = -u_i(t,\cdot,-x_i,\cdot), \ i = 1,2,3,$$

$$u_i(t,\cdot,x_j,\cdot) = u_i(t,\cdot,-x_j,\cdot) \text{ for } i \neq j,$$

and, similarly,

$$f_i(t,\cdot,x_i,\cdot) = -f_i(t,\cdot,-x_i,\cdot), \ f_i(t,\cdot,x_j,\cdot) = f_i(t,\cdot,-x_j,\cdot) \text{ for } i \neq j,$$

we can check that

- equations (2.7), (2.8) are invariant with respect to the above transformations;

- the velocity field $\mathbf{u}$ satisfies the so-called *complete slip* conditions

$$\mathbf{u} \cdot \mathbf{n} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n} = \mathbf{0} \tag{2.12}$$

  on the boundary of the spatial block $[0, a_1] \times [0, a_2] \times [0, a_3]$.

We remark that the most commonly used boundary conditions for *viscous* fluids confined to a general spatial domain Ω (not necessarily a flat torus) are the *no-slip*

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}. \tag{2.13}$$

We will focus on this type of the boundary condition in Part III of this lecture series. As a matter of fact, the problem of the choice of correct boundary conditions in the real world applications is rather complex, some parts of the boundaries may consist of a different fluid in motion, or the fluid domain is not *a priori* known (free boundary problems). The interested reader may consult Priezjev and Troian [17] for relevant discussion.

2.4 Initial conditions

Given the initial state at a reference time t_0, say $t_0 = 0$, the time evolution of the fluid is determined as a solution of the Navier–Stokes system (2.7), (2.8). It is convenient to introduce the initial density

$$\varrho(0, x) = \varrho_0(x), \ x \in \Omega, \tag{2.14}$$

together with the initial distribution of the momentum,

$$(\varrho\mathbf{u})(0, x) = (\varrho\mathbf{u})_0(x), \ x \in \Omega, \tag{2.15}$$

as, strictly speaking, the momentum balance (2.8) is an evolutionary equation for $\varrho\mathbf{u}$ rather than $\mathbf{u}$. Such a difference will become clear in the so-called *weak* formulation of the problem discussed in the forthcoming section.

Chapter 3

Weak solutions

A vast class of *non-linear* evolutionary problems arising in mathematical fluid mechanics is not known to admit classical (differentiable, smooth) solutions for all choices of data and on an arbitrary time interval. On the other hand, most of the real world problems call for solutions defined in-the-large approached in the numerical simulations. In order to perform a rigorous analysis, we have to introduce a concept of *generalized* or *weak* solutions, for which derivatives are interpreted in the sense of distributions. The dissipation represented by viscosity should provide a strong regularizing effect. Another motivation, at least in the case of the compressible Navier–Stokes system (2.7), (2.8), is the possibility to study the fluid dynamics emanating from irregular initial state, for instance, the density ϱ_0 may not be continuous. As shown by Hoff [11], the singularities incorporated initially will "survive" in the system at any time; thus the weak solutions are necessary in order to describe the dynamics.

3.1 Equation of continuity – weak formulation

We consider equation (2.7) on the space-time cylinder $(0,T) \times \Omega$, where Ω is the flat torus introduced in Section 2.3. Multiplying (2.7) on $\varphi \in C_c^\infty((0,T) \times \Omega)$, integrating the resulting expression over $(0,T) \times \Omega$, and performing by parts integration, we obtain

$$\int_0^T \int_\Omega \Big(\varrho(t,x) \partial_t \varphi(t,x) + \varrho(t,x) \mathbf{u}(t,x) \cdot \nabla_x \varphi(t,x) \Big) \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{3.1}$$

Definition 3.1 *We say that a pair of functions* ϱ, $\mathbf{u}$ *is a* weak solution *to equation (2.7) in the space-time cylinder* $(0,T)\times\Omega$ *if* ϱ, $\varrho\mathbf{u}$ *are locally integrable in* $(0,T)\times\Omega$ *and the integral identity (3.1) holds for any test function* $\varphi \in C_c^\infty((0,T)\times\Omega)$.

3.1.1 Weak-strong compatibility

It is easy to see that any classical (smooth) solution of equation (2.7) is also a weak solution. Similarly, any weak solution that is continuously differentiable satisfies (2.7) pointwise. Such a property is called *weak-strong compatibility*.

3.1.2 Weak continuity

Up to now, we have left apart the problem of satisfaction of the initial condition (2.14). Obviously, some kind of *weak continuity* is needed for (2.14) to make sense. To this end, we make extra hypotheses, namely,

$$\varrho \in L^1(0,T;L^1_{\rm loc}(\Omega)), \quad \varrho\mathbf{u} \in L^1(0,T;L^1_{\rm loc}(\Omega;\mathbb{R}^3)). \tag{3.2}$$

Taking

$$\varphi(t,x) = \psi(t)\phi(x),\ \psi \in C_c^\infty(0,T),\ \phi \in C_c^\infty(\Omega)$$

as a test function in (3.1) we may infer, by virtue of (3.2), that the function

$$t \mapsto \int_\Omega \varrho(t,x)\phi(x)\ {\rm d}x \text{ is absolutely continuous in } [0,T] \tag{3.3}$$

for any $\phi \in C_c^\infty(\Omega)$. In particular, the initial condition (2.14) may be satisfied in the sense that

$$\lim_{t\to 0+} \int_\Omega \varrho(t,x)\phi(x)\ {\rm d}x = \int_\Omega \varrho_0(x)\phi(x)\ {\rm d}x \text{ for any } \phi \in C_c^\infty(\Omega).$$

To this aim, take

$$\varphi_\varepsilon(t,x) = \psi_\varepsilon(t)\varphi(t,x),\ \varphi \in C_c^\infty([0,T]\times\Omega),$$

where $\psi_\varepsilon \in C_c^\infty(0,\tau)$,

$$0 \leq \psi_\varepsilon \leq 1,\ \psi_\varepsilon \nearrow 1_{[0,\tau]} \text{ as } \varepsilon \to 0.$$

Taking φ_ε as a test function in (3.1) and letting $\varepsilon \to 0$, we conclude, making use of (3.3), that

$$\int_\Omega \varrho(\tau,x)\varphi(\tau,x)\ \mathrm{d}x - \int_\Omega \varrho_0(x)\varphi(0,x)\ \mathrm{d}x \tag{3.4}$$
$$= \int_0^\tau \int_\Omega \Big(\varrho(t,x)\partial_t\varphi(t,x) + \varrho(t,x)\mathbf{u}(t,x)\cdot\nabla_x\varphi(t,x)\Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any $\tau \in [0,T]$ and any $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Formula (3.4) can be alternatively used as a definition of *weak solution* to problem (2.7), (2.14). It is interesting to compare (3.4) with the original integral formulation of the principle of mass conservation stated in (2.1). To this end, we take

$$\varphi_\varepsilon(t,x) = \phi_\varepsilon(x),$$

with $\phi_\varepsilon \in C_c^\infty(B)$ such that

$$0 \leq \phi_\varepsilon \leq 1,\ \phi_\varepsilon \nearrow 1_B \text{ as } \varepsilon \to 0.$$

It is easy to see that

$$\int_\Omega \varrho(\tau,x)\varphi_\varepsilon(\tau,x)\ \mathrm{d}x - \int_\Omega \varrho_0(x)\varphi_\varepsilon(0,x)\ \mathrm{d}x \to \int_B \varrho(\tau,x)\ \mathrm{d}x - \int_B \varrho_0(x)\ \mathrm{d}x$$

as $\varepsilon \to 0$, which coincides with the expression on the left-hand side of (2.1). Consequently, the right-hand side of (3.4) must posses a limit and we set

$$\int_0^\tau \int_\Omega \varrho(t,x)\mathbf{u}(t,x)\cdot\nabla_x\phi_\varepsilon(x)\ \mathrm{d}x\ \mathrm{d}t \to -\int_0^\tau \int_{\partial B} \varrho(t,x)\mathbf{u}(t,x)\cdot\mathbf{n}\ \mathrm{dS}_x\ \mathrm{d}t.$$

In other words, the weak solutions possess a *normal trace* on the boundary of the cylinder $(0,\tau)\times B$ that satisfies (2.1), see Chen and Frid [2] for more elaborate treatment of the normal traces of solutions to conservation laws.

3.1.3 Total mass conservation

Taking $\varphi = 1$ for $t \in [0,\tau]$ in (3.4) we obtain

$$\int_\Omega \varrho(\tau,x)\ \mathrm{d}x = \int_\Omega \varrho_0(x)\ \mathrm{d}x = M_0 \text{ for any } \tau \geq 0, \tag{3.5}$$

meaning, the total mass M_0 of the fluid is a constant of motion.

3.2 Balance of momentum — weak formulation

Similarly to the preceding part, we introduce a weak formulation of the balance of momentum (2.8):

Definition 3.2 *The functions ϱ, $\mathbf{u}$ represent a* weak solution *to the momentum equation (2.8) in the set $(0,T)\times\Omega$ if the integral identity*

$$\int_0^T \int_\Omega \Big((\varrho\mathbf{u})(t,x)\cdot\partial_t\boldsymbol{\varphi}(t,x) + (\varrho\mathbf{u}\otimes\mathbf{u})(t,x) : \nabla_x\boldsymbol{\varphi}(t,x) \tag{3.6}$$

$$+p(\varrho)(t,x)\mathrm{div}_x\boldsymbol{\varphi}(t,x)\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$= \int_0^T \int_\Omega \Big(\mu\nabla_x\mathbf{u}(t,x) : \nabla_x\boldsymbol{\varphi}(t,x)$$

$$+(\lambda+\mu)\mathrm{div}_x\mathbf{u}(t,x)\mathrm{div}_x\boldsymbol{\varphi}(t,x) - \varrho(t,x)\mathbf{f}(t,x)\cdot\boldsymbol{\varphi}(t,x)\Big)\ \mathrm{d}x\ \mathrm{d}t$$

is satisfied for any test function $\boldsymbol{\varphi}\in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$.

Of course, we have tacitly assumed that all quantities appearing in (3.6) are at least locally integrable in $(0,T)\times\Omega$. In particular, as (3.6) contains explicitly $\nabla_x\mathbf{u}$, we have to assume integrability of this term. As we shall see in the following section, one can expect, given the available *a priori* bounds, $\nabla_x\mathbf{u}$ to be square integrable, specifically,

$$\mathbf{u}\in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)).$$

If $\Omega\subset\mathbb{R}^3$ is a (bounded) domain with a non-void boundary, we can enforce several kinds of boundary conditions by means of the properties of the test functions. Thus, for instance, the *no-slip* boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \tag{3.7}$$

require the integral identity (3.6) to be satisfied for any *compactly supported* test function $\boldsymbol{\varphi}$, while

$$\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)),$$

where $W_0^{1,2}(\Omega;\mathbb{R}^3)$ is the Sobolev space obtained as the closure of $C_c^\infty(\Omega;\mathbb{R}^3)$ in the $W^{1,2}$-norm. On the other hand, for the periodic boundary conditions, we can allow test functions $\boldsymbol{\varphi} \in C_c^\infty((0,T)\times\overline{\Omega};\mathbb{R}^3)$ and the velocity $\mathbf{u} \in L^2(0,T;W_{\mathrm{per}}^{1,2}(\Omega;\mathbb{R}^3))$.

Remark 3.1 We may get the weak-strong compatibility as in the case of continuity equation.

Chapter 4

A priori bounds

A priori bounds are natural constraints imposed on the set of (hypothetical) smooth solutions by the data as well as by the differential equations satisfied. *A priori* bounds determine the function spaces framework the (weak) solutions are looked for. By definition, they are formal, derived under the principal hypothesis of smoothness of all quantities in question.

4.1 Total mass conservation

The fluid density ϱ satisfies the equation of continuity that may be written in the form

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho = -\varrho \mathrm{div}_x \mathbf{u}. \tag{4.1}$$

This is a transport equation with the characteristic field defined

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{X}(t, x_0) = \mathbf{u}(t, \mathbf{X}), \ \mathbf{X}(0, x_0) = x_0.$$

Accordingly, (4.1) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \varrho(t, \mathbf{X}(t, \cdot)) = -\varrho(t, \mathbf{X}(t, \cdot)) \mathrm{div}_x \mathbf{u}(t, \mathbf{X}(t, \cdot)).$$

Consequently, we obtain

$$\begin{aligned} &\inf_{x\in\Omega} \varrho(0,x) \exp\left(-t\|\mathrm{div}_x\mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\right) \\ &\leq \varrho(t,x) \leq \\ &\leq \sup_{x\in\Omega} \varrho(0,x) \exp\left(t\|\mathrm{div}_x\mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\right) \end{aligned} \tag{4.2}$$

for any $t \in [0,T]$.

Unfortunately, the bounds established in (4.2) depend on $\|\mathrm{div}_x\mathbf{u}\|_{L^\infty}$ on which we have no information. Thus we may infer only that

$$\varrho(t,x) \geq 0, \tag{4.3}$$

provided $\varrho(0,x) \geq 0$ in Ω.

Relation (4.3) combined with the total mass conservation (3.5) yields

$$\|\varrho(t,\cdot)\|_{L^1(\Omega)} = \|\varrho_0\|_{L^1(\Omega)},\ \varrho(0,\cdot) = \varrho_0. \tag{4.4}$$

4.2 Energy balance

Taking the scalar product of the momentum equation (2.4) with $\mathbf{u}$ we deduce the *kinetic energy balance equation*

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right)+\mathrm{div}_x\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}\right)+\mathrm{div}_x(p(\varrho)\mathbf{u})-p(\varrho)\mathrm{div}_x\mathbf{u}-\mathrm{div}_x(\mathbb{S}\mathbf{u})+\mathbb{S}:\nabla_x\mathbf{u} = \varrho\mathbf{f}\cdot\mathbf{u}. \tag{4.5}$$

Our goal is to integrate (4.5) by parts in order to deduce *a priori* bounds. Imposing the no-slip boundary condition (2.13) or the space-periodic boundary condition (2.11) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2\right)\ \mathrm{d}x - \int_\Omega p(\varrho)\mathrm{div}_x\mathbf{u}\ \mathrm{d}x + \int_\Omega \mathbb{S}:\nabla_x\mathbf{u}\ \mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x,$$

where, in accordance with (2.6),

$$\int_\Omega \mathbb{S}:\nabla_x\mathbf{u}\ \mathrm{d}x = \mu\int_\Omega |\nabla_x\mathbf{u}|^2\ \mathrm{d}x + (\lambda+\mu)\int_\Omega |\mathrm{div}_x\mathbf{u}|^2\ \mathrm{d}x \geq c\int_\Omega |\nabla_x\mathbf{u}|^2\ \mathrm{d}x, \tag{4.6}$$

$c > 0$, provided $\mu > 0$ and $\lambda + 2/3\mu \geq 0$.

Seeing that

$$\int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \ \mathrm{d}x \leq \int_\Omega |\mathbf{f}| \sqrt{\varrho} \sqrt{\varrho} |\mathbf{u}| \ \mathrm{d}x$$

$$\leq \frac{1}{2} \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \left(\int_\Omega \varrho \ \mathrm{d}x + \int_\Omega \varrho |\mathbf{u}|^2 \ \mathrm{d}x \right),$$

we focus on the integral

$$\int_\Omega p(\varrho) \mathrm{div}_x \mathbf{u} \ \mathrm{d}x.$$

Multiplying the equation of continuity (4.1) by $b'(\varrho)$ we obtain the *renormalized equation of continuity*

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0. \tag{4.7}$$

Consequently, in particular, the choice

$$b(\varrho) = P(\varrho) \equiv \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z$$

leads to

$$b'(\varrho)\varrho - b(\varrho) = p(\varrho).$$

Thus

$$- \int_\Omega p(\varrho) \mathrm{div}_x \mathbf{u} \ \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega P(\varrho) \ \mathrm{d}x,$$

and we deduce the *total energy* balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \ \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \ \mathrm{d}x. \tag{4.8}$$

We conclude with

Energy estimates:

$$\sup_{t\in[0,T]} \|\sqrt{\varrho}\mathbf{u}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^3)} \leq c(E_0,T,\mathbf{f}), \tag{4.9}$$

$$\sup_{t\in[0,T]} \int_\Omega P(\varrho)(t,\cdot)\ \mathrm{d}x \leq c(E_0,T,\mathbf{f}), \tag{4.10}$$

$$\int_0^T \|\mathbf{u}(t,\cdot)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\ \mathrm{d}t \leq c(E_0,T,\mathbf{f}), \tag{4.11}$$

where E_0 denotes the initial energy

$$E_0 = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0)\right)\ \mathrm{d}x.$$

Note that in case of the Dirichlet boundary conditions we have $\|\mathbf{u}\|_{1,2} \leq C\|\nabla_x\mathbf{u}\|_2$ while for the space periodic conditions, due to the fact that we control the total mass, we also control $\int_\Omega \varrho|\mathbf{u}|\ \mathrm{d}x$ which yields $\|\mathbf{u}\|_{1,2} \leq C(\|\nabla_x\mathbf{u}\|_2 + \|\varrho\mathbf{u}\|_1)$.

4.3 Pressure estimates

A seemingly direct way to pressure estimates is to "compute" the pressure in the momentum balance (2.8):

$$p(\varrho) = -\Delta^{-1}\mathrm{div}_x\partial_t(\varrho\mathbf{u}) - \Delta^{-1}\mathrm{div}_x\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \Delta^{-1}\mathrm{div}_x\mathrm{div}_x\mathbb{S} + \Delta^{-1}\mathrm{div}_x(\varrho\mathbf{f}),$$

where Δ^{-1} is an "inverse" of the Laplacean. In order to justify this formal step, we use the so-called *Bogovskii operator* $\mathcal{B} \approx \mathrm{div}_x^{-1}$.

We multiply equation (2.8) on

$$\mathbf{B}[\varrho] = \mathcal{B}\left[b(\varrho) - \frac{1}{|\Omega|}\int_\Omega b(\varrho)\ \mathrm{d}x\right]$$

and integrate by parts to obtain

$$\int_0^T \int_\Omega p(\varrho)b(\varrho)\ \mathrm{d}x\ \mathrm{d}t \tag{4.12}$$

$$= \frac{1}{|\Omega|} \int_0^T \int_\Omega p(\varrho)\ \mathrm{d}x \int_\Omega b(\varrho)\ \mathrm{d}x\ \mathrm{d}t + \int_0^T \int_\Omega \mathbb{S} : \nabla_x \mathbf{B}[\varrho]\ \mathrm{d}x\ \mathrm{d}t$$

$$- \int_0^T \int_\Omega \varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla_x \mathbf{B}[\varrho]\ \mathrm{d}x - \int_0^T \int_\Omega \varrho \mathbf{u} \cdot \partial_t \mathbf{B}[\varrho]\ \mathrm{d}x\ \mathrm{d}t - \int_0^T \int_\Omega \varrho \mathbf{f} \cdot \mathbf{B}[\varrho]\ \mathrm{d}x\ \mathrm{d}t$$

$$+ \int_\Omega \left(\varrho \mathbf{u} \cdot \mathbf{B}[\varrho](T, \cdot) - \varrho_0 \mathbf{u}_0 \cdot \mathbf{B}[\varrho_0] \right)\ \mathrm{d}x.$$

Furthermore, we have

$$\partial_t \mathbf{B}[\varrho] = -\mathcal{B} \Big[\mathrm{div}_x (b(\varrho) \mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} \tag{4.13}$$

$$- \frac{1}{|\Omega|} \int_\Omega \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u}\ \mathrm{d}x \Big].$$

We recall the basic properties of the Bogovskii operator:

BOGOVSKII OPERATOR:

$$\mathrm{div}_x \mathcal{B}[h] = h \text{ for any } h \in L^p(\Omega),\ \int_\Omega h\ \mathrm{d}x = 0,\ 1 < p < \infty,\ \mathcal{B}[h]|_{\partial\Omega} = 0, \tag{4.14}$$

$$\|\mathcal{B}[h]\|_{W_0^{1,p}(\Omega;\mathbb{R}^3)} \leq c(p) \|h\|_{L^p(\Omega)},\ 1 < p < \infty, \tag{4.15}$$

$$\|\mathcal{B}[h]\|_{L^q(\Omega;\mathbb{R}^3)} \leq \|\mathbf{g}\|_{L^q(\Omega;\mathbb{R}^3)} \tag{4.16}$$

for $h \in L^p(\Omega),\ h = \mathrm{div}_x \mathbf{g},\ \mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ 1 < q < \infty.$

As will be seen in the last part (Chapter 8), we can show that for $b(\varrho) = \varrho^\theta$ the right-hand side is possible to estimate provided

$$\theta \leq \min \left\{ \frac{\gamma}{2}, \frac{2}{3}\gamma - 1 \right\}. \tag{4.17}$$

Note that for $\gamma \leq 6$ the restriction comes from the second term in (4.17) while for $\gamma > 6$ the first term is more restrictive. In fact, working a bit harder, we can remove also the limit $\theta \leq \frac{\gamma}{2}$ for $\gamma > 6$. Furthermore, to remove the term at $t := T$ we may use a suitable cut-off function in time.

Chapter 5

Complete weak formulation

A *complete* weak formulation of the (compressible) *Navier–Stokes* system takes into account both the renormalized equation of continuity and the energy inequality. Here and hereafter we assume that $\Omega \subset \mathbb{R}^3$ is either a bounded domain with Lipschitz boundary or a periodic box. For the sake of definiteness, we take the pressure in the form

$$p(\varrho) = a\varrho^\gamma, \text{ with } a > 0 \text{ and } \gamma > 3/2. \tag{5.1}$$

In Part II we restrict ourselves to the case when γ is "sufficiently" large, Part III will contain the proof only under restriction (5.1).

5.1 Equation of continuity

Let us introduce a class of (nonlinear) functions b such that

$$b \in C^1([0,\infty)),\ b(0) = 0,\ b'(r) = 0 \text{ whenever } r \geq M_b. \tag{5.2}$$

We say that ϱ, $\mathbf{u}$ is a (renormalized) solution of the equation of continuity (2.3), supplemented with the initial condition,

$$\varrho(0,\cdot) = \varrho_0,$$

if $\varrho \in C_{\text{weak}}([0,T];L^\gamma(\Omega))$, $\varrho \geq 0$, $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, and the integral identity

$$\int_0^T \int_\Omega \Big((\varrho + b(\varrho))\, \partial_t \varphi + (\varrho + b(\varrho))\, \mathbf{u} \cdot \nabla_x \varphi + (b(\varrho) - b'(\varrho)\varrho)\, \mathrm{div}_x \mathbf{u} \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$
$$= - \int_\Omega (\varrho_0 + b(\varrho_0))\, \varphi(0, \cdot) \ \mathrm{d}x \tag{5.3}$$

is satisfied for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$ and any b belonging to the class specified in (5.2).

In particular, taking $b \equiv 0$ we deduce the standard weak formulation of (2.3) in the form (we use the considerations from Section 3 to include also the term at $t := \tau$; we avoid this term in (5.3) due to certain technical complications connected with nonlinearity of $b(\cdot)$)

$$\int_\Omega \Big(\varrho(\tau, \cdot)\varphi(\tau, \cdot) - \varrho_0 \varphi(0, \cdot) \Big) \ \mathrm{d}x \tag{5.4}$$
$$= \int_0^\tau \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$

for any $\tau \in [0,T]$ and any $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$.

In case of space-periodic boundary conditions we assume ϱ and $\mathbf{u}$ space-periodic, while for the homogeneous Dirichlet boundary conditions we assume $\mathbf{u} \in L^2(0,T; W_0^{1,2}(\mathbb{R}^3; \mathbb{R}^3))$. Note that (5.4) actually holds on the whole physical space $\mathbb{R}^3$ provided (in case of the Dirichlet boundary conditions) ϱ, $\mathbf{u}$ were extended to be zero outside Ω. Note also that (5.4) implies that the initial condition $\varrho(0, \cdot) = \varrho_0(\cdot)$ is fulfilled. In case of the space periodic boundary conditions we may extend the functions outside Ω due to the periodicity.

5.2 Momentum equation

In addition to the previous assumptions we suppose that

$$\varrho \mathbf{u} \in C_{\mathrm{weak}}([0,T]; L^q(\Omega; \mathbb{R}^3)) \text{ for a certain } q > 1, \ p(\varrho) \in L^1((0,T) \times \Omega).$$

The weak formulation of the momentum equation reads:

$$\int_\Omega \Big(\varrho \mathbf{u}(\tau,\cdot)\cdot \boldsymbol{\varphi}(\tau,\cdot) - (\varrho\mathbf{u})_0 \cdot \boldsymbol{\varphi}(0,\cdot) \Big)\ \mathrm{d}x \tag{5.5}$$

$$= \int_0^\tau \int_\Omega \Big(\varrho \mathbf{u}\cdot \partial_t \boldsymbol{\varphi} + \varrho (\mathbf{u}\otimes\mathbf{u}) : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \Big(\mu \nabla_x \mathbf{u} : \nabla_x \boldsymbol{\varphi} + (\lambda+\mu) \mathrm{div}_x \mathbf{u} \mathrm{div}_x \boldsymbol{\varphi} - \varrho \mathbf{f}\cdot\boldsymbol{\varphi} \Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any $\tau \in [0,T]$ and for any test function $\boldsymbol{\varphi} \in C_c^\infty([0,T]\times\Omega;\mathbb{R}^3)$.

Note that (5.5) already includes the satisfaction of the initial condition

$$\varrho\mathbf{u}(0,\cdot) = (\varrho\mathbf{u})_0.$$

5.3 Energy inequality

The weak solutions are not known to be uniquely determined by the initial data. Therefore it is desirable to introduce as much physically grounded conditions as allowed by the construction of the weak solutions. One of them is

Energy inequality:

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \Big(\mu |\nabla_x \mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}_x \mathbf{u}|^2 \Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$\le \int_\Omega \left(\frac{1}{2\varrho_0} |(\varrho\mathbf{u})_0|^2 + P(\varrho_0) \right)\ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho \mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \tag{5.6}$$

for a.a. $\tau \in (0,T)$, where

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z.$$

Some remarks are in order. To begin, given the specific choice of the pressure $p(\varrho) = a\varrho^\gamma$ and the fact that the total mass of the fluid is a constant of motion, the function $P(\varrho)$ in (5.6) can be taken as

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma.$$

Next, we need a kind of *compatibility* condition between ϱ_0 and $(\varrho\mathbf{u})_0$ provided we allow the initial density to vanish on a nonempty set:

$$(\varrho\mathbf{u})_0 = \mathbf{0} \text{ a.a. on the "vacuum" set } \{x \in \Omega \mid \varrho_0(x) = 0\}. \tag{5.7}$$

Part II

Weak sequential stability for large γ

Chapter 6

Weak sequential stability

The problem of *weak sequential stability* may be stated as follows:

WEAK SEQUENTIAL STABILITY:

Given a family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of weak solutions of the compressible Navier–Stokes system, emanating from the initial data

$$\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon},\ (\varrho\mathbf{u})_\varepsilon(0,\cdot) = (\varrho\mathbf{u})_{0,\varepsilon},$$

we want to show that

$$\varrho_\varepsilon \to \varrho,\ \mathbf{u}_\varepsilon \to \mathbf{u} \text{ as } \varepsilon \to 0$$

in a certain sense and at least for suitable subsequences, where ϱ, $\mathbf{u}$ is another weak solution of the same system.

Although showing *weak sequential stability* does not provide an explicit proof of *existence* of the weak solutions, its verification represents one of the prominent steps towards a rigorous *existence theory* for a given system of equations.

6.1 Uniform bounds

To begin the analysis, we need *uniform* bounds in terms of the data. To this end, we choose the initial data in such a way that

$$\int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}} |(\varrho \mathbf{u})_{0,\varepsilon}|^2 + P(\varrho_{0,\varepsilon}) \right) \, \mathrm{d}x \leq E_0, \tag{6.1}$$

where the constant E_0 is independent of ε. Moreover, the main and most difficult steps of the proof of weak sequential stability remain basically the same under the simplifying assumption

$$\mathbf{f} \equiv \mathbf{0}.$$

In accordance with the energy inequality (5.6), we get for any $T < \infty$

$$\sup_{t \in (0,T)} \|\varrho_\varepsilon(t,\cdot)\|_{L^\gamma(\Omega)} \leq c \tag{6.2}$$

and

$$\sup_{t \in (0,T)} \|\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^3))} \leq c, \tag{6.3}$$

together with

$$\int_0^T \|\mathbf{u}_\varepsilon(t,\cdot)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \, \mathrm{d}t \leq c, \tag{6.4}$$

where the symbol c stands for a generic constant independent of ε.

Interpolating (6.2), (6.3), we get

$$\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^q(\Omega;\mathbb{R}^3)} = \|\sqrt{\varrho_\varepsilon}\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^q(\Omega;\mathbb{R}^3)} \leq \|\sqrt{\varrho_\varepsilon}\|_{L^{2\gamma}(\Omega)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)},$$

with

$$q = \frac{2\gamma}{\gamma+1} > 1 \text{ provided } \gamma > 1.$$

We conclude that

$$\sup_{t\in[0,T]} \|\varrho_\varepsilon \mathbf{u}_\varepsilon(t,\cdot)\|_{L^q(\Omega;\mathbb{R}^3)} \leq C, \ q = \frac{2\gamma}{\gamma+1}. \tag{6.5}$$

Next, applying a similar treatment to the convective term in the momentum equation, we have

$$\|\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\|_{L^q(\Omega;\mathbb{R}^{3\times 3})} = \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{2\gamma/(\gamma+1)}(\Omega;\mathbb{R}^3)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega;\mathbb{R}^3)}, \text{ with } q = \frac{6\gamma}{4\gamma+3}.$$

Using the standard *embedding relation*

$$W^{1,2}(\Omega) \hookrightarrow L^6(\Omega), \tag{6.6}$$

we may therefore conclude that

$$\int_0^T \|\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\|^2_{L^q(\Omega;\mathbb{R}^{3\times 3})} \ \mathrm{d}t \le c, \ q = \frac{6\gamma}{4\gamma+3}. \tag{6.7}$$

Note that

$$\frac{6\gamma}{4\gamma+3} > 1 \text{ as long as } \gamma > \frac{3}{2}.$$

Finally, we have the pressure estimates (see Chapter 8 for the proof):

$$\int_0^T \int_\Omega p(\varrho_\varepsilon)\varrho_\varepsilon^\alpha \ \mathrm{d}x \ \mathrm{d}t = a \int_0^T \int_\Omega \varrho_\varepsilon^{\gamma+\alpha} \ \mathrm{d}x \ \mathrm{d}t \le c \tag{6.8}$$

for $\alpha = \min\{\frac{\gamma}{2}, \frac{2}{3}\gamma - 1\}$.

6.2 Limit passage

In view of the uniform bounds established in the previous section, we may assume that

$$\varrho_\varepsilon \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)), \tag{6.9}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \tag{6.10}$$

passing to suitable subsequences as the case may be. Moreover, since ϱ_ε satisfies the equation of continuity (5.4), relation (6.9) may be strengthened to (see Chapter 7, in particular Lemma 7.4 and Theorem 7.2)

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\mathrm{weak}}([0,T];L^\gamma(\Omega)). \tag{6.11}$$

Let us recall that, in view of (6.9), relation (6.11) simply means

$$\left\{ t \mapsto \int_\Omega \varrho_\varepsilon(t,\cdot)\varphi \ \mathrm{d}x \right\} \to \left\{ t \mapsto \int_\Omega \varrho(t,\cdot)\varphi \ \mathrm{d}x \right\} \text{ in } C[0,T]$$

for any $\varphi \in C_c^\infty(\Omega)$.

6.3 Compactness of the convective term

Our next goal is to establish convergence of the convective terms. Recall that, in view of the estimate (6.5), we may suppose that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u}} \text{ weakly-(*) in } L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega;\mathbb{R}^3))$$

and even

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u}} \text{ in } C_{\text{weak}}([0,T];L^{2\gamma/(\gamma+1)}(\Omega;\mathbb{R}^3)), \tag{6.12}$$

where the bar denotes (and will always denote in the future) a weak limit of a composition.

Our goal is to show that

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}.$$

This can be observed in several ways. Seeing that

$$W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \text{ compactly for } 1 \leq q < 6,$$

we deduce that

$$L^p(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega) \text{ compactly whenever } p > \frac{6}{5}. \tag{6.13}$$

In particular, relation (6.11) yields (for $\gamma > \frac{6}{5}$, cf. Theorem 7.2)

$$\varrho_\varepsilon \to \varrho \text{ in } C([0,T];W^{-1,2}(\Omega))),$$

which, combined with (6.10) and (6.12), gives rise to the desired conclusion

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}.$$

For more details see again Chapter 7.

6.4 Passing to the limit — step 1

Now, due to the fact that $\gamma > 3/2$ and due to estimate (6.7), we may infer that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u} \otimes \mathbf{u}} \text{ weakly in } L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3}) \text{ for a certain } q > 1. \tag{6.14}$$

Recalling (6.12), compactness of the embedding (6.13) and using that $\frac{2\gamma}{\gamma+1} > \frac{6}{5}$ for $\gamma > \frac{6}{5}$ we conclude that

$$\overline{\varrho\mathbf{u}\otimes\mathbf{u}} = \varrho\mathbf{u}\otimes\mathbf{u}.$$

Summing up the previous discussion we deduce that the limit functions ϱ, $\mathbf{u}$ satisfy the equation of continuity

$$\int_\Omega \Big(\varrho(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_0\varphi(0,\cdot)\Big)\ \mathrm{d}x \tag{6.15}$$
$$= \int_0^\tau\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any $\tau\in[0,T]$ and any $\varphi\in C_c^\infty([0,T]\times\overline{\Omega})$, together with a relation for the momentum

$$\int_\Omega \Big(\varrho\mathbf{u}(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot) - (\varrho\mathbf{u})_0\cdot\boldsymbol{\varphi}(0,\cdot)\Big)\ \mathrm{d}x \tag{6.16}$$
$$= \int_0^\tau\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + \overline{p(\varrho)}\mathrm{div}_x\boldsymbol{\varphi}\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$- \int_0^\tau\int_\Omega \Big(\mu\nabla_x\mathbf{u} : \nabla_x\boldsymbol{\varphi} + (\lambda+\mu)\mathrm{div}_x\mathbf{u}\,\mathrm{div}_x\boldsymbol{\varphi}\Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any test function $\boldsymbol{\varphi}\in C_c^\infty([0,T]\times\Omega;\mathbb{R}^3)$.

Here, we have also to assume at least weak convergence of the initial data, specifically,

$$\varrho_{0,\varepsilon}\to\varrho_0 \text{ weakly in } L^\gamma(\Omega), \tag{6.17}$$
$$(\varrho\mathbf{u})_{0,\varepsilon}\to(\varrho\mathbf{u})_0 \text{ weakly in } L^1(\Omega;\mathbb{R}^3).$$

Thus it remains to show the crucial relation

$$\overline{p(\varrho)} = p(\varrho)$$

or, equivalently,

$$\varrho_\varepsilon\to\varrho \text{ a.a. in } (0,T)\times\Omega. \tag{6.18}$$

This will be carried over in a series of steps specified in the remaining part of this chapter.

6.5 Strong convergence of the densities

In order to simplify presentation and to highlight the leading ideas, we assume that

$$\gamma \geq 5,$$

in particular

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}(0,T;L^\gamma(\Omega)),\ \gamma \geq 5.$$

6.5.1 Compactness via Div-Curl lemma

Div-Curl lemma, developed by Murat and Tartar [15], [18], represents an efficient tool for handling compactness in non-linear problems, where the classical Rellich–Kondraschev argument is not applicable.

Div-Curl lemma:

Lemma 6.1 *Let $B \subset \mathbb{R}^M$ be an open set. Suppose that*

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^p(B;\mathbb{R}^M),$$

$$\mathbf{w}_n \to \mathbf{w} \text{ weakly in } L^q(B;\mathbb{R}^M)$$

as $n \to \infty$, where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Let, moreover,

$$\{\mathrm{div}[\mathbf{v}]\}_{n=1}^\infty \text{ be precompact in } W^{-1,s}(B),$$

$$\{\mathrm{curl}[\mathbf{w}]\}_{n=1}^\infty \text{ be precompact in } W^{-1,s}(B;\mathbb{R}^{M\times M})$$

for a certain $s > 1$.

Then

$$\mathbf{v}_n \cdot \mathbf{w}_n \to \mathbf{v} \cdot \mathbf{w} \text{ weakly in } L^r(B).$$

We give the proof only for a very special case that will be needed in the future, namely, we assume that

$$\mathrm{div}\, \mathbf{v}_n = 0, \ \mathbf{w}_n = \nabla_x \Phi_n, \ \int_{\mathbb{R}^M} \Phi_n \, \mathrm{d}y = 0. \tag{6.19}$$

Moreover, given the local character of the weak convergence, it is enough to show the result for $B = \mathbb{R}^M$. By the same token, we may assume that all functions are compactly supported. We recall that a (scalar) sequence $\{g_n\}_{n=1}^\infty$ is precompact in $W^{-1,s}(\mathbb{R}^M)$ if

$$g_n = \mathrm{div}[\mathbf{h}_n], \text{ with } \{\mathbf{h}_n\}_{n=1}^\infty \text{ precompact in } L^s(\mathbb{R}^M;\mathbb{R}^M).$$

Now, it follows from the standard compactness arguments that

$$\Phi_n \to \Phi \text{ (strongly) in } L^q(\mathbb{R}^M), \ \nabla_x \Phi = \mathbf{w}.$$

Taking $\varphi \in C_c^\infty(\mathbb{R}^M)$ we have

$$\int_{\mathbb{R}^M} \mathbf{v}_n \cdot \mathbf{w}_n \varphi \, \mathrm{d}y = \int_{\mathbb{R}^M} \mathbf{v}_n \cdot \nabla_x \Phi_n \varphi \, \mathrm{d}y$$

$$= - \int_{\mathbb{R}^M} \mathbf{v}_n \cdot \nabla_x \varphi \Phi_n \, \mathrm{d}y \to - \int_{\mathbb{R}^M} \mathbf{v} \cdot \nabla_x \varphi \Phi \, \mathrm{d}y$$

$$= \int_{\mathbb{R}^M} \mathbf{v} \cdot \mathbf{w} \varphi \, \mathrm{d}y,$$

which completes the proof under the simplifying hypothesis (6.19).

6.5.2 Renormalized equation

We start with the renormalized equation (5.3) with $b(\varrho) = \varrho \log(\varrho) - \varrho$:

$$\int_0^T \int_\Omega \Big((\varrho_\varepsilon \log(\varrho_\varepsilon) \, \partial_t \psi - \varrho_\varepsilon \mathrm{div}_x \mathbf{u}_\varepsilon \psi \Big) \, \mathrm{d}x \, \mathrm{d}t = - \int_\Omega \varrho_{0,\varepsilon} \log(\varrho_{0,\varepsilon}) \, \mathrm{d}x \quad (6.20)$$

for any $\psi \in C_c^\infty[0,T)$, $\psi(0) = 1$. Clearly, relation (6.20) is a direct consequence of (5.3). Repeating the procedure from Chapter 2 we can get

$$\int_\Omega \varrho_\varepsilon \log(\varrho_\varepsilon)(t,\cdot) \, \mathrm{d}x + \int_0^T \int_\Omega \varrho_\varepsilon \mathrm{div}_x \mathbf{u}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t = \int_\Omega \varrho_{0,\varepsilon} \log(\varrho_{0,\varepsilon}) \, \mathrm{d}x. \quad (6.21)$$

Passing to the limit for $\varepsilon \to 0$ in (6.21) and assuming $\varrho_{0,\varepsilon} \to \varrho_0$ in $L^p(\Omega)$ for some $p > 1$ we get

$$\int_\Omega \overline{\varrho \log \varrho(t,\cdot)} \, \mathrm{d}x + \int_0^t \int_\Omega \overline{\varrho \mathrm{div}_x \mathbf{u}} \, \mathrm{d}x \, \mathrm{d}\tau = \int_\Omega \varrho_0 \log(\varrho_0) \, \mathrm{d}x. \quad (6.22)$$

Our next goal is to show that the limit functions ϱ, $\mathbf{u}$, besides (6.15), satisfy also its renormalized version. To this end, we use the procedure proposed by DiPerna and Lions [4], specifically, we regularize (6.15) by a family of regularizing kernels $\kappa_\delta(x)$ to obtain:

$$\partial_t \varrho_\delta + \mathrm{div}_x(\varrho_\delta \mathbf{u}) = \mathrm{div}_x(\varrho_\delta \mathbf{u}) - [\mathrm{div}_x(\varrho \mathbf{u})]_\delta,$$

with

$$v_\delta = \kappa_\delta * v, \text{ where } * \text{ stands for spatial convolution.}$$

We easily deduce that

$$\partial_t b(\varrho_\delta) + \mathrm{div}_x(b(\varrho_\delta)\mathbf{u}) + \Big(b'(\varrho_\delta)\varrho_\delta - b(\varrho_\delta)\Big)\mathrm{div}_x\mathbf{u}$$

$$= b'(\varrho_\delta)\Big(\mathrm{div}_x(\varrho_\delta\mathbf{u}) - [\mathrm{div}_x(\varrho\mathbf{u})]_\delta\Big).$$

Taking the limit $\delta \to 0$ and using Friedrich's lemma (see Chapter 7; here we need that $\varrho \in L^2((0,T)\times\Omega))$ and the procedure from Chapter 2 we get

$$\int_\Omega \varrho\log(\varrho)(t,\cdot)\ \mathrm{d}x + \int_0^t \int_\Omega \varrho\mathrm{div}_x\mathbf{u}\ \mathrm{d}x\ \mathrm{d}\tau = \int_\Omega \varrho_0\log(\varrho_0)\ \mathrm{d}x;$$

whence, in combination with (6.22),

$$\int_\Omega \Big(\overline{\varrho\log(\varrho)} - \varrho\log(\varrho)\Big)(t,\cdot)\ \mathrm{d}x + \int_0^t \int_\Omega \Big(\overline{\varrho\mathrm{div}_x\mathbf{u}} - \varrho\mathrm{div}_x\mathbf{u}\Big)\ \mathrm{d}x\ \mathrm{d}\tau = 0. \quad (6.23)$$

Assume, for a moment, that we can show

$$\int_0^\tau \int_\Omega \overline{\varrho\mathrm{div}_x\mathbf{u}}\ \mathrm{d}x\ \mathrm{d}t \geq \int_0^\tau \int_\Omega \varrho\mathrm{div}_x\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \text{ for any } \tau > 0, \quad (6.24)$$

which, together with lower semi-continuity of convex functionals, yields

$$\overline{\varrho\log(\varrho)} = \varrho\log(\varrho). \quad (6.25)$$

In order to continue, we need the following (standard) result:

Lemma 6.2 *Suppose that*

$$\varrho_\varepsilon \to \varrho \textit{ weakly in } L^2(Q),$$

where

$$\overline{\varrho\log(\varrho)} = \varrho\log(\varrho).$$

Then

$$\varrho_\varepsilon \to \varrho \textit{ in } L^1(Q).$$

Proof: Suppose that
$$0 < \delta \leq \varrho.$$
Consequently, because of convexity of $z \mapsto z\log(z)$, we have a.e. in $Q_{M,\delta} = \{(t,x) \in Q; \varrho(t,x) \leq M, \varrho(t,x) \geq \delta\}$

$$\varrho_\varepsilon \log(\varrho_\varepsilon) - \varrho\log(\varrho) = (\log(\varrho)+1)\,(\varrho_\varepsilon - \varrho) + \alpha(\varrho,\varrho_\varepsilon)|\varrho_\varepsilon - \varrho|^2$$
$$\geq \begin{cases} (\log(\varrho)+1)(\varrho_\varepsilon - \varrho) + \dfrac{1}{4M}|\varrho_\varepsilon - \varrho|^2 & \text{if } \varrho_\varepsilon(t,x) \leq 2M, \\ (\log(\varrho)+1)(\varrho_\varepsilon - \varrho) + \dfrac{|\varrho_\varepsilon - \varrho|^2}{4|\varrho_\varepsilon - \varrho|} & \text{if } \varrho_\varepsilon(t,x) > 2M. \end{cases}$$

Therefore
$$\int_{\{\delta\leq\varrho\}\cap Q_M} |\varrho_\varepsilon - \varrho| \ \mathrm{d}x\ \mathrm{d}t$$
$$\leq C(M,|Q|)\Big(\Big|\int_{\{\delta\leq\varrho\}\cap Q_M} (\log(\varrho)+1)(\varrho_\varepsilon - \varrho)\ \mathrm{d}x\ \mathrm{d}t\Big|$$
$$+\Big|\int_{\{\delta\leq\varrho\}\cap Q_M} (\varrho_\varepsilon\log(\varrho_\varepsilon) - \varrho\log(\varrho))\ \ \mathrm{d}x\ \mathrm{d}t\Big|\Big)$$
$$+C(M,|Q|)\Big(\Big|\int_{\{\delta\leq\varrho\}\cap Q_M} (\log(\varrho)+1)(\varrho_\varepsilon - \varrho)\ \mathrm{d}x\ \mathrm{d}t\Big|$$
$$+\Big|\int_{\{\delta\leq\varrho\}\cap Q_M} (\varrho_\varepsilon\log(\varrho_\varepsilon) - \varrho\log(\varrho))\ \ \mathrm{d}x\ \mathrm{d}t\Big|\Big)^{\frac{1}{2}}.$$

Since $\int_{\{(t,x)\in Q;\varrho(t,x)>M\}} |\varrho_\varepsilon - \varrho|\ \mathrm{d}x\ \mathrm{d}t \to 0$ for $M \to \infty$ uniformly in $\varepsilon \in (0,1)$, we conclude that
$$\varrho_\varepsilon \to \varrho \text{ a.a. on the set } \{\varrho \geq \delta\} \text{ for any } \delta > 0.$$

Now, since
$$\varrho_\varepsilon \to \varrho \text{ a.a. on the set } \{\varrho = 0\}$$
and
$$|\{0 < \varrho < \delta\}| \to 0 \text{ as } \delta \to 0,$$
we obtain the desired conclusion. □

In accordance with the previous discussion, the proof of strong (pointwise) convergence of $\{\varrho_\varepsilon\}_{\varepsilon>0}$ reduces to showing (6.24). This will be done in the next section.

6.5.3 The effective viscous flux

The effective viscous flux

$$(2\mu + \lambda)\mathrm{div}_x \mathbf{u} - p(\varrho)$$

is a remarkable quantity that enjoys better regularity and compactness properties than its components separately. To see this, we start with the momentum equation

$$\int_\Omega \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon(\tau,\cdot) \cdot \boldsymbol{\varphi}(\tau,\cdot) - (\varrho \mathbf{u})_{0,\varepsilon} \cdot \boldsymbol{\varphi}(0,\cdot) \Big)\ \mathrm{d}x \tag{6.26}$$
$$= \int_0^\tau \int_\Omega \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \boldsymbol{\varphi} + p(\varrho_\varepsilon)\mathrm{div}_x \boldsymbol{\varphi} \Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$- \int_0^\tau \int_\Omega \Big(\mu \nabla_x \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi} + (\lambda + \mu)\mathrm{div}_x \mathbf{u}_\varepsilon \mathrm{div}_x \boldsymbol{\varphi} \Big)\ \mathrm{d}x\ \mathrm{d}t,$$

together with its weak limit

$$\int_\Omega \Big(\varrho \mathbf{u}(\tau,\cdot) \cdot \boldsymbol{\varphi}(\tau,\cdot) - (\varrho \mathbf{u})_0 \cdot \boldsymbol{\varphi}(0,\cdot) \Big)\ \mathrm{d}x \tag{6.27}$$
$$= \int_0^\tau \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \boldsymbol{\varphi} + \overline{p(\varrho)}\mathrm{div}_x \boldsymbol{\varphi} \Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$- \int_0^\tau \int_\Omega \Big(\mu \nabla_x \mathbf{u} : \nabla_x \boldsymbol{\varphi} + (\lambda + \mu)\mathrm{div}_x \mathbf{u} \mathrm{div}_x \boldsymbol{\varphi} \Big)\ \mathrm{d}x\ \mathrm{d}t.$$

Our goal is to take

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_\varepsilon = \phi \nabla_x \Delta^{-1}[1_\Omega \varrho_\varepsilon],\ \phi \in C_c^\infty(\Omega)$$

as a test function in (6.26), and

$$\boldsymbol{\varphi} = \phi \nabla_x \Delta^{-1}[1_\Omega \varrho],\ \phi \in C_c^\infty(\Omega),$$

in (6.27).

Here, Δ^{-1} represents the inverse of the Laplacean for space-periodic functions for space periodic boundary conditions. For homogeneous Dirichlet boundary conditions it is possible to use the inverse Laplace operator introduced in (8.34). Since $\Omega \subset \mathbb{R}^3$ is a bounded domain, we have

$$\nabla_x \Delta^{-1}[1_\Omega \varrho_\varepsilon] \text{ bounded in } L^\infty(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^3)),\ \gamma > 3.$$

Moreover, as $1_\Omega \varrho_\varepsilon$ as well as $1_\Omega \varrho$ satisfy the equation of continuity, we have

$$\partial_t \nabla_x \Delta^{-1}[1_\Omega \varrho_\varepsilon] = -\nabla_x \Delta^{-1} \mathrm{div}_x [\varrho_\varepsilon \mathbf{u}_\varepsilon],\ \partial_t \nabla_x \Delta^{-1}[1_\Omega \varrho] = -\nabla_x \Delta^{-1} \mathrm{div}_x [\varrho \mathbf{u}].$$

Step 1: As

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\mathrm{weak}}([0,T]; L^\gamma(\Omega)),$$

we have, in accordance with the standard Sobolev embedding relation

$$W^{1,\gamma}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega}),$$

$$\nabla_x \Delta^{-1}[1_\Omega \varrho_\varepsilon] \to \nabla_x \Delta^{-1}[1_\Omega \varrho] \text{ in } C([0,T] \times \overline{\Omega}).$$

In particular, we deduce from (6.26), (6.27),

$$\lim_{\varepsilon \to 0} \Big[\int_0^\tau \int_\Omega \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi}_\varepsilon + \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi}_\varepsilon + p(\varrho_\varepsilon) \mathrm{div}_x \boldsymbol{\varphi}_\varepsilon \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \Big(\mu \nabla_x \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi}_\varepsilon + (\lambda + \mu) \mathrm{div}_x \mathbf{u}_\varepsilon \mathrm{div}_x \boldsymbol{\varphi}_\varepsilon \Big) \ \mathrm{d}x \Big] \ \mathrm{d}t$$

$$= \int_0^\tau \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \boldsymbol{\varphi} + \overline{p(\varrho)} \mathrm{div}_x \boldsymbol{\varphi} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \Big(\mu \nabla_x \mathbf{u} : \nabla_x \boldsymbol{\varphi} + (\lambda + \mu) \mathrm{div}_x \mathbf{u} \mathrm{div}_x \boldsymbol{\varphi} \Big) \ \mathrm{d}x \ \mathrm{d}t,$$

with

$$\boldsymbol{\varphi} = \phi \nabla_x \Delta^{-1}[1_\Omega \varrho],$$

similarly for $\boldsymbol{\varphi}_\varepsilon$. Therefore

$$\lim_{\varepsilon \to 0} \Big[\int_0^\tau \int_\Omega \Big(\phi p(\varrho_\varepsilon) \varrho_\varepsilon + p(\varrho_\varepsilon) \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega \varrho_\varepsilon] \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{6.28}$$

$$- \int_0^\tau \int_\Omega \phi \Big(\mu \nabla_x \mathbf{u}_\varepsilon : \nabla_x^2 \Delta^{-1}[1_\Omega \varrho_\varepsilon] + (\lambda + \mu) \mathrm{div}_x \mathbf{u}_\varepsilon \varrho_\varepsilon \Big) \ \mathrm{d}x \ \mathrm{d}t \Big]$$

$$- \lim_{\varepsilon \to 0} \int_0^\tau \int_\Omega \Big(\mu \nabla_x \mathbf{u}_\varepsilon \cdot \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega \varrho_\varepsilon]$$

$$+ (\lambda + \mu) \mathrm{div}_x \mathbf{u}_\varepsilon \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega \varrho_\varepsilon] \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \int_0^\tau \int_\Omega \Big(\phi \overline{p(\varrho)} \varrho + \overline{p(\varrho)} \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega \varrho] \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \phi \Big(\mu \nabla_x \mathbf{u} : \nabla_x^2 \Delta^{-1}[1_\Omega \varrho] + (\lambda + \mu) \mathrm{div}_x \mathbf{u} \varrho \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \Big(\mu \nabla_x \mathbf{u} \cdot \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega \varrho] + (\lambda + \mu) \mathrm{div}_x \mathbf{u} \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega \varrho] \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \lim_{\varepsilon \to 0} \int_0^\tau \int_\Omega \Big(\phi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Delta^{-1}[\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon)]$$

$$- \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \left(\phi \nabla_x \Delta^{-1}[1_\Omega \varrho_\varepsilon] \right) \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \Big(\phi \varrho \mathbf{u} \cdot \nabla_x \Delta^{-1}[\mathrm{div}_x(\varrho \mathbf{u})] - \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \left(\phi \nabla_x \Delta^{-1}[1_\Omega \varrho] \right) \Big) \, \mathrm{d}x \, \mathrm{d}t.$$

Step 2: We have

$$\int_\Omega \phi \nabla_x \mathbf{u}_\varepsilon : \nabla_x^2 \Delta^{-1}[1_\Omega \varrho_\varepsilon] \, \mathrm{d}x = \int_\Omega \phi \sum_{i,j=1}^3 \left(\partial_{x_j} u_\varepsilon^i [\partial_{x_i} \Delta^{-1} \partial_{x_j}][1_\Omega \varrho_\varepsilon] \right) \, \mathrm{d}x$$

$$= \int_\Omega \sum_{i,j=1}^3 \left(\partial_{x_j} (\phi u_\varepsilon^i) [\partial_{x_i} \Delta^{-1} \partial_{x_j}][1_\Omega \varrho_\varepsilon] \right) \, \mathrm{d}x$$

$$- \int_\Omega \sum_{i,j=1}^3 \left(\partial_{x_j} \phi u_\varepsilon^i [\partial_{x_i} \Delta^{-1} \partial_{x_j}][1_\Omega \varrho_\varepsilon] \right) \, \mathrm{d}x$$

$$= \int_\Omega \phi \mathrm{div}_x \mathbf{u}_\varepsilon \varrho_\varepsilon \, \mathrm{d}x + \int_\Omega \nabla_x \phi \cdot \mathbf{u}_\varepsilon \varrho_\varepsilon \, \mathrm{d}x - \int_\Omega \sum_{i,j=1}^3 \left(\partial_{x_j} \phi u_\varepsilon^i [\partial_{x_i} \Delta^{-1} \partial_{x_j}][1_\Omega \varrho_\varepsilon] \right) \, \mathrm{d}x.$$

Consequently, going back to (6.28) and dropping the compact terms, we obtain

$$\lim_{\varepsilon \to 0} \int_0^\tau \int_\Omega \phi \Big(p(\varrho_\varepsilon) \varrho_\varepsilon - (\lambda + 2\mu) \mathrm{div}_x \mathbf{u}_\varepsilon \varrho_\varepsilon \Big) \, \mathrm{d}x \, \mathrm{d}t \tag{6.29}$$

$$- \int_0^\tau \int_\Omega \phi \Big(\overline{p(\varrho)} \varrho - (\lambda + 2\mu) \mathrm{div}_x \mathbf{u} \varrho \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$= \lim_{\varepsilon \to 0} \int_0^\tau \int_\Omega \phi \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Delta^{-1}[\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon)]$$

$$-\varrho_\varepsilon(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \Delta^{-1} \nabla_x [1_\Omega \varrho_\varepsilon]\Big) \, dx \, dt$$

$$-\int_0^\tau \int_\Omega \phi\Big(\varrho \mathbf{u} \cdot \nabla_x \Delta^{-1} [\mathrm{div}_x(\varrho \mathbf{u})] - \varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla_x \Delta^{-1} \nabla_x [1_\Omega \varrho]\Big) \, dx \, dt.$$

Step 3: Our ultimate goal is to show that the right-hand side of (6.29) vanishes. To this end, we write

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Delta^{-1} [\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon)] - \varrho_\varepsilon(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \Delta^{-1} \nabla_x [1_\Omega \varrho_\varepsilon]$$

$$= \mathbf{u}_\varepsilon \cdot \Big(\varrho_\varepsilon \nabla_x \Delta^{-1} [\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon)] - \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Delta^{-1} \nabla_x [1_\Omega \varrho_\varepsilon]\Big).$$

Consider the bilinear form

$$[\mathbf{v}, \mathbf{w}] = \sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{i,j}[w^j] - w^i \mathcal{R}_{i,j}[v^j]\Big), \ \mathcal{R}_{i,j} = \partial_{x_i} \Delta^{-1} \partial_{x_j},$$

where we may write

$$\sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{i,j}[w^j] - w^i \mathcal{R}_{i,j}[v^j]\Big)$$

$$= \sum_{i,j=1}^{3} \Big((v^i - \mathcal{R}_{i,j}[v^j])\mathcal{R}_{i,j}[w^j] - (w^i - \mathcal{R}_{i,j}[w^j])\mathcal{R}_{i,j}[v^j]\Big)$$

$$= \mathbf{U} \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{Z},$$

where

$$U^i = \sum_{j=1}^{3} (v^i - \mathcal{R}_{i,j}[v^j]), \ W^i = \sum_{j=1}^{3} (w^i - \mathcal{R}_{i,j}[w^j]), \ \mathrm{div}_x \mathbf{U} = \mathrm{div}_x \mathbf{W} = 0,$$

and

$$V^i = \partial_{x_i} \left(\sum_{j=1}^{3} \Delta^{-1} \partial_{x^j} w^j \right), \ Z^i = \partial_{x_i} \left(\sum_{j=1}^{3} \Delta^{-1} \partial_{x^j} v^j \right), \ i = 1, 2, 3.$$

Thus a direct application of Div-Curl lemma (Lemma 6.1) yields

$$[\mathbf{v}_\varepsilon, \mathbf{w}_\varepsilon] \to [\mathbf{v}, \mathbf{w}] \text{ weakly in } L^s(\mathbb{R}^3)$$

whenever $\mathbf{v}_\varepsilon \to \mathbf{v}$ weakly in $L^p(\mathbb{R}^3;\mathbb{R}^3)$, $\mathbf{w}_\varepsilon \to \mathbf{w}$ weakly in $L^q(\mathbb{R}^3;\mathbb{R}^3)$, and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.$$

Seeing that

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T];L^\gamma(\Omega)),\ \ \varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho\mathbf{u} \text{ in } C_{\rm weak}([0,T];L^{2\gamma/(\gamma+1)}(\Omega;\mathbb{R}^3))$$

we conclude that (we use $\mathbf{v}$ with $v_i = \delta_{ik}\varrho$ for $k = 1,2,3$ and $\mathbf{w} = \varrho\mathbf{u}$; similarly for $\mathbf{v}_\varepsilon$ and $\mathbf{w}_\varepsilon$)

$$1_\Omega \varrho_\varepsilon(t,\cdot)\nabla_x\Delta^{-1}[{\rm div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon)(t,\cdot)] - (\varrho_\varepsilon\mathbf{u}_\varepsilon)(t,\cdot)\cdot\nabla_x\Delta^{-1}\nabla_x[1_\Omega\varrho_\varepsilon(t,\cdot)] \tag{6.30}$$
$$\to$$
$$\varrho(t,\cdot)\nabla_x\Delta^{-1}[{\rm div}_x(\varrho\mathbf{u})(t,\cdot)] - (\varrho\mathbf{u})(t,\cdot)\cdot\nabla_x\Delta^{-1}\nabla_x[1_\Omega\varrho(t,\cdot)]$$
$$\text{weakly in } L^s(\Omega;\mathbb{R}^3) \text{ for all } t \in [0,T],$$

with

$$s = \frac{2\gamma}{\gamma+3} > \frac{6}{5} \text{ since } \gamma \geq 5.$$

Thus we conclude that the convergence in (6.30) takes place in the space

$$L^q(0,T;W^{-1,2}(\Omega;\mathbb{R}^3)) \text{ for any } 1 \leq q < \infty;$$

whence, going back to (6.29), we conclude

$$\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega \phi\Big(p(\varrho_\varepsilon)\varrho_\varepsilon - (\lambda+2\mu){\rm div}_x\mathbf{u}_\varepsilon\varrho_\varepsilon\Big)\ {\rm d}x\ {\rm d}t \tag{6.31}$$
$$= \int_0^\tau\int_\Omega \phi\Big(\overline{p(\varrho)}\varrho - (\lambda+2\mu){\rm div}_x\mathbf{u}\varrho\Big)\ {\rm d}x\ {\rm d}t.$$

As a matter of fact, using exactly same method and localizing also in the space variable, we could prove that a.e. in $(0,T)\times\Omega$

$$\overline{p(\varrho)\varrho} - (\lambda+2\mu)\overline{{\rm div}_x\mathbf{u}\varrho} = \overline{p(\varrho)}\varrho - (\lambda+2\mu){\rm div}_x\mathbf{u}\varrho, \tag{6.32}$$

which is the celebrated relation on "weak continuity" of the effective viscous pressure discovered by Lions [14].

Since p is a non-decreasing function, we have

$$\int_0^\tau \int_\Omega \Big(p(\varrho_\varepsilon) - p(\varrho) \Big) (\varrho_\varepsilon - \varrho) \ {\rm d}x \ {\rm d}t \geq 0;$$

now relation (6.31) yields the desired conclusion (6.24), namely

$$\int_0^\tau \int_\Omega \Big(\overline{{\rm div}_x \mathbf{u} \varrho} - {\rm div}_x \mathbf{u} \varrho \Big) \ {\rm d}x \ {\rm d}t \geq 0.$$

Thus we get (6.25); whence

$$\varrho_\varepsilon \to \varrho \text{ a.a. in } (0,T) \times \Omega \tag{6.33}$$

and in $L^q((0,T) \times \Omega)$ for any $q < \gamma + \min\left\{ \frac{\gamma}{2}, \frac{2}{3}\gamma - 1 \right\}$.

Part III

Existence of weak solutions for small γ

Last part of the lecture series is devoted to the proof of existence of weak solutions to the compressible Navier–Stokes system provided $p(\varrho) \sim \varrho^\gamma$ with $\gamma > \frac{3}{2}$. The proof is technically much more complicated than the previous part, however, there are several places which are quite similar to it. Moreover, in the following chapter we also prove several facts (renormalized solution to the continuity equation, continuity in time, estimates of the density etc.) which we skipped in the previous part due to technical complications we tried to avoid there.

We first show the Friedrichs commutator lemma which plays a central role in the study of renormalized solutions to the continuity equation. Next we consider the continuity in time of the density and the momentum. The last chapter contains the core of the existence proof: the approximate problem, existence of a solution for fixed positive regularizing parameters and finally the limit passages which give us solution to our original problem. Note that the proof is performed for the homogeneous Dirichlet boundary conditions for the velocity. The presentation of this part is mostly based on the material from book [16] by A. Novotný and I. Straškraba. For another approach, based on the construction of the approximative problem via a numerical method, see [9].

Chapter 7

Mathematical tools

7.1 Continuity equation: renormalized solutions and extension

We recall that for a function $f \in L^p(\mathbb{R}; L^q(\mathbb{R}^N))$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, or $f \in C(\mathbb{R}; L^q(\mathbb{R}^N)$ we can define the mollifiers:

- over time
$$T_\varepsilon(f)(t,x) = \int_{\mathbb{R}} \omega_\varepsilon(t-\tau) f(\tau,x)\,\mathrm{d}\tau;$$
we have for $1 \leq q \leq \infty$
$$\begin{aligned}&T_\varepsilon(f) \in C^\infty(\mathbb{R}; L^q(\mathbb{R}^N)),\\ &T_\varepsilon(f) \to f \quad \text{in } L^p(\mathbb{R}; L^q(\mathbb{R}^N)) \text{ if } f \in L^p(\mathbb{R}; L^q(\mathbb{R}^N)), \quad 1 \leq p < \infty,\\ &T_\varepsilon(f) \to f \quad \text{in } C(\mathbb{R}; L^q(\mathbb{R}^N)) \text{ if } f \in C_B(\mathbb{R}; L^q(\mathbb{R}^N));\end{aligned}$$
moreover
$$\|T_\varepsilon(f)\|_{L^p(\mathbb{R};L^q(\mathbb{R}^N))} \leq \|f\|_{L^p(\mathbb{R};L^q(\mathbb{R}^N))}, \quad 1 \leq p \leq \infty$$

- over space
$$S_\varepsilon(f)(t,x) = \int_{\mathbb{R}^N} \omega_\varepsilon(x-y) f(t,y)\,\mathrm{d}y;$$
then we have for $1 \leq p \leq \infty$
$$\begin{aligned}&S_\varepsilon(f) \in L^p(\mathbb{R}; C^\infty(\mathbb{R}^N)),\\ &S_\varepsilon(f) \to f \quad \text{in } L^p(\mathbb{R}; L^q(\mathbb{R}^N)) \text{ if } f \in L^p(\mathbb{R}; L^q(\mathbb{R}^N)), \quad 1 \leq q < \infty;\end{aligned}$$

moreover

$$\|S_\varepsilon(f)\|_{L^p(\mathbb{R};L^q(\mathbb{R}^N))} \leq \|f\|_{L^p(\mathbb{R};L^q(\mathbb{R}^N))}, \quad 1 \leq q \leq \infty$$

A central technical result is the Friedrichs commutator lemma.

Lemma 7.1 *Let $N \geq 2$, $1 \leq q, \beta \leq \infty$, $(q,\beta) \neq (1,\infty)$, $\frac{1}{q} + \frac{1}{\beta} \leq 1$. Let $1 \leq \alpha \leq \infty$, $\frac{1}{p} + \frac{1}{\alpha} \leq 1$. Assume for $I \subset \mathbb{R}$ a bounded time interval*

$$\varrho \in L^\alpha(I; L^\beta_{\text{loc}}(\mathbb{R}^N)), \quad \mathbf{u} \in L^p(I; W^{1,q}_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)).$$

Then

$$S_\varepsilon(\mathbf{u}\cdot\nabla_x\varrho) - \mathbf{u}\cdot\nabla_x S_\varepsilon(\varrho) \to 0 \quad \text{in } L^s(I; L^r_{\text{loc}}(\mathbb{R}^N)).$$

Here, $\frac{1}{s} = \frac{1}{\alpha} + \frac{1}{p}$ and $r \in [1,q)$ if $\beta = \infty$ and $q \in (1,\infty]$, $\frac{1}{q} + \frac{1}{\beta} \leq \frac{1}{r} \leq 1$ otherwise, where

$$\mathbf{u}\cdot\nabla_x\varrho := \operatorname{div}_x(\varrho\mathbf{u}) - \varrho\operatorname{div}_x(\mathbf{u}) \quad (\text{in } \mathcal{D}'(\mathbb{R}^N)).$$

Proof: To simplify, we consider only the case $\beta, q < \infty$ which is enough for our purpose.
Step 1: We have

$$\begin{aligned}
&\Big\langle S_\varepsilon(\mathbf{u}\cdot\nabla_x\varrho), \varphi\Big\rangle \\
&\qquad = \int_0^T \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \varrho(t,y)\mathbf{u}(t,y)\cdot\nabla_x\omega_\varepsilon(x-y)\,\mathrm{d}y\Big)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad - \int_0^T \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \varrho(t,y)\operatorname{div}\mathbf{u}(t,y)\omega_\varepsilon(x-y)\,\mathrm{d}y\Big)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

$$\Big\langle \mathbf{u}\cdot\nabla_x S_\varepsilon(\varrho), \varphi\Big\rangle = \int_0^T \int_{\mathbb{R}^N} \mathbf{u}(t,x)\cdot\Big(\int_{\mathbb{R}^N} \nabla_x\omega_\varepsilon(x-y)\varrho(t,y)\,\mathrm{d}y\Big)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Therefore

$$\Big\langle S_\varepsilon(\mathbf{u}\cdot\nabla_x\varrho) - \mathbf{u}\cdot\nabla_x S_\varepsilon(\varrho), \varphi\Big\rangle = \int_0^T \int_{\mathbb{R}^N} \Big(I_\varepsilon(t,x) - J_\varepsilon(t,x)\Big)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

with

$$I_\varepsilon(t,x) = \int_{\mathbb{R}^N} \varrho(t,y)\Big(\mathbf{u}(t,y) - \mathbf{u}(t,x)\Big)\cdot\nabla_x\omega_\varepsilon(x-y)\,\mathrm{d}y,$$

and

$$J_\varepsilon(t,x) = \int_{\mathbb{R}^N} \varrho(t,y)\mathrm{div}_x\mathbf{u}(t,y)\omega_\varepsilon(x-y)\,\mathrm{d}y.$$

We define r_0 as

$$\frac{1}{r_0} = \frac{1}{\beta} + \frac{1}{q}$$

and get

$$J_\varepsilon \to \varrho\mathrm{div}_x\mathbf{u} \quad \text{strongly in } L^s(I; L^{r_0}_{\mathrm{loc}}(\mathbb{R}^N)).$$

In Steps 2, 3 and 4 we show that

$$I_\varepsilon \to \varrho\mathrm{div}_x\mathbf{u} \quad \text{strongly in } L^s(I; L^{r_0}_{\mathrm{loc}}(\mathbb{R}^N))$$

which will finish the proof of this lemma.

Step 2: We aim at proving

$$\|I_\varepsilon\|_{L^{r_0}(B_R)} \le C\|\varrho(t)\|_{L^\beta(B_{R+1})}\|\nabla_x\mathbf{u}(t)\|_{L^q(B_{R+2};\mathbb{R}^{N\times N})} \quad \text{for a.a. } t\in(0,T).$$

We have

$$
\begin{aligned}
&\|I_\varepsilon\|^{r_0}_{L^{r_0}(B_R)} \\
&= \int_{B_R} \left| \int_{|x-y|\le\varepsilon} \varrho(t,y)\Big(\mathbf{u}(t,y)-\mathbf{u}(t,x)\Big)\frac{1}{\varepsilon^{N+1}}\cdot\nabla\omega\left(\frac{x-y}{\varepsilon}\right)\mathrm{d}y\right|^{r_0}\mathrm{d}x \\
&= \int_{B_R} \left| \int_{|z|\le 1} \varrho(t,x-\varepsilon z)\frac{\mathbf{u}(t,x-\varepsilon z)-\mathbf{u}(t,x)}{\varepsilon}\cdot\nabla\omega(z)\,\mathrm{d}z\right|^{r_0}\mathrm{d}x \\
&\le \left(\int_{|z|\le 1}|\nabla\omega(z)|^{r_0'}\,\mathrm{d}z\right)^{\frac{r_0}{r_0'}}\times \\
&\times \int_{B_R}\int_{|z|\le 1}|\varrho(t,x-\varepsilon z)|^{r_0}\left|\frac{\mathbf{u}(t,x-\varepsilon z)-\mathbf{u}(t,x)}{\varepsilon}\right|^{r_0}\mathrm{d}z\,\mathrm{d}x \\
&\le C(\omega)\int_{B_{R+1}}\int_{|z|\le 1}|\varrho(t,\xi)|^{r_0}\left|\frac{\mathbf{u}(t,\xi+\varepsilon z)-\mathbf{u}(t,\xi)}{\varepsilon}\right|^{r_0}\mathrm{d}z\,\mathrm{d}\xi \quad (\text{for } \varepsilon<1) \\
&\le C(\omega)|B_1|^{\frac{r_0}{\beta}}\int_{B_{R+1}}|\varrho(t,\xi)|^{r_0}\left(\int_{|z|\le 1}\left|\frac{\mathbf{u}(t,\xi+\varepsilon z)-\mathbf{u}(t,\xi)}{\varepsilon}\right|^{q}\mathrm{d}z\right)^{\frac{r_0}{q}}\mathrm{d}\xi \\
&\le C(\omega,N,q,\beta)\left(\int_{B_{R+1}}|\varrho(t,\xi)|^{\beta}\,\mathrm{d}\xi\right)^{\frac{r_0}{\beta}}\times \\
&\times\left(\int_{B_{R+1}}\int_{|z|\le 1}\left|\frac{\mathbf{u}(t,\xi+\varepsilon z)-\mathbf{u}(t,\xi)}{\varepsilon}\right|^{q}\mathrm{d}z\,\mathrm{d}\xi\right)^{\frac{r_0}{q}} \\
&\le C\|\varrho(t)\|_{L^\beta(B_{R+1})}\|\nabla_x\mathbf{u}(t)\|_{L^q(B_{R+2};\mathbb{R}^{N\times N})}.
\end{aligned}
$$

Step 3: Let us show the strong convergence, first for a.a. $t\in(0,T)$, in $L^{r_0}_{\mathrm{loc}}(\mathbb{R}^N)$. Due to Step 2 it is enough to verify that the strong convergence holds for any $\varrho\in C^\infty_c(\mathbb{R}^N)$, $t\in(0,T)$ fixed. Indeed, let $\varrho_n\in C^\infty_c(\mathbb{R}^N)$, $\varrho_n\to\varrho(t,\cdot)$ in $L^\beta(B_{R+1})$. Then

$$
\begin{aligned}
&\|(I_\varepsilon-\varrho\,\mathrm{div}_x\mathbf{u})(t,\cdot)\|_{L^{r_0}(B_R)} \\
&\le \left\|\int_{\mathbb{R}^N}\Big(\varrho(t,y)-\varrho_n(y)\Big)\Big(\mathbf{u}(t,y)-\mathbf{u}(t,\cdot)\Big)\cdot\nabla_x\omega_\varepsilon(\cdot-y)\,\mathrm{d}y\right\|_{L^{r_0}(B_R)} \\
&+ \left\|\int_{\mathbb{R}^N}\varrho_n(y)\Big(\mathbf{u}(t,y)-\mathbf{u}(t,\cdot)\Big)\cdot\nabla_x\omega_\varepsilon(\cdot-y)\,\mathrm{d}y-(\varrho_n\mathrm{div}_x\mathbf{u}(t,\cdot))\right\|_{L^{r_0}(B_R)} \\
&+ \left\|\Big(\varrho_n-\varrho(t,\cdot)\Big)\mathrm{div}_x\mathbf{u}(t,\cdot)\right\|_{L^{r_0}(B_R)}.
\end{aligned}
$$

The first term is bounded by (see also the treatment of the second term, below)

$$C\|\varrho_n - \varrho(t,\cdot)\|_{L^\beta(B_{R+1})}\|\nabla_x \mathbf{u}(t,\cdot)\|_{L^q(B_{R+2};\mathbb{R}^{N\times N})} \to 0 \quad \text{for } n\to\infty,$$

the third is bounded by

$$C\|\varrho_n - \varrho(t,\cdot)\|_{L^\beta(B_R)}\|\mathrm{div}_x \mathbf{u}(t,\cdot)\|_{L^q(B_R;\mathbb{R}^{N\times N})} \to 0 \quad \text{for } n\to\infty.$$

To conclude, let ϱ be a smooth function. Using the change of variables $z = \frac{x-y}{\varepsilon}$, as above,

$$\widetilde{I}_\varepsilon(t,x) = \int_{|z|\le 1} \varrho(t,x-\varepsilon z)\left(\frac{\mathbf{u}(t,x-\varepsilon z)-\mathbf{u}(t,x)}{\varepsilon}\right)\cdot\nabla\omega(z)\,\mathrm{d}z.$$

As $\mathbf{u}\in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ for a.a. $t\in(0,T)$,

$$\frac{\mathbf{u}(t,x-\varepsilon z)-\mathbf{u}(t,x)}{\varepsilon} = -z\cdot\int_0^1 \nabla_x\mathbf{u}(t,x-\varepsilon\tau z)\,\mathrm{d}\tau \to -z\cdot\nabla_x\mathbf{u}(t,x)$$

for a.a. $t\in(0,T)$ and a.a. $(x,z)\in\mathbb{R}^N\times B_1$ (a.a. points are Lebesgue points). Moreover, as ϱ is smooth, $\varrho(t,x-\varepsilon z)\to\varrho(t,x)$, $(x,z)\in B_{R+1}\times B_1$, $t\in(0,T)$. Therefore, by Vitali's theorem

$$\begin{aligned}\int_{\mathbb{R}^N}(\widetilde{I}_\varepsilon\varphi)(t,x)\,\mathrm{d}x \quad &\to \quad -\int_{B_1} z_i\partial_j\omega(z)\,\mathrm{d}z\int_{\mathbb{R}^N}\varrho(t,x)\partial_i\mathbf{u}_j(t,x)\varphi(t,x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}(\varrho\,\mathrm{div}_x\mathbf{u})(t,x)\varphi(t,x)\,\mathrm{d}x.\end{aligned}$$

Step 4: We have

$$\begin{aligned}\|I_\varepsilon\|^s_{L^s(I;L^{r_0}(B_R))} &\le C\int_0^T \|\varrho(t,\cdot)\|^s_{L^\beta(B_{R+1})}\|\nabla_x\mathbf{u}(t,\cdot)\|^s_{L^q(B_{R+2};\mathbb{R}^{N\times N})}\,\mathrm{d}t\\ &\le C\|\varrho\|^s_{L^\alpha(I;L^\beta(B_{R+1}))}\|\nabla_x\mathbf{u}\|^s_{L^p(I;L^q(B_{R+2};\mathbb{R}^{N\times N}))}.\end{aligned}$$

Due to this and the fact that

$$I_\varepsilon \to \varrho\,\mathrm{div}_x\mathbf{u} \quad \text{in } L^{r_0}_{\mathrm{loc}}(\mathbb{R}^N) \text{ for a.a. } t\in(0,T),$$

we get due to Step 2 by the Lebesgue dominated convergence theorem the claim of the lemma. □

Next we show that we may extend (sufficiently regular) solution to the continuity equation outside a Lipschitz domain in such a way that the extension (by zero in the case of our boundary conditions) solves the continuity equation in the full $\mathbb{R}^N$. In particular, this shows that we may use as test functions smooth functions up to the boundary.

Lemma 7.2 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geq 2$, $I \subset \mathbb{R}$ be an open interval and $Q_T = I \times \Omega$. Let $\varrho \in L^2(Q_T)$, $\mathbf{u} \in L^2(I; W_0^{1,2}(\Omega; \mathbb{R}^N))$ and $f \in L^1(Q_T)$ satisfy*

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = f \quad \textit{in } \mathcal{D}'(Q_T).$$

Extending $(\varrho, \mathbf{u}, f)$ by $(0, \mathbf{0}, 0)$ outside Ω,

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = f \quad \textit{in } \mathcal{D}'(I \times \mathbb{R}^N).$$

Proof: We have to show (after the extension by $(0, \mathbf{0}, 0)$)

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^N} \varrho \partial_t \eta \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}^N} \varrho \mathbf{u} \cdot \nabla_x \eta \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^N} f \eta \,\mathrm{d}x\,\mathrm{d}t \quad \forall \eta \in C_c^\infty((0,T) \times \mathbb{R}^N). \end{aligned}$$

Denote

$$\Phi_m \in C_c^\infty(\Omega),\ m \in \mathbb{N},\ 0 \leq \Phi_m \leq 1,$$

$$\Phi_m(x) = 1 \text{ for } x \in \left\{ y \in \Omega;\ \mathrm{dist}(y, \partial\Omega) \geq \frac{1}{m} \right\},$$

$$|\nabla_x \Phi_m(x)| \leq 2m \text{ for } x \in \Omega.$$

Evidently, $\Phi_m \to 1$ pointwise in Ω and for any fixed compact $K \subset \Omega$,

$$\mathrm{supp}\, \nabla_x \Phi_m \subset \Omega \setminus K \text{ for } m \geq m_0(K) \in \mathbb{N}, \quad |\mathrm{supp}\, \nabla_x \Phi_m| \to 0.$$

We can write

$$\int_0^T \int_{\mathbb{R}^N} f \eta \,\mathrm{d}x\ \mathrm{d}t = \int_0^T \int_{\mathbb{R}^N} f \Phi_m \eta \,\mathrm{d}x\ \mathrm{d}t + \underbrace{\int_0^T \int_{\mathbb{R}^N} f \eta (1 - \Phi_m) \,\mathrm{d}x\ \mathrm{d}t}_{\to 0},$$

$$\int_0^T \int_{\mathbb{R}^N} \varrho \partial_t \eta \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_{\mathbb{R}^N} \varrho \partial_t (\Phi_m \eta) \, \mathrm{d}x \, \mathrm{d}t + \underbrace{\int_0^T \int_{\mathbb{R}^N} \varrho \partial_t \eta (1 - \Phi_m) \, \mathrm{d}x \, \mathrm{d}t}_{\to 0},$$

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} \varrho \mathbf{u} \cdot \nabla_x \eta \, \mathrm{d}x \, \mathrm{d}t &= \int_0^T \int_{\mathbb{R}^N} \varrho \mathbf{u} \cdot \nabla_x (\Phi_m \eta) \, \mathrm{d}x \, \mathrm{d}t \\ &+ \underbrace{\int_0^T \int_{\mathbb{R}^N} \varrho \mathbf{u} \cdot \nabla_x \eta (1 - \Phi_m) \, \mathrm{d}x \, \mathrm{d}t}_{\to 0} - \int_0^T \int_{\mathbb{R}^N} \varrho \mathbf{u} \cdot \nabla_x \Phi_m \eta \, \mathrm{d}x \, \mathrm{d}t.\end{aligned}$$

We know that

$$\int_0^T \int_{\mathbb{R}^N} \mathbf{f} \eta \Phi_m \, \mathrm{d}x \, \mathrm{d}t = - \int_0^T \int_{\mathbb{R}^N} \varrho \partial_t (\Phi_m \eta) \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{\mathbb{R}^N} \varrho \mathbf{u} \cdot \nabla_x (\Phi_m \eta) \, \mathrm{d}x \, \mathrm{d}t$$

as $\Phi_m \eta$ has support in Ω. Therefore we have to show that

$$I_m = \int_0^T \int_{\mathbb{R}^N} \varrho \mathbf{u} \cdot \nabla_x \Phi_m \eta \, \mathrm{d}x \, \mathrm{d}t \to 0.$$

But due to the Hardy inequality

$$\begin{aligned} |I_m| &\leq \int_0^T \int_{\mathbb{R}^N} |\varrho| |\mathbf{u}| |\nabla_x \Phi_m| |\eta| \, \mathrm{d}x \, \mathrm{d}t \\ &\leq 2 \sup_{t,x} |\eta(t,x)| \int_0^T \|\varrho\|_{L^2(\{\operatorname{supp} \nabla_x \Phi_m\})} \left\| \frac{\mathbf{u}}{\operatorname{dist}(x, \partial\Omega)} \right\|_{L^2(\Omega;\mathbb{R}^N)} \mathrm{d}t \\ &\leq C(\eta, \Omega) \int_0^T \|\varrho\|_{L^2(\{\operatorname{supp} \nabla_x \Phi_m\})} \|\mathbf{u}\|_{W_0^{1,2}(\Omega;\mathbb{R}^N)} \, \mathrm{d}t \\ &\leq C(\eta, \Omega) \|\varrho\|_{L^2(0,T;L^2(\{\operatorname{supp} \nabla_x \Phi_m\}))} \|\mathbf{u}\|_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^N))} \to 0 \end{aligned}$$

as $m \to \infty$. The lemma is proved. $\square$

Remark 7.1 Hence, in case $\varrho \in L^2((0,T) \times \Omega)$ (as $\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3))$ will be satisfied), we further have

$$\int_\Omega \varrho(t,x) \, \mathrm{d}x = \int_\Omega \varrho(s,x) \, \mathrm{d}x \quad \text{for any } t, s \in [0,T].$$

It is enough to take $\eta \equiv 1$ in $[s,t] \times \overline{\Omega}$, provided ϱ is weakly continuous in $L^1(\Omega)$ (which will be proved later). On the other hand, if only $\varrho \in L^p(Q_T)$, $1 \leq p < 2$, the mass may not be conserved.

An explicit counterexample (due to E. Feireisl and H. Petzeltová) shows this, even in 1D. Let $\Omega = (0,1)$ and

$$u(x) = \Big(x(1-x)\Big)^{\alpha}, \quad \varrho(t,x) = \frac{1}{u(x)} h\left(t - \int_0^x \frac{1}{u(y)}\,\mathrm{d}y\right), \quad \frac{1}{2} < \alpha < 1,$$

with $h \in C^1(\mathbb{R})$, $h(s) = 0$ for $s \leq 0$. Evidently

$$u \in W_0^{1,2}(0,1), \qquad \begin{cases} u' \sim x^{\alpha-1}, & x \to 0 \implies \alpha > \frac{1}{2}, \\ u' \sim (1-x)^{\alpha-1}, & x \to 1 \implies \alpha > \frac{1}{2}, \end{cases}$$

$$\varrho \in C([0,T]; L^p(0,1)),\ 1 \leq p < \frac{1}{\alpha} \quad \left(\frac{1}{u} \in L^p,\ p < \frac{1}{\alpha}\right) \quad \Rightarrow \alpha < 1,$$

and

$$\left.\begin{array}{l} \partial_t \varrho = \frac{1}{u(x)} h'\left(t - \int_0^x \frac{1}{u(y)}\,\mathrm{d}y\right) \\ \partial_x(\varrho u) = h'\left(t - \int_0^x \frac{1}{u(y)}\,\mathrm{d}y\right) \frac{-1}{u(x)} \end{array}\right\} \Rightarrow \partial_t \varrho + \partial_x(\varrho u) = 0.$$

But

$$\int_\Omega \varrho(t,x)\,\mathrm{d}x \text{ is not constant, as}$$

$$\int_\Omega \varrho(0,x)\,\mathrm{d}x = 0, \text{ but for } h \text{ suitably chosen } \int_\Omega \varrho(t,x)\,\mathrm{d}x \neq 0\,\forall t > 0.$$

This example can also be generalized to higher space dimensions.

We finish this section by showing that, under certain regularity assumptions, a weak solution to the continuity equation is also a renormalized solution. This fact will be important in the proof of the existence of weak solution in the last chapter.

Due to Lemma 7.1 we have

Lemma 7.3 *Let $N \geq 2$, $2 \leq \beta < \infty$, $\lambda_0 < 1$, $-1 < \lambda_1 \leq \frac{\beta}{2} - 1$ and*

$$b \in C([0,\infty)) \cap C^1((0,\infty)), \qquad |b'(t)| \leq ct^{-\lambda_0},\ t \in [0,1], \tag{7.1}$$

$$|b'(t)| \leq ct^{\lambda_1},\ t \geq 1. \tag{7.2}$$

Let $\varrho \in L^\beta(I; L^\beta_{\mathrm{loc}}(\mathbb{R}^N))$, $\varrho \geq 0$ a.e. in $I \times \mathbb{R}^N$, $\mathbf{u} \in L^2(I; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N))$ and $f \in L^z(I; L^z_{\mathrm{loc}}(\mathbb{R}^N))$, $z = \frac{\beta}{\beta-\lambda_1}$ if $\lambda_1 > 0$, $z = 1$ if $\lambda_1 \leq 0$. Suppose that

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = f \quad \text{in } \mathcal{D}'(I \times \mathbb{R}^N). \tag{7.3}$$

(i) Then for any $b \in C^1([0,\infty))$ satisfying (7.2) we have

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + \{\varrho b'(\varrho) - b(\varrho)\}\mathrm{div}_x\mathbf{u} = f b'(\varrho) \quad \text{in } \mathcal{D}'(I \times \mathbb{R}^N). \tag{7.4}$$

(ii) If $f = 0$, then (7.4) holds for any b satisfying (7.1) and (7.2).

Proof: We consider only case (ii), leaving (i) as possible exercise for the reader.

We regularize (7.3) over space variable and get

$$\partial_t S_\varepsilon(\varrho) + \mathrm{div}_x(S_\varepsilon(\varrho)\mathbf{u}) = r_\varepsilon(\varrho,\mathbf{u}) \quad \text{a.e. in } I \times \mathbb{R}^N, \tag{7.5}$$

where

$$r_\varepsilon(\varrho,\mathbf{u}) = \mathrm{div}_x(S_\varepsilon(\varrho)\mathbf{u}) - \mathrm{div}_x(S_\varepsilon(\varrho\mathbf{u})).$$

But

$$\begin{aligned} r_\varepsilon(\varrho,\mathbf{u}) &= \mathbf{u}\cdot\nabla_x S_\varepsilon(\varrho) + S_\varepsilon(\varrho)\mathrm{div}_x\mathbf{u} - S_\varepsilon(\mathrm{div}_x(\varrho\mathbf{u})) \\ &= \mathbf{u}\cdot\nabla_x S_\varepsilon(\varrho) - S_\varepsilon(\mathbf{u}\cdot\nabla_x\varrho) + S_\varepsilon(\varrho)\mathrm{div}_x\mathbf{u} - S_\varepsilon(\varrho\mathrm{div}_x\mathbf{u}), \end{aligned}$$

hence by Lemma 7.1 and an easy observation

$$r_\varepsilon(\varrho,\mathbf{u}) \to 0 \text{ in } L^r(I;L^r_{\mathrm{loc}}(\mathbb{R}^N)), \quad \frac{1}{r} = \frac{1}{\beta} + \frac{1}{2}(\le 1).$$

To avoid singularity at $\varrho = 0$, we multiply (7.5) by $b_h'(S_\varepsilon(\varrho))$ with $b_h(\cdot) = b(h+\cdot)$, $h > 0$, and obtain

$$\begin{aligned} \partial_t b_h(S_\varepsilon(\varrho)) + \mathrm{div}_x(b_h(S_\varepsilon(\varrho))\mathbf{u}) + \Big(S_\varepsilon(\varrho)b_h'(S_\varepsilon(\varrho)) - b_h(S_\varepsilon(\varrho))\Big)\mathrm{div}_x\mathbf{u} \\ = \ r_\varepsilon b_h'(S_\varepsilon(\varrho)) \quad \text{a.e. in } I \times \mathbb{R}^N. \end{aligned}$$

Now we pass with $\varepsilon \to 0^+$. As $S_\varepsilon(\varrho) \to \varrho$ in $L^\beta(I;L^\beta_{\mathrm{loc}}(\mathbb{R}^N))$ (i.e. for a subsequence a.e. in $I \times \mathbb{R}^N$), we get by Vitali's (convergence) theorem that $b_h(S_\varepsilon(\varrho)) \to b_h(\varrho)$,

$$S_\varepsilon(\varrho)b_h'(S_\varepsilon(\varrho)) - b_h(S_\varepsilon(\varrho)) \to \varrho b_h'(\varrho) - b_h(\varrho) \quad \text{in } L^p_{\mathrm{loc}}(I \times \mathbb{R}^N),\ 1 \le p < 2$$

($S_\varepsilon(\varrho)b_h'(S_\varepsilon(\varrho)) \le C S_\varepsilon(\varrho)^{1+\frac{\beta}{2}-1}$ for $S_\varepsilon(\varrho) \gg 1$). As this term is bounded also in $L^2(I \times \Omega')$ for Ω' bounded subset of $\mathbb{R}^N$, then the convergence holds also in the weak sense in $L^2(I \times \Omega')$. Therefore, passing with $\varepsilon \to 0$, we have

$$\partial_t b_h(\varrho) + \mathrm{div}_x(b_h(\varrho)\mathbf{u}) + \Big(\varrho b_h'(\varrho) - b_h(\varrho)\Big)\mathrm{div}_x\mathbf{u} = 0,$$

as

$$\Big|\int_0^T\int_{\Omega'} r_\varepsilon b_h'(S_\varepsilon(\varrho))\,\mathrm{d}x\,\mathrm{d}t\Big| \le \int_0^T \|r_\varepsilon\|_{L^r(\Omega')}\|b_h'(S_\varepsilon(\varrho))\|_{L^{r'}(\Omega')}\,\mathrm{d}t \to 0,$$

where $\frac{1}{r'} = 1-\frac{1}{r} = \frac{1}{2}-\frac{1}{\beta} = \frac{\beta-2}{2\beta}$ and $\|b_h'(S_\varepsilon(\varrho))\|_{L^r(\Omega')} \le \|S_\varepsilon(\varrho)\|_{L^\beta(\Omega')}$.

Finally we aim to pass with $h\to 0^+$. Recall that

$$|\{(t,x);\varrho\ge k\}\cap(I\times\Omega')| \le k^{-\beta}\|\varrho\|^\beta_{L^\beta((I\times\Omega')\cap\{\varrho\ge k\})}.$$

Then we write for $\psi\in C_c^\infty(I\times\mathbb{R}^N)$

$$\begin{aligned}\int_{I\times\mathbb{R}^N}&\Big(\varrho b_h'(\varrho)-b_h(\varrho)\Big)\mathrm{div}_x\mathbf{u}\psi\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{(I\times\mathbb{R}^N)\cap\{\varrho\le k\}\cap\mathrm{supp}\,\psi}\Big(\varrho b_h'(\varrho)-b_h(\varrho)\Big)\mathrm{div}_x\mathbf{u}\psi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{(I\times\mathbb{R}^N)\cap\{\varrho> k\}\cap\mathrm{supp}\,\psi}\Big(\varrho b_h'(\varrho)-b_h(\varrho)\Big)\mathrm{div}_x\mathbf{u}\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Now, passing with $h\to 0^+$, the first term on the right-hand side goes to

$$\int_{I\times\mathbb{R}^N}\Big(\varrho b'(\varrho)-b(\varrho)\Big)\mathrm{div}_x\mathbf{u}\psi 1_{\{\varrho\le k\}}\,\mathrm{d}x\,\mathrm{d}t,$$

due to the Lebesgue dominated convergence theorem. The second term can be controlled by

$$\begin{aligned}C\int_{\{\varrho>k\}}&\Big(\varrho(\varrho+h)^{\frac{\beta}{2}-1}+\varrho^{\frac{\beta}{2}}\Big)\,|\mathrm{div}_x\mathbf{u}|\,|\psi|\,\mathrm{d}x\,\mathrm{d}t\\ &\le C\int_{\{\varrho\ge k\}}\Big(\varrho^{\frac{\beta}{2}}+\varrho\Big)\,|\mathrm{div}_x\mathbf{u}||\psi|\,\mathrm{d}x\,\mathrm{d}t\\ &\le C\Big(\|\varrho\|^{\frac{\beta}{2}}_{L^\beta((I\times\Omega')\cap\{\varrho\ge k\})}\|\mathrm{div}_x\mathbf{u}\|_{L^2(I\times\Omega')}\\ &\qquad+\|\varrho\|^{\frac{\beta}{2}}_{L^\beta((I\times\Omega')\cap\{\varrho\ge k\})}k^{1-\frac{\beta}{2}}\|\mathrm{div}_x\mathbf{u}\|_{L^2(I\times\Omega')}\Big)\to_{k\to\infty}0.\end{aligned}$$

Further,

$$\begin{aligned}\int_{I\times\mathbb{R}^N}&\Big(\varrho b'(\varrho)-b(\varrho)\Big)\mathrm{div}_x\mathbf{u}\psi 1_{\{\varrho\le k\}}\,\mathrm{d}x\,\mathrm{d}t\\ &\to_{k\to\infty}\int_{I\times\mathbb{R}^N}\Big(\varrho b'(\varrho)-b(\varrho)\Big)\mathrm{div}_x\mathbf{u}\psi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

by the Lebesgue dominated convergence theorem. The other terms can be controlled similarly. The lemma is proved. □

7.2 Continuity in time

First, we have

Definition 7.1 *The function g belongs to $C_{\rm weak}([0,T];L^q(\Omega))$, $1 \le q < \infty$, if $\int_\Omega g\varphi \,{\rm d}x \in C([0,T])$ for all $\varphi \in L^{q'}(\Omega)$.*

We have the following easy result:

Lemma 7.4 *Let $1 < q < \infty$, $\Omega \subset \mathbb{R}^N$ be a domain and $I \subset \mathbb{R}$ be an open and bounded interval. Let $f \in L^\infty(I;L^q(\Omega))$ and $\partial_t \int_\Omega f\eta \,{\rm d}x \in L^1(I)$ for all $\eta \in C_c^\infty(\Omega)$. Then there exists $g \in C_{\rm weak}(\overline{I};L^q(\Omega))$ such that for a.a. $t \in I$ $f(t,\cdot) = g(t,\cdot)$ (in the sense of $L^q(\Omega)$).*

Proof: Take any $\eta \in C_c^\infty(\Omega)$. As $\int_\Omega f\eta \,{\rm d}x \in W^{1,1}(I)$, we know that there exists $w_\eta \in {\rm AC}([0,T])$ such that $w_\eta(t) = \int_\Omega f(t,\cdot)\eta \,{\rm d}x$ for a.e. $t \in I$. Furthermore, by virtue of the theorem on Lebesgue points we know that there exists $N \subset I$ with zero one-dimensional Lebesgue measure such that for any $t \in I \setminus N$ and any $\eta \in C_c^\infty(\Omega)$

$$\lim_{h\to 0^+} \int_t^{t+h} \Big(\int_\Omega f\eta \,{\rm d}x\Big)\,{\rm d}s = \int_\Omega f\eta \,{\rm d}x;$$

whence we see that $w_\eta(t) = \int_\Omega f(t,\cdot)\eta \,{\rm d}x$ for any $\eta \in C_c^\infty(\Omega)$ and any $t \in I\setminus N$ (the set N is in particular the same for all functions η) and

$$\lim_{h\to 0^+} \int_t^{t+h} \Big(\int_\Omega f\eta \,{\rm d}x\Big)\,{\rm d}s = w_\eta(t) \quad \text{ for all } t \in \overline{I}$$

(at the endpoints the limits are one-sided). It implies that

$$|w_\eta(t)| \le \|f\|_{L^\infty(0,T;L^q(\Omega))}\|\eta\|_{L^{q'}(\Omega)}.$$

Thus by the Riesz representation theorem,

$$w_\eta(t) = \int_\Omega g(t,\cdot)\eta \,{\rm d}x \quad \text{ for all } \eta \in C_c^\infty(\Omega)$$

and $g(t,\cdot) \in L^q(\Omega)$. We will show that $g \in C_{\rm weak}([0,T];L^q(\Omega))$. To this aim choose $\varepsilon > 0$ and take arbitrary $\varphi \in L^{q'}(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $L^{q'}(\Omega)$,

$1 \leq q' < \infty$, we have

$$\left|\int_\Omega \Big(g(t+\delta,\cdot) - g(t,\cdot)\Big)\varphi \,\mathrm{d}x\right| \leq \left|\int_\Omega \Big(g(t+\delta,\cdot) - g(t,\cdot)\Big)\eta \,\mathrm{d}x\right| + \left|\int_\Omega \Big(g(t+\delta,\cdot) - g(t,\cdot)\Big)(\varphi - \eta) \,\mathrm{d}x\right|,$$

where $\eta \in C_c^\infty(\Omega)$ is suitably chosen in such a way that the second integral is less than $\varepsilon/2$. Now, due to the continuity of $\int_\Omega g\eta \,\mathrm{d}x$, we can choose δ_0 sufficiently small that for $0 \leq |\delta| \leq \delta_0$ the first integral is bounded by $\varepsilon/2$. The lemma is proved. $\Box$.

Remark 7.2 Looking at the weak formulation of the continuity equation, as $\varrho \in L^\infty(0,T;L^\gamma(\Omega))$ and $\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$, we immediately see (at least for $\gamma > \frac{6}{5}$) that $\varrho \in C_{\mathrm{weak}}([0,T];L^\gamma(\Omega))$, as

$$\partial_t \int_\Omega \varrho\eta \,\mathrm{d}x = -\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\eta \,\mathrm{d}x \in L^1(0,T).$$

Remark 7.3 Similarly we have that $\varrho\mathbf{u} \in C_{\mathrm{weak}}([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3))$. Indeed,

$$\int_\Omega (\varrho|\mathbf{u}|)^{\frac{2\gamma}{\gamma+1}} \,\mathrm{d}x = \int_\Omega (\varrho|\mathbf{u}|^2)^{\frac{\gamma}{\gamma+1}} \varrho^{\frac{\gamma}{\gamma+1}} \,\mathrm{d}x \leq \left(\int_\Omega \varrho|\mathbf{u}|^2 \,\mathrm{d}x\right)^{\frac{\gamma}{\gamma+1}} \left(\int_\Omega \varrho^\gamma \,\mathrm{d}x\right)^{\frac{1}{\gamma+1}} \in L^\infty(I).$$

Looking at the weak formulation of the momentum equation, it is an easy task to verify that for $\boldsymbol{\varphi} \in C_c^\infty([0,T]\times\Omega;\mathbb{R}^3)$

$$\partial_t\Big(\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi} \,\mathrm{d}x\Big) \in L^1(I),$$

which finishes the proof.

In what follows we will use the following abstract version of the Arzelà–Ascoli theorem (see [12, Theorem 1.6.9])

Theorem 7.1 *Let X and B be Banach spaces such that $B \hookrightarrow\hookrightarrow X$. Let f_n be a sequence of functions: $\overline{I} \to B$ which is uniformly bounded in B and uniformly continuous in X. Then there exists $f \in C(\overline{I};X)$ such that $f_n \to f$ in $C(\overline{I};X)$ at least for a chosen subsequence.*

Then we have

Theorem 7.2 *Let $1 < p, q < \infty$, Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geq 2$. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of functions: $\overline{I} \to L^q(\Omega)$ such that*

- $g_n \in C_{\text{weak}}(\overline{I}; L^q(\Omega))$ *for all* $n \in \mathbb{N}$
- g_n *is uniformly continuous in* $W^{-1,p}(\Omega)$
- g_n *is uniformly bounded in* $L^q(\Omega)$.

Then at least for a chosen subsequence
(i)

$$g_n \to g \quad \text{in } C_{\text{weak}}(\overline{I}; L^q(\Omega)).$$

(ii) If moreover $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,p}(\Omega)$ (i.e. $1 < p \leq \frac{N}{N-1}$ and $1 < q < \infty$, or $\frac{N}{N-1} < p < \infty$, $\frac{Np}{N+p} < q < \infty$), then

$$g_n \to g \quad \text{in } C(\overline{I}; W^{-1,p}(\Omega)).$$

Proof: (i) As $W^{-1,p}(\Omega) \hookrightarrow W^{-1,s}(\Omega)$ for $s = \min\left\{p, \frac{N}{N-1}\right\}$, the sequence g_n is uniformly continuous in $W^{-1,s}(\Omega)$. As the embedding $L^q(\Omega) \hookrightarrow W^{-1,s}(\Omega)$ is compact, we have by virtue of Theorem 7.1 $g_n \to g$ in $C(\overline{I}; W^{-1,s}(\Omega))$, at least for a chosen subsequence.

Therefore, for a given $\varepsilon > 0$ there exists n_0 such that for m, $n > n_0$:

$$\Big| \int_\Omega (g_n(t,\cdot) - g_m(t,\cdot))\eta \, \mathrm{d}x \Big|$$

$$\leq \|(g_n(t,\cdot) - g_m(t,\cdot)\|_{W^{-1,s}(\Omega)} \|\eta\|_{W^{1,s'}(\Omega)} \leq \varepsilon \|\eta\|_{W^{1,s'}(\Omega)},$$

for all $\eta \in C_c^\infty(\Omega)$, for all $t \in \overline{I}$. Hence for any $\eta \in C_c^\infty(\Omega)$ the mappings $t \mapsto \int_\Omega g_n(t,\cdot)\eta \, \mathrm{d}x$ form a Cauchy sequence in $C(\overline{I})$ which has a limit $A_\eta \in C(\overline{I})$. Similarly as in Lemma 7.4 it is possible to verify that if $\|g_n\|_{L^\infty(0,T;L^q(\Omega))} \leq C$, then $\max_{t \in [0,T]} \|g_n(t,\cdot)\|_{L^q(\Omega)} \leq C$, uniformly in n. Thus

$$\sup_{t \in \overline{I}} |A_\eta(t)| \leq \left| \limsup_{n\to\infty} \int_\Omega g_n(t,\cdot)\eta \, \mathrm{d}x \right| \leq C\|\eta\|_{L^{q'}(\Omega)},$$

$\eta \in C_c^\infty(\Omega)$, we see that $\eta \mapsto A_\eta$ is a linear densely defined bounded operator from $L^q(\Omega)$ to $\mathbb{R}$. Hence

$$A_\eta(t) = \int_\Omega g(t,\cdot)\eta \, \mathrm{d}x \quad \text{with } g(t,\cdot) \in L^q(\Omega).$$

Moreover, $t \mapsto \int_\Omega g(t,\cdot)\eta \, \mathrm{d}x \in C(\overline{I})$ for all $\eta \in C_c^\infty(\Omega)$ and by the density argument also for $\eta \in L^{q'}(\Omega)$. Moreover, again by the density argument

$$\sup_{t\in\overline{I}} \Big| \int_\Omega (g_n(t,\cdot) - g(t,\cdot))\eta \, \mathrm{d}x \Big| \to_{n\to\infty} 0,$$

hence

$$\int_\Omega g_n(t,\cdot)\eta \, \mathrm{d}x \to \int_\Omega g(t,\cdot)\eta \, \mathrm{d}x \quad \text{in } C(\overline{I})$$

for any $\eta \in L^{q'}(\Omega)$.

To prove (ii), recall that $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,p}(\Omega)$ and the result follows from Theorem 7.1. □

Next

Lemma 7.5 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geq 2$, $1 < q < \infty$, $1 \leq p < \infty$. If $g_n \to g$ in $C_{\mathrm{weak}}(\overline{I}; L^q(\Omega))$, then $g_n \to g$ strongly in $L^p(\overline{I}; W^{-1,r}(\Omega))$ provided $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,r}(\Omega)$.*

Proof: As

$$g_n(t,\cdot) \rightharpoonup g(t,\cdot) \quad \text{in } L^q(\Omega),\ t \in \overline{I},$$

and $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,r}(\Omega)$, we have

$$g_n(t,\cdot) \to g(t,\cdot) \quad \text{in } W^{-1,r}(\Omega),\ t \in \overline{I}.$$

As in particular g_n is bounded in $L^\infty(I; L^q(\Omega))$, then also (cf. the proofs of Lemma 7.4 and Theorem 7.2)

$$\sup_{t\in\overline{I}} \|g_n(t,\cdot)\|_{L^q(\Omega)} \leq C$$

and so is bounded $\sup_{t\in\overline{I}} \|g_n(t,\cdot)\|_{W^{-1,r}(\Omega)}$. Thus by the Lebesgue dominated convergence theorem

$$\int_0^T \|g_n(t,\cdot) - g(t,\cdot)\|^p_{W^{-1,r}(\Omega)} \, \mathrm{d}t \to_{n\to\infty} 0.$$

□

We further have

Lemma 7.6 *Let $N \geq 2$, $1 < \beta < \infty$, $\theta \in (0, \frac{\beta}{4})$ and Ω be a bounded domain in $\mathbb{R}^N$. Let the pair $(\varrho, \mathbf{u})$ fulfill*

$$\varrho \geq 0 \text{ a.e. in } (0,T) \times \mathbb{R}^N,\ \varrho \in L^\infty(0,T; L^\beta_{\mathrm{loc}}(\mathbb{R}^N)) \cap C_{\mathrm{weak}}([0,T]; L^\beta(\Omega)),$$
$$\mathbf{u} \in L^2(0,T; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N))$$

and let $(\varrho, \mathbf{u})$ solve the renormalized continuity equation with $b(s) = s^\theta$, i.e.

$$\partial_t \varrho^\theta + \mathrm{div}_x(\varrho^\theta \mathbf{u}) + (\theta - 1)\varrho^\theta \mathrm{div}_x \mathbf{u} = 0 \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^N). \tag{7.6}$$

Then $\varrho \in C([0,T]; L^p(\Omega))$, $1 \leq p < \beta$.

Remark 7.4 In our case of the compressible Navier–Stokes equations with the pressure law $p(\varrho) = \varrho^\gamma$ we have $\varrho \in C([0,T]; L^p(\Omega))$, $1 \leq p < \gamma$.

Proof: Due to (7.6) we know that $\partial_t \int_\Omega \rho^\theta \eta \,\mathrm{d}x \in L^2(0,T)$ for all $\eta \in C^\infty_c(\Omega)$, hence by Lemma 7.4 we know that $\varrho = \tilde{\varrho}$ a.e. in $(0,T) \times \Omega$, where $\tilde{\varrho}^\theta \in C_{\mathrm{weak}}([0,T]; L^{\frac{\beta}{\theta}}(\Omega))$. We now take (7.6) with $\tilde{\varrho}$ and regularize it over the space variable by the mollifier S_ε. Thus

$$\partial_t S_\varepsilon(\tilde{\varrho}^\theta) + \mathrm{div}_x(S_\varepsilon(\tilde{\varrho}^\theta)\mathbf{u}) = (1-\theta) S_\varepsilon(\tilde{\varrho}^\theta \mathrm{div}_x \mathbf{u}) + r_\varepsilon(\tilde{\varrho}^\theta, \mathbf{u}) \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^N), \tag{7.7}$$

where $r_\varepsilon(\tilde{\varrho}^\theta, \mathbf{u}) = \mathrm{div}_x(S_\varepsilon(\tilde{\varrho}^\theta)\mathbf{u}) - \mathrm{div}_x(S_\varepsilon(\tilde{\varrho}^\theta \mathbf{u}))$. Indeed,

$$S_\varepsilon(\tilde{\varrho}^\theta) \in C([0,T] \times \overline{\Omega}), \quad \|S_\varepsilon(\tilde{\varrho}^\theta)(t,\cdot)\|_{L^q(\mathbb{R}^N)} \leq \|\tilde{\varrho}^\theta(t,\cdot)\|_{L^q(\mathbb{R}^N)}$$

by the Hausdorff–Young inequality. Therefore there exists $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \sup_{t \in [0,T]} \|S_\varepsilon(\tilde{\varrho}^\theta)(t,\cdot)\|_{L^q(\mathbb{R}^N)} < \infty, \quad 1 \leq q \leq \frac{\beta}{\theta}.$$

Furthermore,

$$\begin{array}{ll} S_\varepsilon(\tilde{\varrho}^\theta)(t,\cdot) \to \tilde{\varrho}^\theta(t,\cdot) & \text{strongly in } L^q(\Omega), 1 \leq q \leq \frac{\beta}{\theta}, t \in [0,T], \\ S_\varepsilon(\tilde{\varrho}^\theta \mathrm{div}_x \mathbf{u}) \to \tilde{\varrho}^\theta \mathrm{div}_x \mathbf{u} & \text{strongly in } L^2(0,T; L^{\frac{2\beta}{2\theta+\beta}}(\Omega)). \end{array} \tag{7.8}$$

By Lemma 7.1 (Friedrichs commutator lemma)

$$r_\varepsilon(\tilde{\varrho}^\theta, \mathbf{u}) \to 0 \quad \text{in } L^2(I; L^{\frac{2\beta}{2\theta+\beta}}(\Omega)).$$

We now apply Lemma 7.3 (renormalized solution with non-zero right hand side) with $b(s) = (s+1)^2$ to (7.7)

$$\begin{aligned}\partial_t(S_\varepsilon(\tilde\varrho^\theta)+1)^2 + \mathrm{div}_x((S_\varepsilon(\tilde\varrho^\theta)+1)^2\mathbf{u}) + (S_\varepsilon(\tilde\varrho^\theta)^2-1)\mathrm{div}_x\mathbf{u} \\ = 2(1-\theta)(S_\varepsilon(\tilde\varrho^\theta)+1)S_\varepsilon(\tilde\varrho^\theta\mathrm{div}_x\mathbf{u}) + 2(S_\varepsilon(\tilde\varrho^\theta)+1)r_\varepsilon(\tilde\varrho^\theta,\mathbf{u})\end{aligned} \quad (7.9)$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^N)$. We have that $\{\int_\Omega |S_\varepsilon(\tilde\varrho^\theta)|^2\eta\,\mathrm{d}x\}_{\varepsilon>0}$ is uniformly bounded for every $\eta\in C_c^\infty(\Omega)$ on $[0,T]$ and by (7.9) together with assumptions on $\varrho,\mathbf{u}$ also uniformly continuous on $[0,T]$. Now, due to (7.8) and Arzelà–Ascoli theorem

$$\int_\Omega |S_\varepsilon(\tilde\varrho^\theta)|^2\eta\,\mathrm{d}x \to \int_\Omega |\tilde\varrho^\theta|^2\eta\,\mathrm{d}x \quad \text{in } C[0,T], \quad \eta\in C_c^\infty(\Omega).$$

Therefore, by density argument ($\eta_\varepsilon\to 1$), $\int_\Omega |\tilde\varrho^\theta|^2\,\mathrm{d}x \in C([0,T])$. As $\tilde\varrho^\theta \in C_{\mathrm{weak}}([0,T];L^2(\Omega))$, we get

$$\tilde\varrho^\theta \in C([0,T];L^2(\Omega)).$$

Now, due to interpolation of Lebesgue spaces

$$\tilde\varrho \in C([0,T];L^p(\Omega)), \quad 1\le p<\beta.$$

Finally, due to our assumption $\varrho=\tilde\varrho$ (in the sense of the $L^\beta(\Omega)$-space) for every $t\in[0,T]$. $\square$

Chapter 8

Existence proof

8.1 Approximations

Recall that we aim at proving the existence of weak solutions (in the sense as presented in Chapter 5) to the following problem (we set $a = 1$ in the pressure law for the sake of simplicity):

$$\begin{gathered}
\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} - (\mu+\lambda)\nabla_x\mathrm{div}_x\mathbf{u} \\
+\nabla_x\varrho^\gamma = \varrho\mathbf{f} \quad \text{in } (0,T)\times\Omega, \\
\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0 \quad \text{in } (0,T)\times\Omega, \\
\mathbf{u}(t,x) = \mathbf{0} \quad \text{on } (0,T)\times\partial\Omega, \\
\varrho(0,x) = \varrho_0(x), \qquad (\varrho\mathbf{u})(0,x) = (\varrho\mathbf{u})_0(x) \quad \text{in } \Omega.
\end{gathered} \tag{8.1}$$

At the first level we regularize the pressure ($\delta > 0$) and get the regularized system with artificial pressure

$$\begin{gathered}
\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} - (\mu+\lambda)\nabla_x\mathrm{div}_x\mathbf{u} \\
+\nabla_x\varrho^\gamma + \delta\nabla_x\varrho^\beta = \varrho\mathbf{f} \quad \text{in } (0,T)\times\Omega, \\
\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0 \quad \text{in } (0,T)\times\Omega, \\
\mathbf{u}(t,x) = \mathbf{0} \quad \text{on } (0,T)\times\partial\Omega, \\
\varrho(0,x) = \varrho_{0,\delta}(x), \qquad (\varrho\mathbf{u})(0,x) = (\varrho\mathbf{u})_{0,\delta}(x) \quad \text{in } \Omega.
\end{gathered} \tag{8.2}$$

At the next level we regularize the continuity equation ($\varepsilon > 0$) and get the continuity equation with dissipation

$$\begin{aligned} \partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} - (\mu+\lambda)\nabla_x\mathrm{div}_x\mathbf{u} \\ +\nabla_x\varrho^\gamma + \delta\nabla_x\varrho^\beta + \varepsilon(\nabla_x\varrho\cdot\nabla_x)\mathbf{u} = \varrho\mathbf{f} \quad \text{in } (0,T)\times\Omega, \\ \partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) - \varepsilon\Delta\varrho = 0 \quad \text{in } (0,T)\times\Omega, \\ \mathbf{u}(t,x) = \mathbf{0}, \qquad \frac{\partial\varrho}{\partial\mathbf{n}} = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ \varrho(0,x) = \varrho_{0,\delta}(x), \qquad (\varrho\mathbf{u})(0,x) = (\varrho\mathbf{u})_{0,\delta}(x) \quad \text{in } \Omega. \end{aligned} \tag{8.3}$$

The ε-term in the approximate balance of momentum is added in order to obtain a suitable form of the energy equality which will be seen below.

The last level is based on the finite dimensional projection (Galerkin approximation) of the momentum equation. We take a basis in $W_0^{1,2}(\Omega;\mathbb{R}^3)$ (orthogonal) which is orthonormal in $L^2(\Omega;\mathbb{R}^3)$ and is formed by eigenfunctions of the Lamé equation

$$-\mu\Delta\mathbf{\Phi}_j - (\mu+\lambda)\nabla_x\mathrm{div}_x\mathbf{\Phi}_j = \alpha_j\mathbf{\Phi}_j,$$

$0<\alpha_1<\alpha_2\leq\ldots$, $\mathbf{\Phi}_j \in W_0^{1,p}(\Omega;\mathbb{R}^3)\cap W^{2,p}(\Omega;\mathbb{R}^3)$, $1\leq p<\infty$ arbitrary, with the scalar products

$$\begin{aligned} (\mathbf{u},\mathbf{v})_{W_0^{1,2}(\Omega;\mathbb{R}^3)} := \int_\Omega \Big(\mu\nabla_x\mathbf{u}:\nabla_x\mathbf{v} + (\mu+\lambda)\mathrm{div}_x\mathbf{u}\,\mathrm{div}_x\mathbf{v}\Big)\,\mathrm{d}x, \\ (\mathbf{u},\mathbf{v})_{L^2(\Omega;\mathbb{R}^3)} := \int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x. \end{aligned}$$

We first show existence of solutions to the Galerkin approximation (Section 8.2). Then we collect estimates independent of the dimension of the Galerkin approximation and pass in Section 8.3 with $n\to\infty$. We receive system (8.3), i.e. system with continuity equation with dissipation. Next we prove estimates independent of the parameter ε and pass with $\varepsilon\to 0^+$ and get the system with the artificial pressure (8.2) (Section 8.4). In the last section we collect estimates independent of δ and pass with $\delta\to 0^+$ to get a solution to the original system (8.1).

8.2 Existence for the Galerkin approximation

We take δ, $\varepsilon>0$, $n\in\mathbb{N}$ and $\beta>1$ sufficiently large (e.g. $\beta\geq 15$ is enough). Let us denote $\mathbf{X}_n = \mathrm{Lin}\{\mathbf{\Phi}_1,\ldots,\mathbf{\Phi}_n\}$. Our aim is to show:

Theorem 8.1 *Under the assumption of Theorem 8.3, let*

$$0 < \underline{\varrho}(\delta) \leq \varrho_{0,\delta} \leq \overline{\varrho}(\delta) < \infty, \quad \varrho_{0,\delta} \in C^\infty(\overline{\Omega}).$$

Then for any ε, $\delta > 0$ and $n \in \mathbb{N}$ there exists a (unique) couple $(\varrho_n, \mathbf{u}_n)$ such that:

(i) for any $p \in [1,\infty)$, $\varrho_n \in C([0,T]; W^{1,p}(\Omega)) \cap L^p(I; W^{2,p}(\Omega))$, $\partial_t \varrho_n \in L^p(I; L^p(\Omega))$, $\varrho > 0$ a.e. in $(0,T) \times \Omega$, $\mathbf{u}_n \in C^{0,1}([0,T]; \mathbf{X}_n)$

(ii)

$$\begin{aligned}\int_0^T \int_\Omega \Big(\partial_t(\varrho_n \mathbf{u}_n) \cdot \boldsymbol{\Phi} - \varrho_n(\mathbf{u}_n \otimes \mathbf{u}_n) : \nabla_x \boldsymbol{\Phi} + \mu \nabla_x \mathbf{u}_n : \nabla_x \boldsymbol{\Phi} \\ + (\mu + \lambda) \mathrm{div}_x \mathbf{u}_n \mathrm{div}_x \boldsymbol{\Phi} - (\varrho_n^\gamma + \delta \varrho_n^\beta) \mathrm{div}_x \boldsymbol{\Phi} \\ + \varepsilon \nabla_x \varrho_n \nabla_x \mathbf{u}_n \cdot \boldsymbol{\Phi} \Big) \, \mathrm{d}x \ \mathrm{d}t = \int_0^T \int_\Omega \varrho_n \mathbf{f} \cdot \boldsymbol{\Phi} \, \mathrm{d}x \ \mathrm{d}t \quad \forall \boldsymbol{\Phi} \in \mathbf{X}_n\end{aligned}$$

(iii)

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) - \varepsilon \Delta \varrho_n = 0 \quad \textit{a.e. in } (0,T) \times \Omega$$

(iv) $\varrho_n(0) = \varrho_{0,\delta}$, $\mathbf{u}_n(0) = P_n \mathbf{u}_0$, $\frac{\partial \varrho_n}{\partial \mathbf{n}}|_{\partial\Omega} = 0$, where P_n is the projector of $L^2(\Omega; \mathbb{R}^3)$ to $\mathbf{X}_n$ and $\varrho_{0,\delta} \in C^\infty(\overline{\Omega})$ is the regularized initial condition

(v) denoting

$$\mathcal{E}_\delta(\varrho, \mathbf{u})(t) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{\varrho^\gamma}{\gamma - 1} + \frac{\delta \varrho^\beta}{\beta - 1} \right)(t, \cdot) \, \mathrm{d}x,$$

we have

$$\begin{aligned}&\mathcal{E}_\delta(\varrho_n, \mathbf{u}_n)(t) + \int_0^t \int_\Omega \Big(\mu |\nabla_x \mathbf{u}_n|^2 + (\mu + \lambda)(\mathrm{div}_x \mathbf{u}_n)^2 \Big) \, \mathrm{d}x \, \mathrm{d}\tau \\ &+ \varepsilon\gamma \int_0^t \int_\Omega \varrho_n^{\gamma-2} |\nabla_x \varrho_n|^2 \, \mathrm{d}x \, \mathrm{d}\tau + \varepsilon\delta\beta \int_0^t \int_\Omega \varrho_n^{\beta-2} |\nabla_x \varrho_n|^2 \, \mathrm{d}x \, \mathrm{d}\tau \\ &\leq \int_0^t \int_\Omega \varrho_n \mathbf{f} \cdot \mathbf{u}_n \, \mathrm{d}x \, \mathrm{d}\tau + \mathcal{E}_\delta(\varrho_{0,\delta}, P_n \mathbf{u}_0) \qquad \textit{a.e. in } (0,T).\end{aligned} \tag{8.4}$$

Let us first look at the parabolic Neumann problem

$$\begin{aligned} \partial_t \varrho - \varepsilon \Delta \varrho = h \quad \text{in } (0,T) \times \Omega, \\ \varrho(0) = \varrho_0 \quad \text{in } \Omega, \\ \frac{\partial \varrho}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0 \quad \text{in } (0,T) \end{aligned} \tag{8.5}$$

with h and ϱ_0 given sufficiently regular functions. We have the following result (for the proof see e.g. [1])

Lemma 8.1 *Let* $0 < \theta \leq 1$, $1 < p, q < \infty$, Ω *bounded,* $\Omega \in C^{2,\theta}$, $\varrho_0 \in \widetilde{W}^{2-\frac{2}{p},q}(\Omega) = \overline{\{z \in C^\infty(\overline{\Omega}); \frac{\partial z}{\partial \mathbf{n}}|_{\partial\Omega} = 0\}}^{\|\cdot\|_{W^{2-\frac{2}{p},q}(\Omega)}}$, *where* $\|\cdot\|_{W^{2-\frac{2}{p},q}(\Omega)}$ *is the norm in the Sobolev–Slobodetskii space. Let* $h \in L^p(0,T;L^q(\Omega))$. *Then there exists unique* $\varrho \in L^p(0,T;W^{2,q}(\Omega)) \cap C([0,T];W^{2-\frac{2}{p},q}(\Omega))$ *with the time derivative* $\partial_t \varrho \in L^p(0,T;L^q(\Omega))$, *together with the estimates*

$$\begin{aligned} &\varepsilon^{1-\frac{1}{p}} \|\varrho\|_{L^\infty(0,T;W^{2-\frac{2}{p},q}(\Omega))} + \|\partial_t \varrho\|_{L^p(0,T;L^q(\Omega))} + \varepsilon \|\varrho\|_{L^p(0,T;W^{2,q}(\Omega))} \\ &\quad \leq C(p,q,\Omega) \Big(\varepsilon^{1-\frac{1}{p}} \|\varrho_0\|_{W^{2-\frac{2}{p},q}(\Omega)} + \|h\|_{L^p(0,T;L^q(\Omega))} \Big). \end{aligned}$$

If $h = \operatorname{div}_x \mathbf{b}$, $\mathbf{b} \in L^p(0,T;L^q(\Omega;\mathbb{R}^3))$, $\varrho_0 \in L^q(\Omega)$, *then there exists unique* $\varrho \in L^p(0,T;W^{1,q}(\Omega)) \cap C([0,T];L^q(\Omega))$, *solving in* $\mathcal{D}'(0,T)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho \eta \,\mathrm{d}x + \varepsilon \int_\Omega \nabla_x \varrho \cdot \nabla_x \eta \,\mathrm{d}x = -\int_\Omega \mathbf{b} \cdot \nabla_x \eta \,\mathrm{d}x, \quad \forall \eta \in C^\infty(\overline{\Omega})$$

and

$$\begin{aligned} &\varepsilon^{1-\frac{1}{p}} \|\varrho\|_{L^\infty(0,T;L^q(\Omega))} + \varepsilon \|\nabla_x \varrho\|_{L^p(0,T;L^q(\Omega))} \\ &\quad \leq C(p,q,\Omega) \Big(\varepsilon^{1-\frac{1}{p}} \|\varrho_0\|_{L^q(\Omega)} + \|\mathbf{b}\|_{L^p(0,T;L^q(\Omega;\mathbb{R}^3))} \Big). \end{aligned}$$

We now return to

$$\begin{aligned} \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) - \varepsilon \Delta \varrho = 0 \quad \text{in } (0,T) \times \Omega, \\ \varrho(0) = \varrho_{0,\delta} \quad \text{in } \Omega, \\ \frac{\partial \varrho}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0 \quad \text{in } (0,T) \end{aligned} \tag{8.6}$$

We aim at proving

Lemma 8.2 *Let $0 < \theta \le 1$, $\Omega \in C^{2,\theta}$ bounded, $0 < \underline{\varrho} \le \varrho_{0,\delta} \le \overline{\varrho} < \infty$, $\varrho_{0,\delta} \in C^\infty(\overline{\Omega})$ and $\frac{\partial \varrho_{0,\delta}}{\partial \mathbf{n}} = 0$ on $\partial\Omega$. Let $\mathbf{u} \in L^\infty(0,T;\mathcal{W}_0^{1,\infty}(\Omega;\mathbb{R}^3))$, where $\mathcal{W}_0^{1,\infty}(\Omega;\mathbb{R}^3) = \{\mathbf{z} \in W^{1,\infty}(\Omega;\mathbb{R}^3); \mathbf{z}|_{\partial\Omega} = 0\}$. Then there exists unique solution to (8.6) $\varrho = \varrho(\mathbf{u}) \in L^p(0,T;W^{2,p}(\Omega)) \cap C([0,T];W^{1,p}(\Omega))$, $\partial_t \varrho \in L^p(0,T;L^p(\Omega))$, $1 < p < \infty$, arbitrary. Moreover*

$$\underline{\varrho} \mathrm{e}^{-\int_0^t \|\mathbf{u}(\tau)\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\,\mathrm{d}\tau} \le \varrho(t,x) \le \overline{\varrho} \mathrm{e}^{\int_0^t \|\mathbf{u}(\tau)\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\,\mathrm{d}\tau}, \tag{8.7}$$

for $t \in [0,T]$ and a.a. $x \in \Omega$. If $\|\mathbf{u}\|_{L^\infty(I;W^{1,\infty}(\Omega;\mathbb{R}^3))} \le K$, then

$$\begin{gathered}
\|\varrho\|_{L^\infty(0,t;W^{1,2}(\Omega))} \le C\|\varrho_{0,\delta}\|_{W^{1,2}(\Omega)} \mathrm{e}^{\frac{C}{\varepsilon}(K+K^2)t}, \\
\|\nabla_x^2 \varrho\|_{L^2((0,t)\times\Omega;\mathbb{R}^{3\times3})} \le \frac{C}{\varepsilon}\sqrt{t}\|\varrho_{0,\delta}\|_{W^{1,2}(\Omega)} K \mathrm{e}^{\frac{C}{\varepsilon}(K+K^2)t} + \frac{C}{\varepsilon}\|\varrho_{0,\delta}\|_{W^{1,2}(\Omega)}, \\
\|\partial_t \varrho\|_{L^2((0,t)\times\Omega)} \le C\sqrt{t}\|\varrho_{0,\delta}\|_{W^{1,2}(\Omega)} K \mathrm{e}^{\frac{C}{\varepsilon}(K+K^2)t} + \|\varrho_{0,\delta}\|_{W^{1,2}(\Omega)}, \\
\|(\varrho(\mathbf{u}_1) - \varrho(\mathbf{u}_2))\|_{L^2((0,t)\times\Omega)} \le \\
C(K,\varepsilon,T)t\|\varrho_{0,\delta}\|_{W^{1,2}(\Omega)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(0,t;W^{1,\infty}(\Omega;\mathbb{R}^3))},
\end{gathered} \tag{8.8}$$

where $t \in [0,T]$.

Proof:
Step 1: First, if $\Omega \in C^2$, $\mathbf{u} \in L^\infty(0,T;\mathcal{W}_0^{1,\infty}(\Omega;\mathbb{R}^3))$, $\varrho_{0,\delta} \in W^{1,2}(\Omega)$, there exists unique $\varrho \in C([0,T];W^{1,2}(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega))$, $\partial_t \varrho \in L^2((0,T)\times\Omega)$ solution to (8.6).

- We construct the solution by the Galerkin method, with the orthonormal (in L^2) and orthogonal (in $W^{1,2}$) basis of the Laplace equation with the Neumann boundary condition at $\partial\Omega$.

- For $n \in \mathbb{N}$, testing by ϱ_n, $\Delta\varrho_n$, $\partial_t \varrho_n$ we get (note that $-\int_\Omega \nabla_x \varrho_n \cdot \nabla_x \Delta \varrho_n \,\mathrm{d}x = \int_\Omega \nabla_x^2 \varrho_n : \nabla_x^2 \varrho_n \,\mathrm{d}x$, as $\int_{\partial\Omega} \nabla_x \varrho_n \cdot \nabla_x \frac{\partial \varrho_n}{\partial \mathbf{n}} \,\mathrm{d}S = 0$ due to the boundary conditions)

$$\begin{aligned}
\|\varrho_n(t)\|_{L^\infty(0,T;W^{1,2}(\Omega))} &\le C(T, \|\mathbf{u}\|_{L^\infty(I;W^{1,\infty}(\Omega;\mathbb{R}^3))}, \varepsilon), \\
\|\nabla_x \varrho_n\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^3))} &\le C(T, \|\mathbf{u}\|_{L^\infty(I;W^{1,\infty}(\Omega;\mathbb{R}^3))}, \varepsilon), \\
\|\partial_t \varrho_n\|_{L^2((0,T)\times\Omega)} &\le C(T, \|\mathbf{u}\|_{L^\infty(I;W^{1,\infty}(\Omega;\mathbb{R}^3))}, \varepsilon).
\end{aligned}$$

- Letting $n \to \infty$ in

$$\int_0^T \left(\int_\Omega \partial_t \varrho_n \psi \, \mathrm{d}x \right) z \, \mathrm{d}t + \varepsilon \int_0^T \int_\Omega \nabla_x \varrho_n \cdot \nabla_x \psi z \, \mathrm{d}x \ \mathrm{d}t$$
$$= - \int_0^T \int_\Omega \mathrm{div}_x (\varrho_n \mathbf{u}) \psi z \, \mathrm{d}x \ \mathrm{d}t \quad \forall \psi \in \mathrm{Lin}\{h_1, \dots, h_n\}, z \in C_c^\infty(0,T)$$

leads to

$$\int_0^T \left(\int_\Omega \partial_t \varrho \psi \, \mathrm{d}x \right) z \, \mathrm{d}t + \varepsilon \int_0^T \int_\Omega \nabla_x \varrho \cdot \nabla_x \psi z \, \mathrm{d}x \ \mathrm{d}t$$
$$= - \int_0^T \int_\Omega \mathrm{div}_x (\varrho \mathbf{u}) \psi z \, \mathrm{d}x \ \mathrm{d}t$$

for any $z \in C_c^\infty(0,T)$ and $\psi \in \mathrm{Lin}\{h_1, h_2, \dots\}$, where $\{h_i\}_{i=1}^\infty$ is the basis formed by the eigenfunctions of the Laplace equations with the homogeneous Neumann boundary condition on $\partial\Omega$.

- By the density argument

$$\int_0^T \int_\Omega \partial_t \varrho \eta \, \mathrm{d}x \, \mathrm{d}t + \varepsilon \int_0^T \int_\Omega \nabla_x \varrho \cdot \nabla_x \eta \, \mathrm{d}x \ \mathrm{d}t$$
$$= - \int_0^T \int_\Omega \mathrm{div}_x (\varrho \mathbf{u}) \eta \, \mathrm{d}x \ \mathrm{d}t \quad \forall \eta \in L^2(0,T; W^{1,2}(\Omega)).$$

- Finally, the continuity in $W^{1,2}(\Omega)$ follows by standard arguments.

Step 2: Now, let $\Omega \in C^{2,\theta}$. We apply Lemma 8.1 (with the right-hand side $h := -\mathrm{div}_x(\varrho \mathbf{u}) \in L^2(0,T; L^6(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ and get (by bootstrapping argument) that $\varrho \in L^p(0,T; W^{2,p}(\Omega)) \cap C([0,T]; W^{1,p}(\Omega))$ with $\partial_t \varrho \in L^p((0,T) \times \Omega)$ for any $1 < p < \infty$.
Step 3: Consider $R(t) = \overline{\varrho} \mathrm{e}^{\int_0^t \|\mathrm{div}_x \mathbf{u}(\tau, \cdot)\|_{L^\infty(\Omega)} \, \mathrm{d}\tau}$. Then

$$R'(t) - \|\mathrm{div}_x \mathbf{u}(t, \cdot)\|_{L^\infty(\Omega)} R(t) = 0, \quad R(0) = \overline{\varrho}$$

and

$$R' + \mathrm{div}_x(R\mathbf{u}) \geq 0 \quad \text{a.e. in } Q_T.$$

Denote $\omega(t,x) = \varrho(t,x) - R(t)$. Then

$$\begin{aligned} \partial_t \omega + \mathrm{div}_x(\omega \mathbf{u}) - \varepsilon \Delta \omega \leq 0 \quad \text{a.e. in } Q_T, \\ \omega(0,x) = \varrho_0 - \overline{\varrho} \leq 0, \quad \frac{\partial \omega}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0. \end{aligned} \tag{8.9}$$

Test (8.9) by $\omega^+ = \max\{\omega, 0\}$

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^+\|^2_{L^2(\Omega)} + \varepsilon \int_\Omega |\nabla_x \omega^+|^2 \,\mathrm{d}x \\ \leq -\frac{1}{2}\int_\Omega |\omega^+|^2 \mathrm{div}_x \mathbf{u} \,\mathrm{d}x &\leq \frac{1}{2}\|\mathrm{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \|\omega^+\|^2_{L^2(\Omega)} \end{aligned}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^+\|^2_{L^2(\Omega)} \leq \|\mathrm{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \|\omega^+\|^2_{L^2(\Omega)}.$$

By the Gronwall inequality

$$\|\omega^+(t,\cdot)\|_{L^2(\Omega)} \leq \|\omega^+(0,\cdot)\|_{L^2(\Omega)} \mathrm{e}^{\int_0^t \|\mathrm{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \,\mathrm{d}\tau} = 0$$

and thus

$$\varrho(t,x) - R(t) \leq 0 \quad \text{a.e. in } Q_T.$$

Analogously, denoting $r(t) = \underline{\varrho} \mathrm{e}^{-\int_0^t \|\mathrm{div}_x \mathbf{u}(\tau)\|_{L^\infty(\Omega)} \,\mathrm{d}\tau}$, $\omega(t,x) = \varrho(t,x) - r(t)$

$$\begin{aligned} \partial_t \omega + \mathrm{div}_x(\omega \mathbf{u}) - \varepsilon \Delta \omega \geq 0 \quad \text{a.e. in } Q_T, \\ \omega(0) = \varrho_0 - \underline{\varrho} \geq 0, \quad \frac{\partial \omega}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0. \end{aligned} \tag{8.10}$$

Testing by ω^- implies $\|\omega^-(t,\cdot)\|_{L^2(\Omega)} = 0$ and thus

$$\varrho(t,x) - r(t) \geq 0 \quad \text{a.e. in } Q_T.$$

Whence (8.7).
Step 4: (L^2 bounds):

a) Test (8.6) by ϱ

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|^2_{L^2(\Omega)} + 2\varepsilon \int_\Omega |\nabla_x \varrho|^2 \,\mathrm{d}x = -\int_\Omega \varrho^2 \mathrm{div}_x \mathbf{u} \,\mathrm{d}x \\ &\qquad \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|^2_{L^2(\Omega)} \leq \|\mathbf{u}\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)} \|\varrho\|^2_{L^2(\Omega)}. \end{aligned}$$

b) Test (8.6) by $-\Delta\varrho$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla_x\varrho|^2\,\mathrm{d}x + 2\varepsilon\int_\Omega |\Delta\varrho|^2\,\mathrm{d}x$$
$$= 2\int_\Omega \varrho\mathrm{div}_x\mathbf{u}\Delta\varrho\,\mathrm{d}x + 2\int_\Omega \mathbf{u}\cdot\nabla_x\varrho\Delta\varrho\,\mathrm{d}x$$
$$\leq C\|\mathbf{u}\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\|\varrho\|_{W^{1,2}(\Omega)}\|\Delta\varrho\|_{L^2(\Omega)}$$
$$\leq \frac{C'}{\varepsilon}\|\mathbf{u}\|^2_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\|\varrho\|^2_{W^{1,2}(\Omega)} + \varepsilon\|\Delta\varrho\|^2_{L^2(\Omega)}.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|^2_{W^{1,2}(\Omega)} \leq \frac{C}{\varepsilon}\Big(\|\mathbf{u}\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)} + \|\mathbf{u}\|^2_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\Big)\|\varrho\|^2_{W^{1,2}(\Omega)},$$

i.e.

$$\|\varrho\|_{L^\infty(0,t;W^{1,2}(\Omega))} \leq \|\varrho_{0,\delta}\|_{W^{1,2}(\Omega)}\mathrm{e}^{\frac{C}{\varepsilon}\left(\|\mathbf{u}\|_{L^\infty(0,t;W^{1,2}(\Omega;\mathbb{R}^3))}+\|\mathbf{u}\|^2_{L^\infty(0,t;W^{1,2}(\Omega;\mathbb{R}^3))}\right)t}.$$

Further

$$\varepsilon\int_0^t \|\Delta\varrho\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \leq \|\varrho_0\|_{W^{1,2}(\Omega)} + C\|\mathbf{u}\|_{L^\infty(0,t;W^{1,\infty}(\Omega;\mathbb{R}^3))}\|\varrho\|_{L^\infty(0,t;W^{1,2}(\Omega))}\int_0^t \|\Delta\varrho\|_{L^2(\Omega)}\,\mathrm{d}\tau$$

which gives $(8.8)_2$. Similarly, testing (8.6) by $\partial_t\varrho$

$$\int_0^t \|\partial_t\varrho\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \frac{\varepsilon}{2}\|\nabla_x\varrho(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)}$$
$$\leq \frac{\varepsilon}{2}\|\nabla_x\varrho_{0,\delta}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t\int_\Omega \mathrm{div}_x(\varrho\mathbf{u})\partial_t\varrho\,\mathrm{d}x\ \mathrm{d}t \leq \frac{\varepsilon}{2}\|\nabla_x\varrho_{0,\delta}\|^2_{L^2(\Omega;\mathbb{R}^3)}$$
$$+ C\Big(\|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))}, \|\varrho\|_{L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}, \sqrt{t}\|\partial_t\varrho\|_{L^2(0,T;L^2(\Omega))}\Big),$$

which yields $(8.8)_3$.

Step 5: (Uniqueness) First, to get $(8.8)_4$, take $\mathbf{u}_1$, $\mathbf{u}_2$ and subtract from the equation for $\varrho_1 = \varrho_1(\mathbf{u}_1)$ the equation for $\varrho_2 = \varrho_2(\mathbf{u}_2)$. It reads:

$$\partial_t(\varrho_1 - \varrho_2) - \varepsilon\Delta(\varrho_1 - \varrho_2) = -\varrho_1 \mathrm{div}_x(\mathbf{u}_1 - \mathbf{u}_2) - \nabla_x\varrho_1 \cdot (\mathbf{u}_1 - \mathbf{u}_2) \\ -(\varrho_1 - \varrho_2)\mathrm{div}_x\mathbf{u}_2 - \nabla_x(\varrho_1 - \varrho_2) \cdot \mathbf{u}_2.$$

Test the above obtained equality by $(\varrho_1 - \varrho_2)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_1 - \varrho_2\|^2_{L^2(\Omega)} + 2\varepsilon\|\nabla_x(\varrho_1 - \varrho_2)\|^2_{L^2(\Omega;\mathbb{R}^3)} = \\ \int_\Omega \big[-\varrho_1\mathrm{div}_x(\mathbf{u}_1 - \mathbf{u}_2) - \nabla_x\varrho_1 \cdot (\mathbf{u}_1 - \mathbf{u}_2) - (\varrho_1 - \varrho_2)\mathrm{div}_x\mathbf{u}_2 - \nabla_x(\varrho_1 - \varrho_2)\cdot\mathbf{u}_2\big]\times \\ \times(\varrho_1 - \varrho_2)\,\mathrm{d}x \leq C(\|\varrho_1\|_{W^{1,2}(\Omega)}\|\varrho_1 - \varrho_2\|_{L^2(\Omega)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)} \\ + \|\mathbf{u}_2\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\|\varrho_1 - \varrho_2\|^2_{L^2(\Omega)})$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_1 - \varrho_2\|_{L^2(\Omega)} \\ \leq C\|\varrho_1\|_{W^{1,2}(\Omega)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)} + C\|\mathbf{u}_2\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\|\varrho_1 - \varrho_2\|_{L^2(\Omega)}.$$

Applying Gronwall's lemma

$$\|(\varrho_1 - \varrho_2)(t,\cdot)\|_{L^2(\Omega)} \leq \\ C\int_0^t \Big(\|\varrho_1\|_{W^{1,2}(\Omega)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\mathrm{e}^{\int_\tau^t C\|\mathbf{u}_2\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)}(s)\,\mathrm{d}s}\Big)\,\mathrm{d}\tau,$$

which proves $(8.8)_4$ and hence also the uniqueness. □

Remark 8.1 We can also show the validity of the renormalized continuity equation. Using the same method as in the proof of the validity of renormalized continuity equation, we have for any b sufficiently smooth, convex

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho))\mathrm{div}_x\mathbf{u} - \varepsilon\Delta b(\varrho) \leq 0.$$

Indeed, formally, multiplying the continuity equation by $b'(\varrho)$

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho))\mathrm{div}_x\mathbf{u} - \varepsilon\Delta b(\varrho) = -\varepsilon b''(\varrho)|\nabla_x\varrho|^2 \leq 0,$$

where we used that

$$\Delta b(\varrho) = \mathrm{div}_x(b'(\varrho)\nabla_x\varrho) = b'(\varrho)\Delta\varrho + b''(\varrho)|\nabla_x\varrho|^2.$$

The details are similar as in the case with $\varepsilon = 0$, the only problematic term has a good sign. Note also that we can integrate the continuity equation over Ω (i.e. use as test function 1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho(t,x)\,\mathrm{d}x = 0 \quad \Rightarrow \quad \int_\Omega \varrho(t,x)\,\mathrm{d}x = \text{const. (in time)}.$$

We now return to the full system with the Galerkin approximation for the velocity. We want to obtain a solution for the Galerkin approximation of

$$\begin{gathered}\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} - (\mu+\lambda)\nabla_x\mathrm{div}_x\mathbf{u} \\ +\nabla_x(\varrho^\gamma + \delta\varrho^\beta) + \varepsilon\nabla_x\varrho\cdot\nabla_x\mathbf{u} = \varrho\mathbf{f}\end{gathered}$$

with ϱ being a solution to the (regularized) continuity equation with the velocity $\mathbf{u}$.

We shall apply the following version of the Schauder fixed point theorem (for the proof see e.g. [6])

Theorem 8.2 *Let $T\colon X \to X$ be continuous and compact, X a Banach space. Let for any $s \in [0,1]$ the fixed points $sT\mathbf{u} = \mathbf{u}$ be bounded. Then T possesses at least one fixed point.*

We define the mapping T as follows. Take $\mathbf{w} \in C([0,T];\mathbf{X}_n)$, where $\mathbf{X}_n$ is the finite dimensional space spanned by the first n eigenvalues of $-\mu\Delta\mathbf{u} - (\mu+\lambda)\nabla_x\mathrm{div}_x\mathbf{u}$ with $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$. We look for $\mathbf{u}_n$, the Galerkin approximation of the linearized momentum equation, i.e. for the solution to

$$\begin{aligned}\int_\Omega \partial_t(\varrho(\mathbf{w})\mathbf{u}_n)\cdot\mathbf{h}_i\,\mathrm{d}x + \int_\Omega \mathrm{div}_x(\varrho(\mathbf{w})\mathbf{w}\otimes\mathbf{u}_n)\cdot\mathbf{h}_i\,\mathrm{d}x \\ +\mu\int_\Omega \nabla_x\mathbf{u}_n : \nabla_x\mathbf{h}_i\,\mathrm{d}x + (\mu+\lambda)\int_\Omega \mathrm{div}_x\mathbf{u}_n\mathrm{div}_x\mathbf{h}_i\,\mathrm{d}x \\ +\int_\Omega(\nabla_x\varrho^\gamma(\mathbf{w}) + \delta\nabla_x\varrho^\beta(\mathbf{w}))\cdot\mathbf{h}_i\,\mathrm{d}x + \int_\Omega \varepsilon\nabla_x\varrho(\mathbf{w})\cdot\nabla_x\mathbf{u}_n\cdot\mathbf{h}_i\,\mathrm{d}x \\ = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{h}_i\,\mathrm{d}x, \qquad \mathbf{u}_n(0) = P_n\mathbf{u}_0, \qquad i = 1,\dots,n.\end{aligned} \tag{8.11}$$

Since for $\mathbf{w} \in C([0,T];\mathbf{X}_n)$ the solution to the regularized continuity equation is bounded away from zero, it is not difficult to see that there exists a solution to (8.11). Moreover, as the problem is linear, $\partial_t \varrho \in L^p((0,T)\times\Omega)$ for any $p<\infty$, by a standard energy method and Gronwall's argument, the solution is unique.

It is also possible to show that T is a continuous and compact mapping from $C([0,T];\mathbf{X}_n)$ to itself. The main point is that we get an estimate of $\partial_t \mathbf{u}_n$, while in the spatial variable the compactness is just a consequence of the fact that $\mathbf{X}_n$ is finite dimensional. What remains to show is the boundedness of the possible fixed points. Take $s \in [0,1]$ and

$$sT(\mathbf{u}_n) = \mathbf{u}_n, \text{ i.e., } T(\mathbf{u}_n) = \frac{\mathbf{u}_n}{s}.$$

Then

$$\begin{aligned}
\int_\Omega \partial_t(\varrho \mathbf{u}_n)\cdot \mathbf{u}_n \,\mathrm{d}x &+ \int_\Omega \operatorname{div}_x(\varrho \mathbf{u}_n \otimes \mathbf{u}_n)\cdot \mathbf{u}_n \,\mathrm{d}x \\
&+ \int_\Omega \varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u}_n \cdot \mathbf{u}_n \,\mathrm{d}x + \int_\Omega \mu |\nabla_x \mathbf{u}_n|^2 \,\mathrm{d}x \\
+ \int_\Omega (\mu+\lambda)(\operatorname{div}_x \mathbf{u}_n)^2 \,\mathrm{d}x &+ s\int_\Omega (\nabla_x \varrho^\gamma + \delta \nabla_x \varrho^\beta)\mathbf{u}_n \,\mathrm{d}x = s\int_\Omega \varrho \mathbf{f}\cdot \mathbf{u}_n \,\mathrm{d}x
\end{aligned}$$

for $s \in [0,1]$. Next, we have

$$\begin{aligned}
\int_\Omega \partial_t(\varrho \mathbf{u}_n)\cdot \mathbf{u}_n \,\mathrm{d}x &= \frac{1}{2}\partial_t \int_\Omega \varrho |\mathbf{u}_n|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \partial_t \varrho |\mathbf{u}_n|^2 \,\mathrm{d}x, \\
\int_\Omega \operatorname{div}_x(\varrho \mathbf{u}_n \otimes \mathbf{u}_n)\cdot \mathbf{u}_n \,\mathrm{d}x &= \frac{1}{2}\int_\Omega \operatorname{div}_x(\varrho \mathbf{u}_n)|\mathbf{u}_n|^2 \,\mathrm{d}x, \\
\int_\Omega \varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u}_n \cdot \mathbf{u}_n \,\mathrm{d}x &= \frac{\varepsilon}{2}\int_\Omega \nabla_x \varrho \nabla_x |\mathbf{u}_n|^2 \,\mathrm{d}x = -\frac{\varepsilon}{2}\int_\Omega \Delta \varrho |\mathbf{u}_n|^2 \,\mathrm{d}x.
\end{aligned}$$

Summing these three integrals we get $\frac{1}{2}\partial_t \int_\Omega \varrho |\mathbf{u}_n|^2 \,\mathrm{d}x$, due to the continuity equation. Further

$$
\begin{aligned}
\int_\Omega \nabla_x \varrho^\gamma \cdot \mathbf{u}_n \,\mathrm{d}x =& \frac{\gamma}{\gamma-1}\int_\Omega \varrho \mathbf{u}_n \cdot \nabla_x \varrho^{\gamma-1}\,\mathrm{d}x \\
=& -\frac{\gamma}{\gamma-1}\int_\Omega \varrho^{\gamma-1}\mathrm{div}_x(\varrho\mathbf{u})\,\mathrm{d}x \\
=& \frac{1}{\gamma-1}\partial_t \int_\Omega \varrho^\gamma \,\mathrm{d}x - \frac{\varepsilon\gamma}{\gamma-1}\int_\Omega \varrho^{\gamma-1}\Delta\varrho\,\mathrm{d}x \\
=& \frac{1}{\gamma-1}\partial_t \int_\Omega \varrho^\gamma \,\mathrm{d}x + \varepsilon\gamma\int_\Omega \varrho^{\gamma-2}|\nabla_x\varrho|^2\,\mathrm{d}x, \\
\int_\Omega \nabla_x \varrho^\beta \cdot \mathbf{u}_n \,\mathrm{d}x =& \frac{1}{\beta-1}\partial_t \int_\Omega \varrho^\beta \,\mathrm{d}x + \varepsilon\beta\int_\Omega \varrho^{\beta-2}|\nabla_x\varrho|^2\,\mathrm{d}x.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\delta^s(\varrho,\mathbf{u}_n) + \mu\int_\Omega |\nabla_x\mathbf{u}_n|^2\,\mathrm{d}x + (\mu+\lambda)\int_\Omega (\mathrm{div}_x\mathbf{u}_n)^2\,\mathrm{d}x \\
+ s\varepsilon\gamma\int_\Omega \varrho^{\gamma-2}|\nabla_x\varrho|^2\,\mathrm{d}x + s\varepsilon\delta\beta\int_\Omega \varrho^{\beta-2}|\nabla_x\varrho|^2\,\mathrm{d}x \le s\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}_n\,\mathrm{d}x,
\end{aligned}
\tag{8.12}
$$

where $\mathcal{E}_\delta^s(\varrho,\mathbf{u}_n) = \frac{1}{2}\int_\Omega \left(\varrho|\mathbf{u}_n|^2 + s\frac{\varrho^\gamma}{\gamma-1} + s\frac{\delta\varrho^\beta}{\beta-1}\right)\mathrm{d}x$. As

$$
\begin{aligned}
\left|\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}_n\,\mathrm{d}x\right| \le& \left|\int_\Omega \sqrt{\varrho}\sqrt{\varrho}\mathbf{u}_n\cdot\mathbf{f}\,\mathrm{d}x\right| \\
&\le \|\varrho\|_{L^\infty(0,T;L_1(\Omega))}^{\frac{1}{2}}\|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)}\|\varrho|\mathbf{u}_n|^2\|_{L^1((0,T)\times\Omega)}^{\frac{1}{2}},
\end{aligned}
$$

we get the $L^\infty(0,T)$ control of the kinetic energy $\int_\Omega \varrho|\mathbf{u}_n|^2\,\mathrm{d}x$ and $L^1(0,T)$ control of $\int_\Omega |\nabla_x\mathbf{u}_n|^2\,\mathrm{d}x$ independently of s. As $\mathbf{X}_n$ is finite dimensional, using (8.12) and (8.7), we see that ϱ is pointwisely controlled independently of s and thus, using once more (8.12), we see that $\|\mathbf{u}_n\|_{C([0,T];\mathbf{X}_n)}$ is also controlled independently of s. Therefore we can apply Theorem 8.2 to finish the proof of Theorem 8.1. Note that (8.4) follows from (8.12) integrating over $(0,T)$, setting $s=1$ and $\varrho := \varrho_n$. $\square$

8.3 Estimates independent of n, limit passage $n \to \infty$

Recall that we have from the energy inequality (8.12)

$$\begin{aligned} &\|\varrho_n |\mathbf{u}_n|^2\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \\ &\|\varrho_n\|_{L^\infty(0,T;L^\beta(\Omega))} \leq C, \\ &\|\mathbf{u}_n\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \leq C, \\ &\|\varrho_n^{\frac{\beta}{2}}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C. \end{aligned} \tag{8.13}$$

Note that

$$\|\varrho_n\|_{L^{\frac{5}{3}\beta}((0,T)\times\Omega)} \leq \|\varrho_n\|^{\frac{2}{5}}_{L^\infty(0,T;L^\beta(\Omega))} \|\varrho_n\|^{\frac{3}{5}}_{L^\beta(0,T;L^{3\beta}(\Omega))} \leq C. \tag{8.14}$$

Next we test the continuity equation by ϱ_n.

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_n\|_2^2 + \varepsilon\|\nabla_x \varrho_n\|_2^2 &= -\int_\Omega \mathrm{div}_x(\varrho_n \mathbf{u}_n)\varrho_n \,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega \mathbf{u}_n \cdot \nabla_x \varrho_n^2 \,\mathrm{d}x = -\frac{1}{2}\int_\Omega \varrho_n^2 \mathrm{div}_x \mathbf{u}_n \,\mathrm{d}x. \end{aligned}$$

Taking $\beta \geq \frac{12}{5}$ ($\frac{5}{3}\beta = 4$) we therefore have

$$\|\varrho_n\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C. \tag{8.15}$$

As we control

$$\varrho_n \mathbf{u}_n = \sqrt{\varrho_n}\sqrt{\varrho_n}\mathbf{u}_n \text{ in } L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\Omega;\mathbb{R}^3))$$

(recall that $\sqrt{\varrho_n}$ is controlled in $L^\infty(0,T;L^{2\beta}(\Omega))$ and $\sqrt{\varrho_n}\mathbf{u}_n$ is controlled in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$), and

$$\varrho_n \mathbf{u}_n \text{ in } L^2(0,T;L^{\frac{6\beta}{\beta+6}}(\Omega;\mathbb{R}^3))$$

(recall that ϱ_n is controlled in $L^\infty(0,T;L^\beta(\Omega))$ and $\mathbf{u}_n$ in $L^2(0,T;L^6(\Omega;\mathbb{R}^3))$), then $\varrho_n\mathbf{u}_n$ is bounded in $L^{\frac{10\beta-6}{3(\beta+1)}}((0,T)\times\Omega;\mathbb{R}^3)$. Hence for $\beta > 3$ we control $\varrho_n\mathbf{u}_n$ in $L^{\tilde{s}}((0,T)\times\Omega;\mathbb{R}^3)$ for some $\tilde{s} > 2$ and by virtue of Lemma 8.1 also $\nabla_x\varrho_n$ in $L^{\tilde{s}}((0,T)\times\Omega;\mathbb{R}^3)$. Thus we know that $\mathrm{div}_x(\varrho_n\mathbf{u}_n)$ is bounded in $L^q((0,T)\times\Omega)$ for some $q > 1$. Whence Lemma 8.1 implies estimates

$$\begin{aligned} \|\nabla_x^2 \varrho_n\|_{L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \leq C, \\ \|\partial_t \varrho_n\|_{L^q((0,T)\times\Omega)} \leq C \end{aligned} \tag{8.16}$$

for some $q > 1$.

We now recall our problem

$$\begin{aligned}\int_\Omega \partial_t(\varrho_n \mathbf{u}_n)\cdot \mathbf{h}_i \,\mathrm{d}x - \int_\Omega \varrho_n(\mathbf{u}_n \otimes \mathbf{u}_n) : \nabla_x \mathbf{h}_i \,\mathrm{d}x + \mu \int_\Omega \nabla_x \mathbf{u}_n : \nabla_x \mathbf{h}_i \,\mathrm{d}x \\ +(\mu+\lambda)\int_\Omega \mathrm{div}_x \mathbf{u}_n \mathrm{div}_x \mathbf{h}_i \,\mathrm{d}x - \int_\Omega (\varrho_n^\gamma + \delta\varrho_n^\beta)\mathrm{div}_x \mathbf{h}_i \,\mathrm{d}x \\ +\varepsilon \int_\Omega \nabla_x \varrho_n \cdot \nabla_x \mathbf{u}_n \cdot \mathbf{h}_i \,\mathrm{d}x = \int_\Omega \varrho_n \mathbf{f}\cdot \mathbf{h}_i \,\mathrm{d}x, \qquad (8.17) \\ \partial_t(\varrho_n) - \varepsilon\Delta\varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = 0, \\ \left.\frac{\partial \varrho_n}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0.\end{aligned}$$

We have (for a chosen subsequence, denoted however again by the same index n)

$$\begin{aligned}&\partial_t \varrho_n \rightharpoonup \partial_t \varrho \quad \text{in } L^q((0,T)\times\Omega), \\ &\nabla_x^2 \varrho_n \rightharpoonup \nabla_x^2 \varrho \quad \text{in } L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3}), \\ &\Rightarrow \quad \nabla_x \varrho_n \to \nabla_x \varrho \quad \text{in } L^r((0,T)\times\Omega;\mathbb{R}^3) \quad \forall r \le 2, \\ &\varrho_n \rightharpoonup^* \varrho \quad \text{in } L^\infty(0,T;L^\beta(\Omega)), \\ &\varrho_n \rightharpoonup \varrho \quad \text{in } L^{\frac{5}{3}\beta}((0,T)\times\Omega), \\ &\Rightarrow \quad \varrho_n \to \varrho \quad \text{in } L^r((0,T)\times\Omega) \quad \forall r < \frac{5}{3}\beta, \\ &\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \\ &\Rightarrow \quad \varrho_n \mathbf{u}_n \rightharpoonup \varrho \mathbf{u} \quad \text{in } L^{\frac{10\beta-6}{3(\beta+1)}}((0,T)\times\Omega).\end{aligned}$$

Next we want to show that in fact $\varrho_n \mathbf{u}_n \to \varrho \mathbf{u}$ strongly. To this aim, let us observe that for P_n the orthogonal projection from $L^2(\Omega;\mathbb{R}^3)$ to $\mathbf{X}_n$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P_n(\varrho_n \mathbf{u}_n)\cdot \mathbf{\Phi} \,\mathrm{d}x = \int_\Omega \varrho_n(\mathbf{u}_n\otimes\mathbf{u}_n) : \nabla_x P_n(\mathbf{\Phi})\,\mathrm{d}x \\ -\mu\int_\Omega \nabla_x \mathbf{u}_n : \nabla_x P_n(\mathbf{\Phi})\,\mathrm{d}x - (\mu+\lambda)\int_\Omega \mathrm{div}_x \mathbf{u}_n \mathrm{div}_x P_n(\mathbf{\Phi})\,\mathrm{d}x \\ +\int_\Omega (\varrho_n^\gamma + \delta\varrho_n^\beta)\mathrm{div}_x P_n(\mathbf{\Phi})\,\mathrm{d}x - \varepsilon\int_\Omega \nabla_x\varrho_n\cdot\nabla_x \mathbf{u}_n\cdot P_n(\mathbf{\Phi})\,\mathrm{d}x + \int_\Omega \varrho_n \mathbf{f}\cdot P_n(\mathbf{\Phi})\,\mathrm{d}x, \\ (8.18)\end{aligned}$$

$t \in (0,T)$ and $\mathbf{\Phi} \in C_c^\infty(\Omega;\mathbb{R}^3)$. We now recall the properties of the projection P_n:

$$
\begin{aligned}
&\int_\Omega P_n(\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}x = \int_\Omega \mathbf{u}\cdot P_n(\mathbf{v})\,\mathrm{d}x, \qquad \forall \mathbf{u},\mathbf{v} \in L^2(\Omega;\mathbb{R}^3),\\
&\lim_{n\to\infty} \|(P_n - I)\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^3)} = 0, \qquad \forall \mathbf{u} \in L^2(\Omega;\mathbb{R}^3),\\
&\|P_n(\mathbf{u})\|_{W^{k,2}(\Omega;\mathbb{R}^3)} \le C\|\mathbf{u}\|_{W^{k,2}(\Omega;\mathbb{R}^3)}, \qquad k=1,2,\\
&\qquad\qquad \mathbf{u} \in W_0^{1,2}(\Omega;\mathbb{R}^3)\cap W^{k,2}(\Omega;\mathbb{R}^3),\\
&\lim_{n\to\infty} \|(P_n - I)\mathbf{u}\|_{W^{1,2}(\Omega;\mathbb{R}^3)} = 0, \qquad \forall \mathbf{u} \in W_0^{1,2}(\Omega;\mathbb{R}^3),\\
&\lim_{n\to\infty}\Big(\sup_{\mathbf{z}\in W^{1,q}(\Omega;\mathbb{R}^3);\mathbf{z}\neq\mathbf{0}}\Big(\frac{\|(P_n-I)\mathbf{z}\|_{L^2(\Omega;\mathbb{R}^3)}}{\|\mathbf{z}\|_{W^{1,q}(\Omega;\mathbb{R}^3)}}\Big)\Big) = 0, \qquad q > \frac{6}{5}.
\end{aligned}
\tag{8.19}
$$

To prove the last statement (the only nontrivial), assume the contrary. Hence there exists $\varepsilon_0 > 0$ and $\{\mathbf{z}_n\}_{n=1}^\infty \subset W^{1,q}(\Omega;\mathbb{R}^3)$ such that $\|\mathbf{z}_n\|_{W^{1,q}(\Omega;\mathbb{R}^3)} = 1$ and $\|(P_n - I)\mathbf{z}_n\|_{L^2(\Omega;\mathbb{R}^3)} \ge \varepsilon_0$. Due to the compact embedding, there exists a subsequence $\mathbf{z}_{n_k} \to \mathbf{z}$ in $L^2(\Omega;\mathbb{R}^3)$; whence

$$
\|(P_n - I)\mathbf{z}_{n_k}\|_{L^2(\Omega;\mathbb{R}^3)} \le \|(P_n - I)(\mathbf{z}_{n_k} - \mathbf{z})\|_{L^2(\Omega;\mathbb{R}^3)} + \|(P_n - I)\mathbf{z}\|_{L^2(\Omega;\mathbb{R}^3)} \to 0
$$

as $k \to \infty$, due to $(8.19)_2$ and $(8.19)_3$.

It is an easy matter to observe that using (8.18) and (8.19) we have for some $a > 1$

$$
\|\partial_t(P_n(\varrho_n\mathbf{u}_n))\|_{L^a(0,T;W^{-2,2}(\Omega;\mathbb{R}^3))} \le C.
$$

Moreover,

$$
\|P_n(\varrho_n\mathbf{u}_n)\|_{L^q(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \le C\|\varrho_n\mathbf{u}_n\|_{L^q(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \le C, \qquad q > 1,
$$

provided $\beta > 15$; if $\frac{10\beta-6}{3(\beta+1)} > 3$, then by Lemma 8.1 $\nabla_x\varrho_n$ is bounded in $L^r((0,T)\times\Omega;\mathbb{R}^3)$ for some $r > 3$ and thus ϱ_n is bounded in $L^r(0,T;L^\infty(\Omega))$. By virtue of the Aubin–Lions lemma,

$$
P_n(\varrho_n\mathbf{u}_n) \to \mathbf{z}, \qquad \text{strongly in } L^q(0,T;L^2(\Omega;\mathbb{R}^3)).
$$

It is not difficult to see, due to the fact that $\varrho_n\mathbf{u}_n$ converges to $\varrho\mathbf{u}$ weakly, that $\mathbf{z} = \varrho\mathbf{u}$; it is enough to note that

$$
\begin{aligned}
\int_0^T\int_\Omega \mathbf{z}\cdot\boldsymbol{\varphi}\eta\,\mathrm{d}x\,\mathrm{d}t &\leftarrow \int_0^T\int_\Omega P_n(\varrho_n\mathbf{u}_n)\cdot\boldsymbol{\varphi}\eta\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^T\int_\Omega \varrho_n\mathbf{u}_n\cdot P_n(\boldsymbol{\varphi})\eta\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi}\eta\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for all $\eta \in C_c^\infty((0,T))$ and all $\boldsymbol{\varphi} \in C_c^\infty(\Omega;\mathbb{R}^3)$.

Then as

$$\begin{aligned}&\|\varrho_n \mathbf{u}_n - \varrho \mathbf{u}\|_{L^q(0,T;L^2(\Omega;\mathbb{R}^3))}\\ &\quad\leq \|\varrho_n \mathbf{u}_n - P_n(\varrho_n \mathbf{u}_n)\|_{L^q(0,T;L^2(\Omega;\mathbb{R}^3))} + \|P_n(\varrho_n \mathbf{u}_n) - \varrho \mathbf{u}\|_{L^q(0,T;L^2(\Omega;\mathbb{R}^3))},\end{aligned}$$

we get due to $(8.19)_5$ that $\varrho_n \mathbf{u}_n \to \varrho \mathbf{u}$ in $L^a(0,T;L^2(\Omega;\mathbb{R}^3))$. Whence (recall that $\beta > 15$)

$$\varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \quad \text{in } L^3((0,T)\times\Omega;\mathbb{R}^3). \tag{8.20}$$

Therefore we have for some $s > 1$

$$\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n \rightharpoonup \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{in } L^s((0,T)\times\Omega;\mathbb{R}^{3\times 3}). \tag{8.21}$$

Finally, as $\nabla_x \varrho_n$ is bounded in $L^{s_1}((0,T)\times\Omega;\mathbb{R}^3)$, $s_1 > 3$, we have that $\nabla_x \varrho_n \to \nabla_x \varrho$ in $L^3((0,T)\times\Omega;\mathbb{R}^3)$, and

$$\varepsilon \nabla_x \varrho_n \cdot \nabla_x \mathbf{u}_n \rightharpoonup \varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u} \quad \text{in } L^{\frac{6}{5}}((0,T)\times\Omega;\mathbb{R}^3).$$

Altogether we can pass to the limit in (8.17) to get

$$\begin{gathered} -\int_0^T \int_\Omega (\varrho \mathbf{u}) \cdot \boldsymbol{\Phi} \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t \\ -\int_0^T \int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x \boldsymbol{\Phi} \psi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \mu \nabla_x \mathbf{u} : \nabla_x \boldsymbol{\Phi} \psi \,\mathrm{d}x\,\mathrm{d}t \\ +(\mu+\lambda)\int_0^T \int_\Omega \mathrm{div}_x \mathbf{u}\,\mathrm{div}_x \boldsymbol{\Phi} \psi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega (\varrho^\gamma + \delta\varrho^\beta)\mathrm{div}_x \boldsymbol{\Phi} \psi \,\mathrm{d}x\,\mathrm{d}t \\ +\varepsilon \int_0^T \int_\Omega \nabla_x \varrho \cdot \nabla_x \mathbf{u} \cdot \boldsymbol{\Phi} \psi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\Phi} \psi \,\mathrm{d}x\,\mathrm{d}t, \end{gathered} \tag{8.22}$$

first for any $\boldsymbol{\Phi} \in \mathrm{Lin}\{\mathbf{h}_1, \mathbf{h}_2, \dots\}$ and $\psi \in C_c^\infty(0,T)$, later due to density argument we could enlarge the space. As we do not need to specify the space now, we will not mention it explicitly. Finally, we may repeat the considerations performed in Chapter 3 connected with the weak continuity of the momentum (note that by Lemma 7.4 we know that $\varrho \mathbf{u} \in C_{\mathrm{weak}}([0,T]; L^{\frac{2\beta}{\beta+1}}(\Omega)))$

and get

$$
\begin{aligned}
\int_\Omega (\varrho\mathbf{u}\cdot)\varphi(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta}\mathbf{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x = \int_0^\tau\int_\Omega (\varrho\mathbf{u})\cdot\partial_t\boldsymbol{\varphi}\,\mathrm{d}x\ \mathrm{d}t \\
+\int_0^\tau\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}):\nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_\Omega \mu\nabla_x\mathbf{u}:\nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\ \mathrm{d}t \\
-(\mu+\lambda)\int_0^\tau\int_\Omega \mathrm{div}_x\mathbf{u}\mathrm{div}_x\boldsymbol{\varphi}\,\mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_\Omega (\varrho^\gamma+\delta\varrho^\beta)\mathrm{div}_x\boldsymbol{\varphi}\,\mathrm{d}x\ \mathrm{d}t \\
+\varepsilon\int_0^\tau\int_\Omega \nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\ \mathrm{d}t
\end{aligned}
\tag{8.23}
$$

for any $\boldsymbol{\varphi}\in C_c^\infty([0,\tau]\times\Omega;\mathbb{R}^3)$ and any $0<\tau\le T$.

After the limit passage in the continuity equation we get

$$
\partial_t\varrho - \varepsilon\Delta\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0, \qquad \frac{\partial\varrho}{\partial\mathbf{n}}\Big|_{\partial\Omega} = 0, \tag{8.24}
$$

satisfied a.e. and in the weak sense.

Finally, we may also pass to the limit in the energy inequality. We take (8.12) with $s=1$ and integrate if over the time interval $[0,t]\subset[0,T]$. To pass to the limit in this inequality, we multiply it by a smooth compactly supported function ψ in the time interval $(0,T)$ and integrate it once more over the time variable, now over $[0,T]$. In this form we may perform the limit passage $n\to\infty$ (in the terms with velocity gradients, we also use Fatou's lemma) and finally we get rid of the function ψ. We have

$$
\begin{aligned}
\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{\varrho^\gamma}{\gamma-1} + \delta\frac{\varrho^\beta}{\beta-1}\right)(\tau,\cdot)\,\mathrm{d}x + \mu\int_0^t\int_\Omega |\nabla_x\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\tau \\
+(\mu+\lambda)\int_0^\tau\int_\Omega(\mathrm{div}_x\mathbf{u})^2\,\mathrm{d}x\,\mathrm{d}\tau + \varepsilon\delta\beta\int_0^\tau\int_\Omega \varrho^{\beta-2}|\nabla_x\varrho|^2\,\mathrm{d}x\,\mathrm{d}\tau \\
+\varepsilon\gamma\int_0^\tau\int_\Omega \varrho^{\gamma-2}|\nabla_x\varrho|^2\,\mathrm{d}x\,\mathrm{d}\tau \\
\le \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega\left(\frac{1}{2}\varrho_{0,\delta}|\mathbf{u}_0|^2 + \frac{\varrho_{0,\delta}^\gamma}{\gamma-1} + \delta\frac{\varrho_{0,\delta}^\beta}{\beta-1}\right)\mathrm{d}x
\end{aligned}
\tag{8.25}
$$

for a.a. $t\in(0,T]$.

8.4 Estimates independent of ε, limit passage $\varepsilon \to 0^+$

Recall that from the energy inequality (8.25) we have

$$\begin{aligned} \|\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^\infty(0,T;L^1(\Omega))} &\leq C, \\ \|\varrho_\varepsilon\|_{L^\infty(0,T;L^\beta(\Omega))} &\leq C, \\ \|\mathbf{u}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} &\leq C. \end{aligned} \tag{8.26}$$

For $\beta \geq 4$ we may test the continuity equation by ϱ_ε and get

$$\sqrt{\varepsilon}\|\varrho_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C. \tag{8.27}$$

However, at this point we need some better (and independent of ε) estimates of the pressure. We recall the properties of the Bogovskii operator (see (4.14)–(4.16) from Chapter 4).

We use as the test function in the momentum equation (8.23)

$$\mathcal{B}\left(\varrho_\varepsilon - \frac{1}{|\Omega|}\int_\Omega \varrho_{0,\delta}\,\mathrm{d}x\right).$$

Recall that ($1 \leq p < \infty$)

$$\left\|\varrho_\varepsilon - \frac{1}{|\Omega|}\int_\Omega \varrho_{0,\delta}\,\mathrm{d}x\right\|_p^p \leq C(p,\Omega)\int_\Omega \varrho_\varepsilon^p\,\mathrm{d}x.$$

Note further that

$$\partial_t\left(\mathcal{B}\left(\varrho_\varepsilon - \frac{1}{|\Omega|}\int_\Omega \varrho_{0,\delta}\,\mathrm{d}x\right)\right) = \mathcal{B}(\partial_t\varrho_\varepsilon) = -\mathcal{B}\big(\mathrm{div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon) + \varepsilon\Delta\varrho_\varepsilon\big).$$

We have

$$\int_0^T\int_\Omega(\varrho_\varepsilon^{\gamma+1} + \delta\varrho_\varepsilon^{\beta+1})\,\mathrm{d}x\ \mathrm{d}t = \sum_{j=1}^9 I_j,$$

where

$$I_1 = \frac{1}{|\Omega|}\int_0^T\Big(\int_\Omega(\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta)\,\mathrm{d}x\int_\Omega \varrho_{0,\delta}\,\mathrm{d}x\Big)\,\mathrm{d}t,$$

$$\begin{aligned} I_2 &= -\int_0^T \int_\Omega (\varrho_\varepsilon \mathbf{u}_\varepsilon) \cdot \partial_t \left(\mathcal{B} \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \right) \mathrm{d}x \ \mathrm{d}t \\ &= -\int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B} \left(\partial_t \varrho_\varepsilon \right) \mathrm{d}x \ \mathrm{d}t \\ &= \int_0^T \int_\Omega \left(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}(\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon)) - \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}(\Delta \varrho_\varepsilon) \right) \psi \, \mathrm{d}x \ \mathrm{d}t \\ &= I_2^1 + I_2^2, \end{aligned}$$

$$I_3 = -\int_0^T \int_\Omega \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B} \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \mathrm{d}x \ \mathrm{d}t,$$

$$I_4 = \int_0^T \int_\Omega \mu \nabla_x \mathbf{u}_\varepsilon : \nabla_x \mathcal{B} \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \mathrm{d}x \ \mathrm{d}t,$$

$$I_5 = \int_0^T \int_\Omega (\mu + \lambda) \mathrm{div}_x \mathbf{u}_\varepsilon \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \mathrm{d}x \ \mathrm{d}t,$$

$$I_6 = \int_0^T \int_\Omega \varepsilon \nabla_x \varrho_\varepsilon \cdot \nabla_x \mathbf{u}_\varepsilon \cdot \mathcal{B} \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \mathrm{d}x \ \mathrm{d}t,$$

$$I_7 = -\int_0^T \int_\Omega \varrho_\varepsilon \mathbf{f} \cdot \mathcal{B} \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \mathrm{d}x \ \mathrm{d}t,$$

$$I_8 = -\int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon (T, \cdot) \cdot \mathcal{B} \left(\varrho_\varepsilon (T, \cdot) - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \mathrm{d}x,$$

$$I_9 = -\int_\Omega \varrho_{0,\delta} \mathbf{u}_0 \cdot \mathcal{B} \left(\varrho_{0,\delta} - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \mathrm{d}x.$$

We estimate each term separately:

$$|I_1| \leq C(\|\varrho_\varepsilon\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega))} + \delta \|\varrho_\varepsilon\|^\beta_{L^\infty(0,T;L^\beta(\Omega))}) \leq C(\mathrm{DATA}),$$

$$\begin{aligned} |I_2^1| &\leq \int_0^T \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon| \cdot |\mathcal{B}(\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon))| \, \mathrm{d}x \ \mathrm{d}t \\ &\leq C \int_0^T \|\varrho_\varepsilon\|_{L^3(\Omega)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega;\mathbb{R}^3)} \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \, \mathrm{d}t \\ &\leq C \int_0^T \|\varrho_\varepsilon\|^2_{L^3(\Omega)} \|\mathbf{u}_\varepsilon\|^2_{L^6(\Omega;\mathbb{R}^3)} \, \mathrm{d}t \leq C(\mathrm{DATA}) \end{aligned}$$

if $\beta \geq 3$,

$$\begin{aligned} |I_2^2| &\leq C \int_0^T \int_\Omega |\varrho_\varepsilon \mathbf{u}_\varepsilon| \Big| \mathcal{B}(\varepsilon \Delta \varrho_\varepsilon) \Big| \, \mathrm{d}x \, \mathrm{d}t \\ &\leq C \int_0^T \varepsilon \|\nabla \varrho_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \|\varrho_\varepsilon\|_{L^3(\Omega)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega;\mathbb{R}^3)} \, \mathrm{d}t \leq C(DATA) \end{aligned}$$

if $\beta \geq 3$,

$$\begin{aligned} |I_3| &\leq C \int_0^T \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \left| \nabla_x \mathcal{B} \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \right| \, \mathrm{d}x \, \mathrm{d}t \\ &\leq C \int_0^T \|\varrho_\varepsilon\|^2_{L^3(\Omega)} \|\mathbf{u}_\varepsilon\|^2_{L^6(\Omega;\mathbb{R}^3)} \, \mathrm{d}t \leq C(\mathrm{DATA}) \end{aligned}$$

if $\beta \geq 3$,

$$\begin{aligned} &|I_4| + |I_5| \leq \\ &\quad C \int_0^T \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \left\| \nabla_x \mathcal{B} \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \mathrm{d}t \\ &\qquad \leq C \int_0^T \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \|\varrho_\varepsilon\|_{L^2(\Omega)} \, \mathrm{d}t \leq C(\mathrm{DATA}) \end{aligned}$$

if $\beta \geq 2$,

$$\begin{aligned} |I_6| \leq C \int_0^T & \varepsilon \|\nabla_x \varrho_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \times \\ & \times \left\| \mathcal{B} \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \right\|_{L^\infty(\Omega;\mathbb{R}^3)} \mathrm{d}t \\ \leq C \int_0^T & \varepsilon \|\nabla_x \varrho_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \|\varrho_\varepsilon\|_{L^\beta(\Omega)} \, \mathrm{d}t \leq C(DATA) \end{aligned}$$

if $\beta > 3$,

$$\begin{aligned} |I_7| &\leq C \int_0^T \|\varrho_\varepsilon\|_{L^\beta(\Omega)} \left\| \mathcal{B} \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x \right) \right\|_{L^{\frac{\beta}{\beta-1}}(\Omega;\mathbb{R}^3)} \mathrm{d}t \\ &\leq C \int_0^T \|\varrho_\varepsilon\|^2_{L^\beta(\Omega)} \, \mathrm{d}t \end{aligned}$$

if $\beta \geq \frac{3}{2}$,

$$|I_8| \leq \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\Omega))} \left\| \mathcal{B}\left(\varrho_\varepsilon - \frac{1}{|\Omega|}\int_\Omega \varrho_{0,\delta}\,\mathrm{d}x\right)\right\|_{C([0,T]\times\overline{\Omega})} \leq C(\mathrm{DATA})$$

for $\beta > 3$; here we use that $\varrho_\varepsilon \mathbf{u}_\varepsilon \in C_{\mathrm{weak}}([0,T]; L^{\frac{2\beta}{\beta+1}}(\Omega;\mathbb{R}^3))$ and the embedding $W^{1,\beta}(\Omega) \hookrightarrow C(\overline{\Omega})$. Finally

$$|I_9| \leq C(\mathrm{DATA}).$$

Therefore,

$$\int_0^T \int_\Omega (\varrho_\varepsilon^{\gamma+1} + \delta \varrho_\varepsilon^{\beta+1})\,\mathrm{d}x\ \mathrm{d}t \leq C(\mathrm{DATA})$$

provided

$$\beta > 3.$$

Hence

$$\|\varrho_\varepsilon\|_{L^{\gamma+1}((0,T)\times\Omega)} + \delta^{\frac{1}{\beta+1}} \|\varrho_\varepsilon\|_{L^{\beta+1}((0,T)\times\Omega)} \leq C. \tag{8.28}$$

Using Theorem 7.2 together with the considerations as in Chapter 3 based on the weak formulation of the continuity equation we have

$$\varrho_\varepsilon \to \varrho \quad \text{in } C_{\mathrm{weak}}(0,T; L^\beta(\Omega)).$$

Using the weak formulation of the momentum equation and the arguments as above and in Chapter 3 we have

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \quad \text{in } C_{\mathrm{weak}}([0,T]; L^{\frac{2\beta}{\beta+1}}(\Omega;\mathbb{R}^3)).$$

Writing the continuity equation (8.24) in the weak form,

$$\begin{aligned} \int_\Omega (\varrho_\varepsilon \Phi)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta}\Phi(0,\cdot)\,\mathrm{d}x \\ = \int_0^\tau \int_\Omega \varrho_\varepsilon \partial_t \Phi\,\mathrm{d}x\ \mathrm{d}t - \int_0^\tau \int_\Omega \varepsilon \nabla_x \varrho_\varepsilon \cdot \nabla_x \Phi\,\mathrm{d}x\ \mathrm{d}t \\ + \int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Phi\,\mathrm{d}x\ \mathrm{d}t \quad \forall \Phi \in C_c^\infty([0,\tau]\times\overline{\Omega}),\ \tau \in (0,T], \end{aligned}$$

we may pass with $\varepsilon \to 0^+$ $(\varepsilon\|\nabla_x \varrho_\varepsilon\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \to 0)$

$$\begin{aligned}\int_\Omega (\varrho\Phi)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta}\Phi(0,\cdot)\,\mathrm{d}x = \int_0^\tau \int_\Omega \varrho_\varepsilon \partial_t \Phi\,\mathrm{d}x\ \mathrm{d}t \\ + \int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Phi\,\mathrm{d}x\ \mathrm{d}t \quad \forall \Phi \in C_c^\infty([0,\tau]\times\overline{\Omega}),\ \tau\in(0,T],\end{aligned} \tag{8.29}$$

whence it also holds for $\Phi \in W^{1,2}((0,\tau)\times\Omega)$ such that $\Phi \in C([0,\tau]; L^{\frac{\beta}{\beta-1}}(\Omega))$ and we recover the weak formulation of the continuity equation.

Next we consider the momentum equation. Since

$$\begin{aligned}\varepsilon \int_0^T \int_\Omega |\nabla_x \varrho_\varepsilon \cdot \nabla_x \mathbf{u}_\varepsilon|\,\mathrm{d}x\ \mathrm{d}t \\ \le \sqrt{\varepsilon}\sqrt{\varepsilon}\|\nabla_x\varrho_\varepsilon\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\|\nabla_x \mathbf{u}_\varepsilon\|_{L^2((0,T)\times\Omega;\mathbb{R}^{3\times3})},\end{aligned}$$

and exactly as in Chapter 6

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \rightharpoonup \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{in } L^q((0,T)\times\Omega;\mathbb{R}^{3\times3}) \text{ for some } q>1,$$

we recover after the limit passage $\varepsilon \to 0^+$

$$\begin{aligned}\int_\Omega (\varrho\mathbf{u}\cdot\mathbf{\Phi})(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta}\mathbf{u}_0\cdot\mathbf{\Phi}(0,\cdot)\,\mathrm{d}x \\ = \int_0^\tau \int_\Omega \varrho\mathbf{u}\cdot\partial_t\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t \\ -\mu\int_0^\tau\int_\Omega \nabla_x\mathbf{u} : \nabla_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t - (\mu+\lambda)\int_0^\tau\int_\Omega \mathrm{div}_x\mathbf{u}\,\mathrm{div}_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t \\ + \int_0^\tau\int_\Omega (\overline{\varrho^\gamma + \delta\varrho^\beta})\mathrm{div}_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t,\end{aligned} \tag{8.30}$$

for any $\mathbf{\Phi} \in C_c^\infty([0,\tau]\times\Omega;\mathbb{R}^3)$ and any $0<\tau\le T$. By density argument, it also holds for $\mathbf{\Phi}$ bounded (continuous) with $\partial_t\mathbf{\Phi} \in L^2(0,\tau;L^{\frac{6\beta}{5\beta-1}}(\Omega;\mathbb{R}^3))$, $\nabla_x\mathbf{\Phi} \in L^2((0,\tau)\times\Omega;\mathbb{R}^{3\times3})$ and $\mathrm{div}_x\mathbf{\Phi} \in L^{\beta+1}((0,\tau)\times\Omega)$ such that the function $\mathbf{\Phi} \in C([0,\tau];L^{\frac{2\beta}{\beta-1}}(\Omega;\mathbb{R}^3))$, $0<\tau\le T$. The last task is to show that $\overline{\varrho^\gamma+\delta\varrho^\beta} = \varrho^\gamma + \delta\varrho^\beta$, i.e., the strong convergence of the density.

8.4.1 Strong convergence of the density

First recall that we have the following renormalized formulation of the continuity equation (note that the equation holds pointwise a.e. in $(0,T)\times\Omega$)

$$\begin{aligned}\partial_t(b(\varrho_\varepsilon)) + \mathrm{div}_x(b(\varrho_\varepsilon)\mathbf{u}_\varepsilon) + (\varrho_\varepsilon b'(\varrho_\varepsilon) - b(\varrho_\varepsilon))\mathrm{div}_x\mathbf{u}_\varepsilon - \varepsilon\Delta b(\varrho_\varepsilon) \\ = -\varepsilon b''(\varrho_\varepsilon)|\nabla_x\varrho_\varepsilon|^2 \le 0\end{aligned} \tag{8.31}$$

with b sufficiently smooth and convex. Due to the fact $\beta > 2$, we also have in the limit (we can prove it directly from the weak formulation of the continuity equation for the limit functions, see Lemma 7.3)

$$\partial_t(b(\varrho)) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho))\mathrm{div}_x\mathbf{u} = 0 \tag{8.32}$$

for b sufficiently smooth; however, now it holds only in the sense of distributions. We will return to the precise formulation below.

We now proceed as in the weak sequential compactness part. The aim is to show the effective viscous flux identity

$$\overline{\varrho^{\gamma+1}} + \delta\overline{\varrho^{\beta+1}} - (2\mu+\lambda)\overline{\varrho\mathrm{div}_x\mathbf{u}} = \overline{\varrho^\gamma}\varrho + \delta\overline{\varrho^\beta}\varrho - (2\mu+\lambda)\varrho\mathrm{div}_x\mathbf{u} \quad \text{a.e. in } \Omega. \tag{8.33}$$

To show (8.33) we proceed exactly as in Chapter 6. One difference is that for $\boldsymbol{\varphi}_\varepsilon = \Phi\nabla_x\Delta^{-1}[\varrho_\varepsilon 1_\Omega]$ with $\Phi \in C_c^\infty([0,T]\times\Omega)$ we have

$$\partial_t\boldsymbol{\varphi}_\varepsilon = \Phi\nabla_x\Delta^{-1}[\partial_t\varrho_\varepsilon 1_\Omega] + \text{l.o.t} = -\nabla_x\Delta^{-1}\mathrm{div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon) + \underbrace{\varepsilon\nabla_x(\varrho_\varepsilon 1_\Omega)}_{\to 0} + \text{l.o.t.},$$

where l.o.t. denotes lower order terms coming from the derivatives of the function Φ. Next, Δ^{-1} represents here the inverse of the Laplacean on $\mathbb{R}^3$, specifically,

$$\partial_{x_j}\Delta^{-1}[v] = \mathcal{F}_{\xi\to x}\left[\frac{\mathrm{i}\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\right].$$

Finally, the term

$$\varepsilon\int_0^T\int_\Omega \nabla_x\varrho_\varepsilon\cdot\nabla_x\mathbf{u}_\varepsilon\cdot\boldsymbol{\varphi}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \to 0 \tag{8.34}$$

for $\varepsilon \to 0^+$. The rest is the same, hence we obtain the effective viscous flux identity.

We intend to use in the renormalized continuity equation $b(\varrho) = \varrho\ln\varrho$. Note that $b''(\varrho) = (\ln\varrho + 1)' = \frac{1}{\varrho} > 0$, i.e. it is a convex function, however, it has superlinear growth at infinity and the derivative is not bounded at zero. However, we can handle the growth by means of Lemma 7.3. Next, due to Lemma 7.6 we know that the density is continuous with values in $L^q(\Omega)$ for any $q < \beta$. We therefore have (we apply the same procedure as we used to the continuity equation in Chapter 3)

$$\int_\Omega (\varrho \ln \varrho\, \varphi)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta} \ln \varrho_{0,\delta} \varphi(0,\cdot)\,\mathrm{d}x$$
$$= \int_0^\tau \int_\Omega \varrho \ln \varrho \partial_t \varphi \,\mathrm{d}x\ \mathrm{d}t + \int_0^\tau \int_\Omega \varrho \ln \varrho \mathbf{u}\cdot\nabla\varphi\,\mathrm{d}x\ \mathrm{d}t - \int_0^\tau \int_\Omega \varrho \mathrm{div}_x \mathbf{u} \varphi\,\mathrm{d}x\ \mathrm{d}t \tag{8.35}$$

for any $\varphi \in C_c^\infty([0,\tau]\times\Omega)$ and any $0<\tau\le T$. However, we may argue as in the proof of Lemma 7.2 to see that we may in fact use test functions from $C_c^\infty([0,\tau]\times\overline{\Omega})$. This finally allows to use as test function $1_\Omega 1_{[0,\tau]}$ to conclude

$$\int_\Omega (\varrho \ln \varrho)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta} \ln \varrho_{0,\delta}\,\mathrm{d}x + \int_0^\tau \int_\Omega \varrho \mathrm{div}_x \mathbf{u}\,\mathrm{d}x\ \mathrm{d}t = 0.$$

Further, for $\varepsilon > 0$ we have a.e. in $(0,T)\times\Omega$

$$\partial_t(\varrho_\varepsilon \ln \varrho_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \ln \varrho_\varepsilon \mathbf{u}_\varepsilon) + \varrho_\varepsilon \mathrm{div}_x \mathbf{u}_\varepsilon - \varepsilon\Delta(\varrho_\varepsilon \ln \varrho_\varepsilon) \le 0.$$

Integrating it over Ω (recall that $\partial_{\mathbf{n}}\varrho_\varepsilon|_{\partial\Omega} = 0$) and over $(0,\tau)$ yields

$$\int_\Omega (\varrho_\varepsilon \ln \varrho_\varepsilon)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta} \ln \varrho_{0,\delta}\,\mathrm{d}x + \int_0^\tau \int_\Omega \varrho_\varepsilon \mathrm{div}_x \mathbf{u}_\varepsilon\,\mathrm{d}x\ \mathrm{d}t \le 0.$$

Passing with $\varepsilon \to 0^+$ (note that $\varrho_\varepsilon \ln \varrho_\varepsilon$ converges in $C_{\mathrm{weak}}([0,T];L^q(\Omega))$ for any $q<\beta$)

$$\int_\Omega (\overline{\varrho \ln \varrho})(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta} \ln \varrho_{0,\delta}\,\mathrm{d}x + \int_0^\tau \int_\Omega \overline{\varrho \mathrm{div}_x \mathbf{u}}\,\mathrm{d}x\ \mathrm{d}t \le 0.$$

Therefore

$$\int_\Omega \big((\overline{\varrho \ln \varrho})(\tau,\cdot) - (\varrho \ln \varrho)(\tau,\cdot)\big)\,\mathrm{d}x \le \int_0^\tau \int_\Omega (\varrho \mathrm{div}_x \mathbf{u} - \overline{\varrho \mathrm{div}_x \mathbf{u}})\,\mathrm{d}x\ \mathrm{d}t$$
$$= \frac{1}{2\mu+\lambda} \int_0^\tau \int_\Omega \Big(\Big(\overline{\varrho^\gamma}\varrho - \overline{\varrho^{\gamma+1}}\Big) + \delta\Big(\overline{\varrho^\beta}\varrho - \overline{\varrho^{\beta+1}}\Big)\Big)\,\mathrm{d}x\ \mathrm{d}t.$$

As

$$\overline{\varrho^{\gamma+1}} - \overline{\varrho^\gamma}\varrho = \lim_{\varepsilon\to 0^+} (\varrho_\varepsilon^{\gamma+1} - \varrho_\varepsilon^\gamma \varrho) = \lim_{\varepsilon\to 0^+} (\varrho_\varepsilon^\gamma - \varrho^\gamma)(\varrho_\varepsilon - \varrho) \ge 0,$$

we have $\overline{\varrho^\gamma}\varrho \le \overline{\varrho^{\gamma+1}}$ and thus

$$\int_\Omega \big((\overline{\varrho \ln \varrho})(\tau,\cdot) - (\varrho \ln \varrho)(\tau,\cdot)\big)\,\mathrm{d}x \le 0.$$

We now apply the following result

Lemma 8.3 *Let $O \subset \mathbb{R}^M$ be a measurable set and $\{v_n\}_{n=1}^{\infty}$ a sequence of functions in $L^1(\Omega)$ such that*

$$v_n \rightharpoonup v \quad \text{in } L^1(O).$$

Let Φ: $\mathbb{R} \to (-\infty, \infty]$ be a continuous convex function. Then

$$\int_\Omega \Phi(v)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega \Phi(v_n)\,\mathrm{d}x.$$

Moreover, if $\Phi(v_n) \rightharpoonup \overline{\Phi(v)}$ in $L^1(O)$, then

$$\Phi(v) \le \overline{\Phi(v)}$$

a.e. in O. If, in addition, Φ is strictly convex on an open convex set $U \subset \mathbb{R}$, and

$$\Phi(v) = \overline{\Phi(v)} \text{ a.e. in } U,$$

then for possibly a subsequence

$$v_n \to v \text{ for a.e. } y \in \{y \in O, v(y) \in U\}.$$

Proof: The proof can be found in [10]. □

This yields

$$\overline{\varrho \ln \varrho} = \varrho \ln \varrho$$

as well as the a.e. in $(0,T) \times \Omega$ the pointwise convergence of the sequence of densities. This, by virtue of Vitali's convergence theorem, implies that $\varrho_\varepsilon \to \varrho$ in $L^p((0,T)\times\Omega)$ for any $p < \beta + 1$.

Hence we get the weak formulation of the momentum equation

$$\begin{aligned} &\int_\Omega (\varrho\mathbf{u}\cdot\mathbf{\Phi})(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta}\mathbf{u}_0\cdot\mathbf{\Phi}(0,\cdot) \\ &= \int_0^\tau \int_\Omega \varrho\mathbf{u}\cdot\partial_t\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t \\ &- \int_0^\tau\int_\Omega \mu\nabla_x\mathbf{u} : \nabla_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t - (\mu+\lambda)\int_0^\tau\int_\Omega \mathrm{div}_x\mathbf{u}\,\mathrm{div}_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t \\ &+ \int_0^\tau\int_\Omega (\varrho^\gamma + \delta\varrho^\beta)\,\mathrm{div}_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t, \end{aligned} \tag{8.36}$$

for any $\mathbf{\Phi} \in C_c^\infty([0,\tau]\times\Omega)$, $0<\tau\leq T$, and the energy inequality (here we proceed exactly as in the limit passage $n\to\infty$)

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{\varrho^\gamma}{\gamma-1} + \delta\frac{\varrho^\beta}{\beta-1}\right)(\tau,\cdot)\,\mathrm{d}x \\ &+\mu\int_0^\tau\int_\Omega |\nabla_x \mathbf{u}|^2\,\mathrm{d}x\ \mathrm{d}t + (\mu+\lambda)\int_0^\tau\int_\Omega (\mathrm{div}_x\mathbf{u})^2\,\mathrm{d}x\ \mathrm{d}t \\ \leq &\int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\ \mathrm{d}t + \int_\Omega\left(\frac{1}{2}\varrho_{0,\delta}|\mathbf{u}_0|^2 + \frac{\varrho_{0,\delta}^\gamma}{\gamma-1} + \delta\frac{\varrho_{0,\delta}^\beta}{\beta-1}\right)\mathrm{d}x\end{aligned} \tag{8.37}$$

for a.e. $\tau\in(0,T]$.

8.5 Estimates independent of δ, limit passage $\delta\to 0^+$

We have as before

$$\begin{gathered}\|\varrho_\delta|\mathbf{u}_\delta|^2\|_{L^\infty(0,T;L^1(\Omega))}\leq C,\\ \|\varrho_\delta\|_{L^\infty(0,T;L^\gamma(\Omega))} + \delta^{\frac{1}{\beta}}\|\varrho_\delta\|_{L^\infty(0,T;L^\beta(\Omega))}\leq C,\\ \|\mathbf{u}_\delta\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\leq C,\end{gathered} \tag{8.38}$$

and

$$\begin{aligned}\varrho_\delta\to\varrho \quad &\text{in } C_{\mathrm{weak}}(0,T;L^\gamma(\Omega)),\\ \varrho_\delta\mathbf{u}_\delta\to\varrho\mathbf{u}\quad &\text{in } C_{\mathrm{weak}}(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3)).\end{aligned} \tag{8.39}$$

We need to estimate the pressure in a better space than just $L^1((0,T)\times\Omega)$. To this aim, we apply similar type of improved pressure estimates as in the previous limit passage in Section 8.4. However, we have to employ a slightly different test function, namely

$$\mathcal{B}\left(\varrho_\delta^\Theta - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^\Theta\,\mathrm{d}x\right).$$

Recall that

$$\left\|\varrho_\delta^\Theta - \frac{1}{|\Omega|}\int_\Omega\varrho_\delta^\Theta\,\mathrm{d}x\right\|_p^p \leq C(p,\Omega)\int_\Omega \varrho_\delta^{p\Theta}\,\mathrm{d}x$$

for any $1 \le p < \infty$, and

$$\partial_t \left(\mathcal{B} \left(\varrho_\delta^\Theta - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\Theta \,\mathrm{d}x \right) \right) = \mathcal{B} \left(\partial_t \varrho_\delta^\Theta - \frac{1}{|\Omega|} \int_\Omega \partial_t \varrho_\delta^\Theta \,\mathrm{d}x \right). \tag{8.40}$$

Due to the renormalized continuity equation, we have in the sense of distributions

$$\partial_t \varrho_\delta^\Theta = -\mathrm{div}_x(\varrho_\delta^\Theta \mathbf{u}_\delta) - (\Theta - 1)\varrho_\delta^\Theta \mathrm{div}_x \mathbf{u},$$

therefore

$$\begin{aligned} &\mathcal{B} \left(\partial_t \varrho_\delta^\Theta - \frac{1}{|\Omega|} \int_\Omega \partial_t \varrho_\delta^\Theta \,\mathrm{d}x \right) \\ &\quad = -\mathcal{B}(\mathrm{div}_x(\varrho_\delta^\Theta \mathbf{u}_\delta)) - (\Theta - 1)\mathcal{B} \left(\varrho_\delta^\Theta \mathrm{div}_x \mathbf{u}_\delta - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\Theta \mathrm{div}_x \mathbf{u}_\delta \,\mathrm{d}x \right). \end{aligned}$$

Thus

$$\int_0^T \int_\Omega (\varrho_\delta^{\gamma+\Theta} + \delta \varrho_\delta^{\beta+\Theta}) \,\mathrm{d}x \,\mathrm{d}t = \sum_{j=1}^8 I_j,$$

where

$$I_1 = \frac{1}{|\Omega|} \int_0^T \Big(\int_\Omega (\varrho_\delta^\gamma + \delta \varrho_\delta^\beta) \,\mathrm{d}x \int_\Omega \varrho_\delta^\Theta \,\mathrm{d}x \Big) \,\mathrm{d}t,$$

$$\begin{aligned} I_2 &= -\int_0^T \int_\Omega (\varrho_\delta \mathbf{u}_\delta) \cdot \partial_t \left(\mathcal{B} \left(\varrho_\delta^\Theta - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\Theta \,\mathrm{d}x \right) \right) \,\mathrm{d}x \,\mathrm{d}t \\ &= -\int_0^T \int_\Omega \mathcal{B} \left(\partial_t \varrho_\delta^\Theta - \frac{1}{|\Omega|} \int_\Omega \partial_t \varrho_\delta^\Theta \,\mathrm{d}x \right) \Bigg] \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_0^T \int_\Omega \Bigg(\varrho_\delta \mathbf{u}_\delta \cdot \mathcal{B}(\mathrm{div}_x(\varrho_\delta \mathbf{u}_\delta)) + (\Theta - 1)\varrho_\delta \mathbf{u}_\delta \cdot \mathcal{B}\Big(\varrho_\delta^\Theta \mathrm{div}_x \mathbf{u}_\delta \\ &\qquad - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\Theta \mathrm{div}_x \mathbf{u}_\delta \,\mathrm{d}x \Big) \Bigg) \,\mathrm{d}x \,\mathrm{d}t \\ &= I_2^1 + I_2^2, \end{aligned}$$

$$I_3 = -\int_0^T \int_\Omega \varrho_\delta (\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla_x \mathcal{B} \left(\varrho_\delta^\Theta - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\Theta \,\mathrm{d}x \right) \,\mathrm{d}x \,\mathrm{d}t,$$

$$I_4 = \int_0^T \int_\Omega \mu \nabla_x \mathbf{u}_\delta : \nabla_x \mathcal{B} \left(\varrho_\delta^\Theta - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\Theta \, \mathrm{d}x \right) \mathrm{d}x \ \mathrm{d}t,$$

$$I_5 = \int_0^T \int_\Omega (\mu + \lambda) \mathrm{div}_x \mathbf{u}_\delta \left(\varrho_\delta^\Theta - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\Theta \, \mathrm{d}x \right) \mathrm{d}x \ \mathrm{d}t,$$

$$I_6 = -\int_0^T \int_\Omega \varrho_\delta \mathbf{f} \cdot \mathcal{B} \left(\varrho_\delta^\Theta - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\Theta \, \mathrm{d}x \right) \mathrm{d}x \ \mathrm{d}t,$$

$$I_7 = -\int_\Omega (\varrho_\delta \mathbf{u}_\delta)(T, \cdot) \cdot \mathcal{B} \left(\varrho_\delta^\Theta(T, \cdot) - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\Theta(T, \cdot) \, \mathrm{d}x \right) \mathrm{d}x,$$

$$I_8 = -\int_\Omega \varrho_{0,\delta}^\Theta \mathbf{u}_0 \cdot \mathcal{B} \left(\varrho_{0,\delta}^\Theta - \frac{1}{|\Omega|} \int_\Omega \varrho_{0,\delta}^\Theta \, \mathrm{d}x \right) \mathrm{d}x.$$

We estimate each term separately:

$$\begin{aligned} |I_1| &\leq C(\|\varrho_\delta\|^{\gamma+\Theta}_{L^\infty(0,T;L^\gamma(\Omega))} \\ &+ \delta \|\varrho_\delta\|^\beta_{L^\infty(0,T;L^\beta(\Omega))} \|\varrho_\delta\|^\Theta_{L^\infty(0,T;L^\gamma(\Omega))}) \leq C(\mathrm{DATA}) \end{aligned}$$

provided $\Theta \leq \gamma$,

$$I_2^1 = C \int_0^T \int_\Omega \varrho_\delta \mathbf{u}_\delta \cdot \mathcal{B}(\mathrm{div}_x(\varrho_\delta \mathbf{u}_\delta)) \, \mathrm{d}x \ \mathrm{d}t$$

$$\leq C \int_0^T \|\varrho_\delta\|_{L^{\frac{3}{2}(1+\Theta)}(\Omega)} \|\mathbf{u}_\delta\|_{L^6(\Omega;\mathbb{R}^3)} \|\varrho_\delta \mathbf{u}_\delta\|_{L^{\frac{6(1+\Theta)}{1+5\Theta}}(\Omega;\mathbb{R}^3)} \, \mathrm{d}t$$

$$\leq C \int_0^T \int_\Omega \|\varrho_\delta\|^{1+\Theta}_{L^{\frac{3}{2}(1+\Theta)}(\Omega)} \|\mathbf{u}_\delta\|^2_{L^6(\Omega;\mathbb{R}^3)} \, \mathrm{d}x \ \mathrm{d}t \leq C(\mathrm{DATA})$$

if $\Theta \leq \frac{2}{3}\gamma - 1$,

$$\begin{aligned} |I_2^2| &\leq C \int_0^T \int_\Omega \varrho_\delta |\mathbf{u}_\delta| \Big| \mathcal{B}\Big(\varrho_\delta^\Theta \mathrm{div}_x \mathbf{u}_\delta - \frac{1}{\Omega|} \int_\Omega \varrho_\delta^\Theta \mathrm{div}_x \mathbf{u}_\delta \, \mathrm{d}x \Big) \Big| \, \mathrm{d}x \ \mathrm{d}t \\ &\leq C \int_0^T \|\varrho_\delta\|_{L^\gamma(\Omega)} \|\mathbf{u}_\delta\|_{L^6(\Omega;\mathbb{R}^3)} \times \\ &\quad \times \Big\| \mathcal{B}\Big(\varrho_\delta^\Theta \mathrm{div}_x \mathbf{u}_\delta - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\Theta \mathrm{div}_x \mathbf{u}_\delta \, \mathrm{d}x \Big) \Big\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega;\mathbb{R}^3)} \mathrm{d}t \\ &\leq C \int_0^T \|\varrho_\delta\|_{L^\gamma(\Omega)} \|\mathbf{u}_\delta\|_{L^6(\Omega;\mathbb{R}^3)} \|\mathrm{div}_x \mathbf{u}_\delta\|_{L^2(\Omega)} \|\varrho_\delta\|^\Theta_{L^{\frac{3\gamma\Theta}{2\gamma-3}}(\Omega)} \leq C(DATA) \end{aligned}$$

if $\Theta \leq \frac{2}{3}\gamma - 1$ and $\gamma \leq 6$,

$$
\begin{aligned}
|I_3| &\leq C \int_0^T \int_\Omega \varrho_\delta |\mathbf{u}_\delta|^2 \left| \nabla_x \mathcal{B}\left(\varrho_\delta^\Theta - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^\Theta \,\mathrm{d}x \right) \right| \mathrm{d}x \;\mathrm{d}t \\
&\leq C \int_0^T \|\varrho_\delta\|_{L^\gamma(\Omega)} \|\mathbf{u}_\delta\|^2_{L^6(\Omega;\mathbb{R}^3)} \|\varrho_\delta\|^\Theta_{L^{\frac{3\gamma\Theta}{2\gamma-3}}(\Omega)} \,\mathrm{d}x \;\mathrm{d}t \leq C(\mathrm{DATA})
\end{aligned}
$$

if $\Theta \leq \frac{2}{3}\gamma - 1$,

$$
\begin{aligned}
&|I_4| + |I_5| \leq \\
&\quad C \int_0^T \|\nabla_x \mathbf{u}_\delta\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \left\| \nabla_x \mathcal{B}\left(\varrho_\delta^\Theta - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^\Theta \,\mathrm{d}x \right) \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \mathrm{d}t \\
&\qquad \leq C \int_0^T \|\nabla_x \mathbf{u}_\delta\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \|\varrho_\delta\|_{L^{2\Theta}(\Omega)} \,\mathrm{d}t \leq C(DATA),
\end{aligned}
$$

if $\Theta \leq \frac{\gamma}{2}$,

$$
\begin{aligned}
|I_6| &\leq C \int_0^T \|\varrho_\delta\|_{L^\gamma(\Omega)} \left\| \mathcal{B}\left(\varrho_\delta^\Theta - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^\Theta \,\mathrm{d}x \right) \right\|_{L^{\frac{\gamma}{\gamma-1}}(\Omega;\mathbb{R}^3)} \mathrm{d}t \\
&\leq C \int_0^T \|\varrho_\delta\|^{1+\Theta}_{L^\gamma(\Omega)} \,\mathrm{d}t
\end{aligned}
$$

if $\Theta \leq \frac{4}{3}\Theta - 1$,

$$
\begin{aligned}
|I_7| &\leq \|\varrho_\delta \mathbf{u}_\delta\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))} \left\| \mathcal{B}\left(\varrho_\delta^\Theta - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^\Theta \,\mathrm{d}x \right) \right\|_{C([0,T];L^{\frac{2\gamma}{\gamma-1}}(\Omega))} \\
&\leq C(\mathrm{DATA})
\end{aligned}
$$

for $\Theta < \frac{5\gamma-1}{6}$; here we use that $\varrho_\delta \mathbf{u}_\delta \in C_{\mathrm{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3))$ and the embedding $W^{1,\frac{\gamma}{\Theta}}(\Omega) \hookrightarrow\hookrightarrow L^{\frac{2\gamma}{\gamma-1}}(\Omega)$. Finally

$$
|I_8| \leq C(\mathrm{DATA}).
$$

Hence

$$
\|\varrho_\delta\|_{L^{\gamma+\Theta}((0,T)\times\Omega)} + \delta \|\varrho_\delta\|_{L^{\beta+\Theta}((0,T)\times\Omega)} \leq C. \tag{8.41}
$$

Note that $\Theta = \min\{\frac{2}{3}\gamma - 1, \frac{\gamma}{2}\}$; for $\gamma = 6$ both values are equal. However, if we proceed once more for $\gamma > 6$ and use, instead of the information ϱ_δ

bounded in $L^\infty(0,T;L^\gamma(\Omega))$, the newly obtained information that the sequence is bounded in $L^{\frac{3}{2}\gamma}((0,T)\times\Omega)$ in the terms coming from the stress tensor, we end up with the fact that ϱ_δ is bounded in $L^{\frac{5}{3}\gamma-1}((0,T)\times\Omega)$. This improvement is not important for us in this situation, so we do not present any details and leave them for interested reader.

As above we can pass to the limit in the weak formulation of the continuity equation to get

$$\begin{aligned} &\int_\Omega (\varrho\Phi)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_0\Phi(0,\cdot)\,\mathrm{d}x \\ &= \int_0^\tau \int_\Omega \varrho\partial_t\Phi\,\mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\Phi\,\mathrm{d}x\ \mathrm{d}t \end{aligned} \tag{8.42}$$

for all $\Phi\in C_c^\infty([0,\tau]\times\overline{\Omega})$ and any $0<\tau\le T$. Moreover, as in the previous section

$$\varrho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta \rightharpoonup \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{in } L^q((0,T)\times\Omega;\mathbb{R}^{3\times3}) \text{ for some } q>1$$

and we may pass to the limit $\delta\to0^+$ in the weak formulation of the momentum equation (note that $\delta\int_0^\tau\int_\Omega \varrho_\delta^\beta \mathrm{div}_x\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t\to 0$)

$$\begin{aligned} &\int_\Omega(\varrho\mathbf{u}\cdot\mathbf{\Phi})(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_0\mathbf{u}_0\cdot\mathbf{\Phi}(0,\cdot)\,\mathrm{d}x \\ &= \int_0^\tau\int_\Omega \varrho\mathbf{u}\cdot\partial_t\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}):\nabla_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t \\ &-\mu\int_0^\tau\int_\Omega\nabla_x\mathbf{u}:\nabla_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t - (\mu+\lambda)\int_0^\tau\int_\Omega \mathrm{div}_x\mathbf{u}\,\mathrm{div}_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t \\ &+\int_0^\tau\int_\Omega \overline{\varrho^\gamma}\,\mathrm{div}_x\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{\Phi}\,\mathrm{d}x\ \mathrm{d}t \end{aligned} \tag{8.43}$$

for all $\mathbf{\Phi}\in C_c^\infty(([0,\tau]\times\overline{\Omega};\mathbb{R}^3)$ and all $0<\tau\le T$. To finish the proof it remains to show that $\overline{\varrho^\gamma}=\varrho^\gamma$. Recall that, due to restriction coming from above, we consider $\gamma>\frac{3}{2}$.

8.5.1 Strong convergence of the density

We will follow a similar strategy as before, i.e., we show

- effective viscous flux identity

- validity of the renormalized continuity equation
- strong convergence of the density

Recall that we control
$$\|\varrho_\delta\|_{L^{\gamma+\theta}((0,T)\times\Omega)} \le C,$$
where $\theta = \min\{\frac{2}{3}\gamma - 1, \frac{\gamma}{2}\}$. Then $\frac{5}{3}\gamma - 1 = 2$ for $\gamma = \frac{9}{5}$, i.e. for $\gamma < \frac{9}{5}$ there is an additional difficulty: for the limit $(\varrho, \mathbf{u})$ we do not have guaranteed the validity of the renormalized continuity equation, we will have to verify it differently.

We denote

$$T(z) = \begin{cases} z & \text{for } z \in [0,1], \\ \in (1,2] \text{ concave} & \text{for } z \in [1,3], \\ 2 & \text{for } z \ge 3 \end{cases}$$

with $T(\cdot) \in C^\infty(\mathbb{R}_0^+)$, and

$$T_k(z) = kT\left(\frac{z}{k}\right),\ k \in \mathbb{N}.$$

We aim at showing

$$\overline{\varrho^\gamma T_k(\varrho)} - (2\mu+\lambda)\overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} = \overline{\varrho^\gamma}\,\overline{T_k(\varrho)} - (2\mu+\lambda)\overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u} \qquad (8.44)$$

a.e. in $(0,T)\times\Omega$ for all $k \in \mathbb{N}$. The proof is based on a similar idea as before; we use a clever test function for approximated momentum equation for $\delta > 0$, then for the limit problem; finally we pass to the limit $\delta \to 0^+$, using certain tools from the compensated compactness theory.

Recall that we have for $\delta > 0$ the renormalized continuity equation in the form (in the sense of distributions in $(0,T)\times\Omega$)

$$\partial_t(T_k(\varrho_\delta)) + \mathrm{div}_x(T_k(\varrho_\delta)\mathbf{u}_\delta) + (\varrho_\delta T_k'(\varrho_\delta) - T_k(\varrho_\delta))\mathrm{div}_x\mathbf{u}_\delta = 0, \qquad (8.45)$$

however, for the limit we only have (in fact, for $\gamma \ge \frac{9}{5}$ the situation is better, however, we are mainly interested in low γ's)

$$\partial_t(\overline{T_k(\varrho)}) + \mathrm{div}_x(\overline{T_k(\varrho)}\mathbf{u}) + \overline{(\varrho T_k'(\varrho) - T_k(\varrho))\mathrm{div}_x\mathbf{u}} = 0 \qquad (8.46)$$

(again in the sense of distributions in $(0,T)\times\Omega$)

We use as the test function in the approximated momentum equation (understood in the weak sense)

$$\partial_t(\varrho_\delta \mathbf{u}_\delta) + \mathrm{div}_x(\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) - \mu\Delta \mathbf{u}_\delta - (\mu+\lambda)\nabla_x \mathrm{div}_x \mathbf{u}_\delta + \nabla_x \left(\varrho_\delta^\gamma + \delta \varrho_\delta^\beta\right) = \varrho_\delta \mathbf{f}$$

the function

$$\boldsymbol{\varphi}_\delta = \phi \nabla_x \Delta^{-1}(1_\Omega T_k(\varrho_\delta)),\ k \in \mathbb{N}$$

and for the limit equation (again, understood in the weak sense)

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu\Delta \mathbf{u} - (\mu+\lambda)\nabla_x \mathrm{div}_x \mathbf{u} + \nabla_x \overline{\varrho^\gamma} = \varrho \mathbf{f}$$

the test function

$$\boldsymbol{\varphi} = \phi \nabla_x \Delta^{-1}(1_\Omega \overline{T_k(\varrho)}),\ k \in \mathbb{N}.$$

Here, Δ^{-1} represents as in the previous section the inverse of the Laplacean on $\mathbb{R}^3$, i.e.

$$\partial_{x_j} \Delta^{-1}[v] = \mathcal{F}_{\xi \to x}\left[\frac{\mathrm{i}\xi_j}{|\xi|^2}\mathcal{F}_{x \to \xi}[v]\right],$$

and $\phi \in C_c^\infty((0,T)\times\Omega)$. Note that for $1 \le p < 3$ we have

$$\|\nabla_x \Delta^{-1}[v]\|_{L^{\frac{3p}{3-p}}(\Omega;\mathbb{R}^3)} \le C\|v\|_{L^p(\Omega)}$$

and for $p > 3$

$$\|\nabla_x \Delta^{-1}[v]\|_{C(\overline{\Omega};\mathbb{R}^3)} \le C\|v\|_{L^p(\Omega)}.$$

Step 1: As

$$\varrho_\delta \to \varrho \text{ in } C_{\mathrm{weak}}([0,T];L^\gamma(\Omega)),$$

we have, in accordance with the standard Sobolev embedding relation

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega}), \qquad p > 3,$$

$$\nabla_x \Delta^{-1}[1_\Omega T_k(\varrho_\delta)] \to \nabla_x \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}] \text{ in } C([0,T]\times\overline{\Omega}).$$

Now for $\phi \in C_c^\infty((0,T)\times\Omega)$

$$\lim_{\delta\to 0^+}\Big[\int_0^T\int_\Omega \Big(\phi p(\varrho_\delta)T_k(\varrho_\delta) + p(\varrho_\delta)\nabla_x\phi\cdot\nabla_x\Delta^{-1}[1_\Omega T_k(\varrho_\delta)]\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{8.47}$$

$$-\int_0^T\int_\Omega \phi\Big(\mu\nabla_x\mathbf{u}_\delta : \nabla_x^2\Delta^{-1}[1_\Omega T_k(\varrho_\delta)] + (\lambda+\mu)\mathrm{div}_x\mathbf{u}_\delta T_k(\varrho_\delta)\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&-\int_0^T \int_\Omega \Big(\mu \nabla_x \mathbf{u}_\delta \cdot \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega T_k(\varrho_\delta)] \\
&+(\lambda+\mu)\mathrm{div}_x \mathbf{u}_\delta \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega T_k(\varrho_\delta)]\Big) \,\mathrm{d}x\,\mathrm{d}t\Big] \\
&= \int_0^T \int_\Omega \Big(\phi \overline{p(\varrho) T_k(\varrho)} - \overline{p(\varrho)} \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}]\Big) \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^T \int_\Omega \phi \Big(\mu \nabla_x \mathbf{u} : \nabla_x^2 \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}] + (\lambda+\mu)\mathrm{div}_x \mathbf{u} \overline{T_k(\varrho)}\Big) \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^T \int_\Omega \Big(\mu \nabla_x \mathbf{u} \cdot \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}] \\
&+(\lambda+\mu)\mathrm{div}_x \mathbf{u} \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}]\Big) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \lim_{\delta \to 0^+} \int_0^T \int_\Omega \Big(\phi \varrho_\delta \mathbf{u}_\delta \cdot \nabla_x \Delta^{-1}[\mathrm{div}_x(T_k(\varrho_\delta)\mathbf{u}_\delta) + (\varrho_\delta T_k'(\varrho_\delta) - T_k(\varrho_\delta))\mathrm{div}_x \mathbf{u}_\delta] \\
&-\varrho_\delta(\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla_x \left(\phi \nabla_x \Delta^{-1}[1_\Omega T_k(\varrho_\delta)]\right)\Big) \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^T \int_\Omega \Big(\phi \varrho \mathbf{u} \cdot \nabla_x \Delta^{-1}[\mathrm{div}_x(\overline{T_k(\varrho)}\mathbf{u}) + \overline{(\varrho T_k'(\varrho) - T_k(\varrho))\mathrm{div}_x \mathbf{u}}] \\
&-\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla_x \left(\phi \nabla_x \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}]\right)\Big) \,\mathrm{d}x\,\mathrm{d}t \\
&- \lim_{\delta \to 0^+} \int_0^T \int_\Omega \partial_t \phi \varrho_\delta \mathbf{u}_\delta \cdot \nabla_x \Delta^{-1}(T_k(\varrho_\delta)) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega \partial_t \phi \varrho \mathbf{u} \cdot \nabla_x \Delta^{-1}(\overline{T_k(\varrho)}) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Step 2: We have

$$
\int_\Omega \phi \nabla_x \mathbf{u}_\delta : \nabla_x^2 \Delta^{-1}[1_\Omega T_k(\varrho_\delta)] \,\mathrm{d}x = \int_\Omega \phi \sum_{i,j=1}^3 \left(\partial_{x_j} u_\delta^i [\partial_{x_i} \Delta^{-1} \partial_{x_j}][1_\Omega T_k(\varrho_\delta)]\right) \,\mathrm{d}x
$$

$$
= \int_\Omega \sum_{i,j=1}^3 \left(\partial_{x_j}(\phi u_\delta^i)[\partial_{x_i} \Delta^{-1} \partial_{x_j}][1_\Omega T_k(\varrho_\delta)]\right) \,\mathrm{d}x
$$

$$-\int_\Omega \sum_{i,j=1}^3 \left(\partial_{x_j}\phi u^i_\delta[\partial_{x_i}\Delta^{-1}\partial_{x_j}][1_\Omega T_k(\varrho_\delta)]\right)\ \mathrm{d}x$$

$$= \int_\Omega \phi \mathrm{div}_x \mathbf{u}_\delta T_k(\varrho_\delta)\ \mathrm{d}x + \int_\Omega \nabla_x\phi\cdot\mathbf{u}_\delta T_k(\varrho_\delta)\ \mathrm{d}x$$

$$-\int_\Omega \sum_{i,j=1}^3 \left(\partial_{x_j}\phi u^i_\delta[\partial_{x_i}\Delta^{-1}\partial_{x_j}][1_\Omega T_k(\varrho_\delta)]\right)\ \mathrm{d}x.$$

Consequently, going back to (8.47) and dropping the compact terms, we obtain

$$\lim_{\delta\to 0^+}\int_0^T\int_\Omega \phi\Big(p(\varrho_\delta)T_k(\varrho_\delta) - (\lambda+2\mu)\mathrm{div}_x\mathbf{u}_\delta T_k(\varrho_\delta)\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{8.48}$$

$$-\int_0^T\int_\Omega \phi\Big(\overline{p(\varrho)T_k(\varrho)} - (\lambda+2\mu)\mathrm{div}_x\mathbf{u}\overline{T_k(\varrho)}\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$= \lim_{\delta\to 0^+}\int_0^T\int_\Omega \phi\Big(\varrho_\delta\mathbf{u}_\delta\cdot\nabla_x\Delta^{-1}[\mathrm{div}_x(T_k(\varrho_\delta)\mathbf{u}_\delta)]$$

$$-\varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \nabla_x\Delta^{-1}\nabla_x[1_\Omega T_k(\varrho_\delta)]\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^T\int_\Omega\Big(\phi\varrho\mathbf{u}\cdot\nabla_x\Delta^{-1}[\mathrm{div}_x(\overline{T_k(\varrho)}\mathbf{u})] - \phi\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\Delta^{-1}\nabla_x[1_\Omega\overline{T_k(\varrho)}]\Big)\ \mathrm{d}x\ \mathrm{d}t.$$

Step 3: Our goal is to show that the right-hand side of (8.48) vanishes. We write

$$\int_\Omega\phi\Big[\varrho_\delta\mathbf{u}_\delta\cdot\nabla_x\Delta^{-1}[1_\Omega\mathrm{div}_x(T_k(\varrho_\delta)\mathbf{u}_\delta)] - \varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \nabla_x\Delta^{-1}\nabla_x[1_\Omega T_k(\varrho_\delta)]\Big]\mathrm{d}x$$

$$= \int_\Omega\phi\mathbf{u}_\delta\cdot\Big[T_k(\varrho_\delta)\nabla_x\Delta^{-1}[\mathrm{div}_x(1_\Omega\varrho_\delta\mathbf{u}_\delta)] - \varrho_\delta\mathbf{u}_\delta\cdot\nabla_x\Delta^{-1}\nabla_x[1_\Omega T_k(\varrho_\delta)]\Big]\mathrm{d}x + \mathrm{l.o.t.},$$

where l.o.t. denotes lower order terms (with derivatives on ϕ). As in Chapter 6, we consider the bilinear form

$$[\mathbf{v},\mathbf{w}] = \sum_{i,j=1}^3\Big(v^i\mathcal{R}_{i,j}[w^j] - w^i\mathcal{R}_{i,j}[v^j]\Big),\ \mathcal{R}_{i,j} = \partial_{x_i}\Delta^{-1}\partial_{x_j},$$

writing

$$\sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{i,j}[w^j] - w^i \mathcal{R}_{i,j}[v^j] \Big)$$

$$= \sum_{i,j=1}^{3} \Big((v^i - \mathcal{R}_{i,j}[v^j]) \mathcal{R}_{i,j}[w^j] - (w^i - \mathcal{R}_{i,j}[w^j]) \mathcal{R}_{i,j}[v^j] \Big)$$

$$= \mathbf{U} \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{Z},$$

where

$$U^i = \sum_{j=1}^{3} (v^i - \mathcal{R}_{i,j}[v^j]), \ W^i = \sum_{j=1}^{3} (w^i - \mathcal{R}_{i,j}[w^j]), \ \mathrm{div}_x \mathbf{U} = \mathrm{div}_x \mathbf{W} = 0,$$

and

$$V^i = \partial_{x_i} \left(\sum_{j=1}^{3} \Delta^{-1} \partial_{x^j} w^j \right), \ Z^i = \partial_{x_i} \left(\sum_{j=1}^{3} \Delta^{-1} \partial_{x^j} v^j \right), \ i = 1, 2, 3.$$

Therefore we may apply the Div-Curl lemma (Lemma 6.1) and using

$$T_k(\varrho_\delta) \to \overline{T_k(\varrho)} \text{ in } C_{\rm weak}([0,T]; L^q(\Omega)), \ 1 \le q < \infty,$$

$$\varrho_\delta \mathbf{u}_\delta \to \varrho \mathbf{u} \text{ in } C_{\rm weak}([0,T]; L^{2\gamma/(\gamma+1)}(\Omega; \mathbb{R}^3)),$$

for $v_i := \overline{T_k(\varrho)} \delta_{il}$, $l = 1, 2, 3$, and $\mathbf{w} := \varrho \mathbf{u}$, similarly for $\mathbf{v}_\delta$ and $\mathbf{w}_\delta$; we conclude that

$$T_k(\varrho_\delta)(t,\cdot) \nabla_x \Delta^{-1} [1_\Omega \mathrm{div}_x (\varrho_\delta \mathbf{u}_\delta)(t,\cdot)] - (\varrho_\delta \mathbf{u}_\delta)(t,\cdot) \cdot \nabla_x \Delta^{-1} \nabla_x [1_\Omega T_k(\varrho_\delta)(t,\cdot)] \tag{8.49}$$

$$\to$$

$$\overline{T_k(\varrho)}(t,\cdot) \nabla_x \Delta^{-1} [1_\Omega \mathrm{div}_x (\varrho \mathbf{u})(t,\cdot)] - (\varrho \mathbf{u})(t,\cdot) \cdot \nabla_x \Delta^{-1} \nabla_x [1_\Omega \overline{T_k(\varrho)}(t,\cdot)]$$

$$\text{weakly in } L^s(\Omega; \mathbb{R}^3) \text{ for all } t \in [0,T],$$

with

$$s < \frac{2\gamma}{\gamma+1}; \text{ but } \frac{2\gamma}{\gamma+1} > \frac{6}{5} \text{ since } \gamma > \frac{3}{2}.$$

Thus we may take $s > \frac{6}{5}$. Then the convergence in (8.49) takes place in the space

$$L^q(0,T; W^{-1,2}(\Omega)) \text{ for any } 1 \le q < \infty;$$

going back to (8.48), we have

$$\lim_{\delta\to 0}\int_0^\tau\int_\Omega \phi\Big(p(\varrho_\delta)T_k(\varrho_\delta)-(\lambda+2\mu)\mathrm{div}_x\mathbf{u}_\delta T_k(\varrho_\delta)\Big)\,\mathrm{d}x\ \mathrm{d}t \tag{8.50}$$
$$=\int_0^\tau\int_\Omega \phi\Big(\overline{p(\varrho)}\ \overline{T_k(\varrho)}-(\lambda+2\mu)\mathrm{div}_x\mathbf{u}\overline{T_k(\varrho)}\Big)\,\mathrm{d}x\ \mathrm{d}t.$$

Therefore, localizing we get as in Chapter 6 the desired form of the *effective viscous flux identity* (8.44).

Next we want to verify that for $\gamma>\frac{3}{2}$ we have the renormalized continuity equation (with $b(\varrho)=T_k(\varrho)$) fulfilled. For $\gamma\ge\frac{9}{5}$ we get this immediately, as ϱ belongs to $L^2((0,T)\times\Omega)$. But for $\gamma<\frac{9}{5}$ additional work is required.

We introduce the quantity *oscillation defect measure*

$$\mathrm{osc}_q(\varrho_\delta-\varrho):=\sup_{k\ge 1}\limsup_{\delta\to 0^+}\|T_k(\varrho_\delta)-T_k(\varrho)\|_{L^q((0,T)\times\Omega)}.$$

Below, we shall show

(i) $\mathrm{osc}_{\gamma+1}(\varrho_\delta-\varrho)<\infty$

(ii)
$$\limsup_{\delta\to 0^+}\int_0^T\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x\ \mathrm{d}t \le \int_0^T\int_\Omega(\overline{\varrho^\gamma T_k(\varrho)}-\overline{\varrho^\gamma}\,\overline{T_k(\varrho)})\,\mathrm{d}x\ \mathrm{d}t, \tag{8.51}$$

(iii) if $\mathrm{osc}_q(\varrho_\delta-\varrho)<\infty$ for some $q>2$, the limit functions $(\varrho,\mathbf{u})$ fulfill the renormalized continuity equation (with $b(\varrho)$ a bounded, smooth function) and hence, by density argument, also for less regular functions.

Lemma 8.4 *We have (ii), i.e. (8.51), and (i).*

Proof: We have

$$\int_0^T\int_\Omega\Big(\overline{\varrho^\gamma T_k(\varrho)}-\overline{\varrho^\gamma}\,\overline{T_k(\varrho)}\Big)\,\mathrm{d}x\ \mathrm{d}t=\lim_{\delta\to 0^+}\int_0^T\int_\Omega\Big(\varrho_\delta^\gamma T_k(\varrho_\delta)-\varrho_\delta^\gamma\overline{T_k(\varrho)}\Big)\,\mathrm{d}x\ \mathrm{d}t$$
$$=\lim_{\delta\to 0^+}\int_0^T\int_\Omega(\varrho_\delta^\gamma-\varrho^\gamma)(T_k(\varrho_\delta)-T_k(\varrho))\,\mathrm{d}x\ \mathrm{d}t$$
$$+\int_0^T\int_\Omega(\overline{\varrho^\gamma}-\varrho^\gamma)(T_k(\varrho)-\overline{T_k(\varrho)})\,\mathrm{d}x\ \mathrm{d}t. \tag{8.52}$$

However, the second term is nonnegative, as

$$\begin{aligned} &\varrho \mapsto \varrho^\gamma \text{ is convex,} \\ &\varrho \mapsto T_k(\varrho) \text{ is concave,} \end{aligned}$$

i.e. $\varrho^\gamma \leq \overline{\varrho^\gamma}$ and $T_k(\varrho) \geq \overline{T_k(\varrho)}$, see Lemma 8.3. Next, as

$$\begin{aligned} |T_k(t) - T_k(s)| &\leq |t - s|, \qquad t, s \geq 0 \\ (t-s)^\gamma &\leq (t^\gamma - s^\gamma), \qquad t \geq s \geq 0, \end{aligned}$$

we get

$$(T_k(t) - T_k(s))(t^\gamma - s^\gamma) \geq |T_k(t) - T_k(s)|^{\gamma+1}, \qquad t, s \geq 0.$$

Hence

$$\begin{aligned} \limsup_{\delta \to 0^+} &\int_0^T \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \, \mathrm{d}x \; \mathrm{d}t \\ &\leq \int_0^T \int_\Omega (\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\,\overline{T_k(\varrho)}) \, \mathrm{d}x \; \mathrm{d}t \end{aligned}$$

which proves (ii). Using now (8.44), we have the identity

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\, \overline{T_k(\varrho)}\Big) \, \mathrm{d}x \; \mathrm{d}t \\ &= (2\mu + \lambda) \lim_{\delta \to 0^+} \int_0^T \int_\Omega \mathrm{div}_x \mathbf{u}_\delta \Big(T_k(\varrho_\delta) - \overline{T_k(\varrho)}\Big) \, \mathrm{d}x \; \mathrm{d}t \\ &= (2\mu + \lambda) \lim_{\delta \to 0^+} \int_0^T \int_\Omega \mathrm{div}_x \mathbf{u}_\delta \Big((T_k(\varrho_\delta) - T_k(\varrho)) + (T_k(\varrho) - \overline{T_k(\varrho)})\Big) \, \mathrm{d}x \; \mathrm{d}t \\ &\leq C \limsup_{\delta \to 0^+} \Big[\|\mathrm{div}_x \mathbf{u}_\delta\|_{L^2((0,T)\times\Omega)} \Big(\|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^2((0,T)\times\Omega)} \\ &\qquad + \|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^2((0,T)\times\Omega)} \Big) \Big]. \end{aligned}$$

Moreover, due to Lemma 8.3

$$\begin{aligned} \|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^2((0,T)\times\Omega)} &\leq \liminf_{\delta \to 0^+} \|T_k(\varrho) - T_k(\varrho_\delta)\|_{L^2((0,T)\times\Omega)} \\ &\leq \limsup_{\delta \to 0^+} \|T_k(\varrho) - T_k(\varrho_\delta)\|_{L^2((0,T)\times\Omega)}. \end{aligned}$$

Hence

$$\begin{aligned}
&\limsup_{\delta\to 0^+} \|T_k(\varrho_\delta) - T_k(\varrho)\|^{\gamma+1}_{L^{\gamma+1}((0,T)\times\Omega)} \\
&= \limsup_{\delta\to 0^+} \int_0^T \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \, \mathrm{d}x \, \mathrm{d}t \\
&\le \int_0^T \int_\Omega \Big(\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\, \overline{T_k(\varrho)}\Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\le C \limsup_{\delta\to 0^+} \|\mathrm{div}_x \mathbf{u}_\delta\|_{L^2((0,T)\times\Omega)} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^2((0,T)\times\Omega)} \\
&\le C \limsup_{\delta\to 0^+} \|\mathrm{div}_x \mathbf{u}_\delta\|_{L^2((0,T)\times\Omega)} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^{\gamma+1}((0,T)\times\Omega)}.
\end{aligned}$$

As we control the L^2-norm of $\mathrm{div}_x \mathbf{u}_\delta$, the proof of (i) is finished. □

We now prove (iii).

Lemma 8.5 *Let $Q \subset \mathbb{R}^4$ be an open set. Let*

$$\begin{aligned}
\varrho_\delta &\rightharpoonup \varrho \quad \text{in } L^1(Q), \\
\mathbf{u}_\delta &\rightharpoonup \mathbf{u} \quad \text{in } L^r(Q;\mathbb{R}^3), \\
\nabla_x \mathbf{u}_\delta &\rightharpoonup \nabla_x \mathbf{u} \quad \text{in } L^r(Q;\mathbb{R}^{3\times 3}),
\end{aligned} \tag{8.53}$$

where $r > 1$. Let

$$\mathrm{osc}_q(\varrho_\delta - \varrho) < \infty, \tag{8.54}$$

$\frac{1}{q} + \frac{1}{r} < 1$, where ϱ_δ, $\mathbf{u}_\delta$ are renormalized solutions to the continuity equation. Then also the limit ϱ, $\mathbf{u}$ is a renormalized solution to the continuity equation.

Remark 8.2 *The claim of the lemma considers the following definition of the renormalized solutions to the continuity equation: for any $b \in C^1([0,\infty))$ such that $b'(z) = 0$ for $z \ge M$ for some $M > 0$ it holds*

$$\partial_t(b(\varrho)) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + (b'(\varrho)\varrho - b(\varrho))\mathrm{div}_x \mathbf{u} = 0$$

in $\mathcal{D}'((0,T)\times\Omega)$. Moreover, using the technique from Lemmas 7.2 and 7.3 and ideas used in Chapter 3, we may end up with the renormalized continuity equation in the time-integrated form with larger class of functions b.

Proof: First of all, note that it is enough to show the result on $J \times K$ with J a bounded time interval, K a ball such that $\overline{J \times K} \subset Q$. Recall that we consider functions $b(z)$ of class $C^1([0,\infty))$ which are constant for large values of z. Due to the assumption of the lemma and results from Chapter 7 we know that

$$\begin{aligned} T_k(\varrho_\delta) \to \overline{T_k(\varrho)} \quad &\text{in } C_{\text{weak}}(\overline{J}; L^\beta(K)) \quad \text{for any } 1 \le \beta < \infty, \\ T_k(\varrho_\delta)\mathbf{u}_\delta \rightharpoonup \overline{T_k(\varrho)}\mathbf{u} \quad &\text{in } L^r(J \times K; \mathbb{R}^3). \end{aligned}$$

Therefore

$$\partial_t \overline{T_k(\varrho)} + \mathrm{div}_x \left(\overline{T_k(\varrho)}\mathbf{u} \right) + \overline{((T_k'(\varrho)\varrho - T_k(\varrho))\mathrm{div}_x \mathbf{u})} = 0 \quad \text{in } \mathcal{D}'(J \times K).$$

Proceeding as in the proof of Lemma 7.3 we can show that

$$\begin{aligned} &\partial_t b(\overline{T_k(\varrho)}) + \mathrm{div}_x \left(b(\overline{T_k(\varrho)})\mathbf{u} \right) + \left((b'(\overline{T_k(\varrho)})\overline{T_k(\varrho)} - b(\overline{T_k(\varrho)}))\mathrm{div}_x \mathbf{u} \right) \\ &\quad = -b'(\overline{T_k(\varrho)})\overline{((T_k'(\varrho)\varrho - T_k(\varrho))\mathrm{div}_x \mathbf{u})} \quad \text{in } \mathcal{D}'(J \times K), \end{aligned}$$

where $b'(z) = 0$ for $z \ge M$. Note that

$$\lim_{k \to \infty} \lim_{\delta \to 0^+} \|\varrho_\delta - T_k(\varrho_\delta)\|_{L^1((0,T) \times \Omega)} = 0$$

as $\varrho_\delta \rightharpoonup \varrho$ in $L^1((0,T) \times \Omega)$ and hence ϱ_δ is equiintegrable. On the other hand,

$$\begin{aligned} \lim_{\delta \to 0^+} \int_0^T \int_\Omega (\varrho_\delta - T_k(\varrho_\delta)) \mathrm{d}x \, \mathrm{d}t &= \int_0^T \int_\Omega (\varrho - \overline{T_k(\varrho)}) \mathrm{d}x \, \mathrm{d}t \\ &= \|\varrho - \overline{T_k(\varrho)}\|_{L^1((0,T) \times \Omega)}. \end{aligned}$$

Therefore it suffices to show that

$$b'(\overline{T_k(\varrho)})\overline{((T_k'(\varrho)\varrho - T_k(\varrho))\mathrm{div}_x \mathbf{u})} \to 0$$

in $L^1(J \times K)$ for $k \to \infty$. Denote

$$Q_{k,M} = \{(t,x) \in J \times K; \, |\overline{T_k(\varrho)}| \le M\}.$$

We have

$$\begin{aligned} &\left\| b'(\overline{T_k(\varrho)})\overline{((T_k'(\varrho)\varrho - T_k(\varrho))\mathrm{div}_x \mathbf{u})} \right\|_{L^1(Q_{k,M})} \\ &\le C \sup\nolimits_{\delta > 0} \|\mathrm{div}_x \mathbf{u}_\delta\|_{L^r(J \times K)} \liminf\nolimits_{\delta \to 0} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^{r'}(Q_{k,M})}. \end{aligned}$$

Clearly,

$$\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^{r'}(Q_{k,M})} \leq \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|^{\alpha}_{L^1(Q_{k,M})}\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|^{1-\alpha}_{L^q(Q_{k,M})},$$

$0 < \alpha < 1$. As the family $\{\varrho_\delta\}_{\delta>0}$ is equiintegrable, due to a similar argument as above

$$\sup_{\delta>0} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^1(J\times K)} \to 0 \quad \text{for } k \to \infty.$$

Now, recalling that $0 \leq T_k'(\varrho_\delta)\varrho_\delta \leq T_k(\varrho_\delta)$, we get

$$\begin{aligned}\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^q(Q_{k,M})} &\leq \Big(\|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^q(Q_{k,M})} \\ &\quad + \|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^q(J\times K)} + \|\overline{T_k(\varrho)}\|_{L^q(Q_{k,M})}\Big) \\ &\leq \Big(\|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^q(Q_{k,M})} + \mathrm{osc}_q(\varrho_\delta - \varrho) + M|J\times K|^{\frac{1}{q}}\Big).\end{aligned}$$

Therefore

$$\limsup_{\delta\to 0^+} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^q(Q_{k,M})} \leq 2\mathrm{osc}_q(\varrho_\delta - \varrho) + M|J\times K|^{\frac{1}{q}} \leq C.$$

The lemma is proved. □

Next we take

$$b_k(\varrho) = \varrho\int_1^\varrho \frac{T_k(z)}{z^2}\,\mathrm{d}z;$$

note that

$$b_k'(\varrho) = \int_1^\varrho \frac{T_k(z)}{z^2}\,\mathrm{d}z + \frac{T_k(\varrho)}{\varrho},$$

i.e. $b_k''(\varrho) > 0$ for $\varrho > 0$. Then $\varrho b_k'(\varrho) - b_k(\varrho) = T_k(\varrho)$ and we have (it follows by the limit passage $\delta \to 0^+$ in the renormalized continuity equation for $\delta > 0$ and the fact that we may extend the density and velocity by zero and use the equations for these extended functions in the whole $\mathbb{R}^3$)

$$\int_\Omega \overline{b_k(\varrho)(\tau,\cdot)}\,\mathrm{d}x - \int_\Omega b_k(\varrho_0)\,\mathrm{d}x + \int_0^\tau\int_\Omega \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}}\,\mathrm{d}x\ \mathrm{d}t = 0$$

for all $\tau \in (0,T]$ and, due to Lemma 8.5

$$\int_\Omega b_k(\varrho)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega b_k(\varrho_0)\,\mathrm{d}x + \int_0^\tau \int_\Omega T_k(\varrho)\mathrm{div}_x\mathbf{u}\,\mathrm{d}x\ \mathrm{d}t = 0$$

for all $\tau \in (0,T]$; we also used that $\varrho_{0,\delta} \to \varrho_0$ in $L^1(\Omega)$ and weakly in $L^\gamma(\Omega)$. Therefore

$$\int_\Omega \Big(\overline{b_k(\varrho(t))} - b_k(\varrho(t))\Big)\,\mathrm{d}x = \int_0^t \int_\Omega \Big(T_k(\varrho)\mathrm{div}_x\mathbf{u} - \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}}\Big)\,\mathrm{d}x\,\mathrm{d}\tau.$$

But b_k is convex and thus

$$\begin{aligned} 0 \le \int_0^T \int_\Omega \Big(T_k(\varrho)\mathrm{div}_x\mathbf{u} - \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}}\Big)\,\mathrm{d}x\ \mathrm{d}t \\ = \int_0^T \int_\Omega \Big(T_k(\varrho) - \overline{T_k(\varrho)}\Big)\mathrm{div}_x\mathbf{u}\,\mathrm{d}x\ \mathrm{d}t \\ + \int_0^T \int_\Omega \Big(\overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u} - \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}}\Big)\,\mathrm{d}x\ \mathrm{d}t. \end{aligned}$$

Now from (8.44) and (8.51)

$$\begin{aligned} \int_0^T \int_\Omega \Big(\overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u}\Big)\,\mathrm{d}x\ \mathrm{d}t \\ = \frac{1}{2\mu+\lambda}\int_0^T \int_\Omega (\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\,\overline{T_k(\varrho)})\,\mathrm{d}x\ \mathrm{d}t \\ \ge \frac{1}{2\mu+\lambda}\limsup_{\delta\to 0^+}\int_0^T \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x\ \mathrm{d}t, \end{aligned}$$

i.e.

$$\begin{aligned} \frac{1}{2\mu+\lambda}\limsup_{\delta\to 0^+}\int_0^T \|T_k(\varrho_\delta) - T_k(\varrho)\|^{\gamma+1}_{L^{\gamma+1}(\Omega)} \\ \le \int_0^T \int_\Omega |T_k(\varrho) - \overline{T_k(\varrho)}||\mathrm{div}_x\mathbf{u}|\,\mathrm{d}x\ \mathrm{d}t \\ \le \|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^2((0,T)\times\Omega)}\|\mathrm{div}_x\mathbf{u}\|_{L^2((0,T)\times\Omega)} \\ \le C\|T_k(\varrho) - \overline{T_k(\varrho)}\|^{\frac{\gamma-1}{2\gamma}}_{L^1((0,T)\times\Omega)}\|T_k(\varrho) - \overline{T_k(\varrho)}\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}((0,T)\times\Omega)}. \end{aligned}$$

Recall that

$$\|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^1((0,T)\times\Omega)} \leq \|T_k(\varrho) - \varrho\|_{L^1((0,T)\times\Omega)} + \|\overline{T_k(\varrho)} - \varrho\|_{L^1((0,T)\times\Omega)}.$$

Hence

$$\lim_{k\to\infty} \|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^1((0,T)\times\Omega)} = 0.$$

As

$$\lim_{k\to\infty} \|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^{\gamma+1}((0,T)\times\Omega)} \leq \mathrm{osc}_{\gamma+1}(\varrho_\delta - \varrho) = C,$$

we also have that

$$\lim_{k\to\infty} \limsup_{\delta\to 0^+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^{\gamma+1}((0,T)\times\Omega)} = 0.$$

Finally, as

$$\limsup_{\delta\to 0^+} \|\varrho_\delta - \varrho\|_{L^1((0,T)\times\Omega)} \leq \limsup_{\delta\to 0^+} \|\varrho_\delta - T_k(\varrho_\delta)\|_{L^1((0,T)\times\Omega)} + \limsup_{\delta\to 0^+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^1((0,T)\times\Omega)} + \limsup_{\delta\to 0^+} \|T_k(\varrho) - \varrho\|_{L^1((0,T)\times\Omega)} = 0,$$

we proved

$$\varrho_\delta \to \varrho \quad \text{in } L^1((0,T)\times\Omega)$$

and therefore also in $L^p((0,T)\times\Omega)$ for every $p < \gamma + \theta$.

To conclude the existence proof, note that we may pass to the limit in the energy inequality as before. We have

Theorem 8.3 *Let $\gamma > \frac{3}{2}$, $0 < \Theta \leq 1$, $\Omega \in C^{2,\Theta}$, $0 < T < \infty$ and $\varrho_0 \in L^\gamma(\Omega)$, $\varrho_0|\mathbf{u}_0|^2 = \frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} \in L^1(\Omega)$ and $\mathbf{f} \in L^\infty((0,T)\times\Omega;\mathbb{R}^3)$. Let $p(\varrho) = \varrho^\gamma$. Then there exists a weak solution to the compressible Navier–Stokes system satisfying the energy inequality, i.e. a weak solution in the sense of Chapter 5.*

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