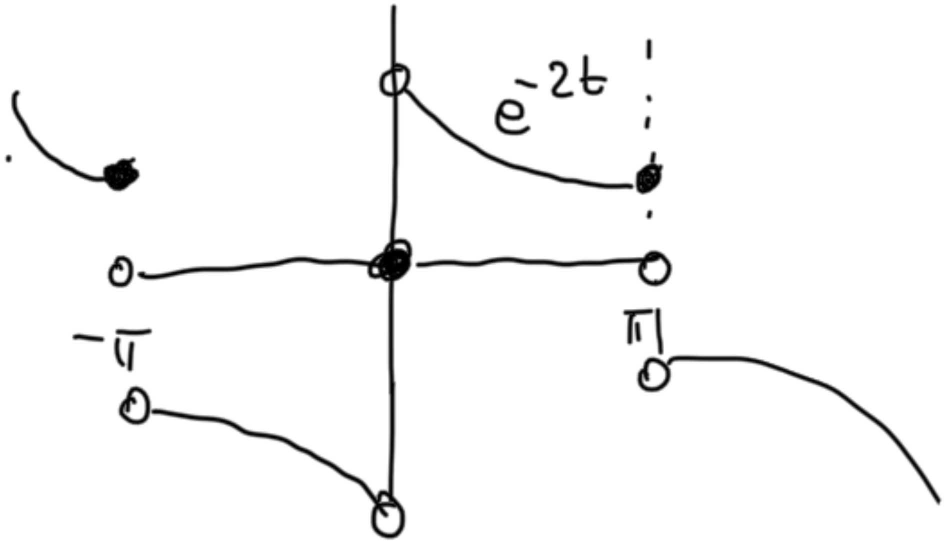
③ $f(t)=e^{-2t}$, $t\in(0,\pi]$ do sinové FŘ, sečtěte $\sum_{k=0}^{\infty}(-1)^k\frac{2k+1}{4+(2k+1)^2}$.

Řešení:



$$f(t)=\begin{cases} e^{-2(t-2k\pi)}, & t\in(2k\pi,(2k+1)\pi],\ k\in\mathbb{Z}\\ 0, & t=2k\pi,\ k\in\mathbb{Z}\\ -e^{2(t-2k\pi)}, & t\in((2k-1)\pi,2k\pi),\ k\in\mathbb{Z}\end{cases}$$

f je lichá na $(-\pi,\pi)$, takže $a_k=0$, $k\in\mathbb{N}\cup\{0\}$, $b_k=\frac{2}{\pi}\int_0^\pi e^{-2t}\sin(kt)\,dt$, $k\in\mathbb{N}$

označme $I_k=\int_0^\pi e^{-2t}\sin(kt)\,dt$, pak

$$I_k=\left[e^{-2t}\frac{\cos(kt)}{k}\right]_\pi^0-\int_0^\pi(-2e^{-2t})\left(-\frac{\cos(kt)}{k}\right)dt=\frac{1}{k}+\frac{(-1)^{k+1}e^{-2\pi}}{k}-\frac{2}{k}\int_0^\pi e^{-2t}\cos(kt)\,dt$$

$$=\frac{1+(-1)^{k+1}e^{-2\pi}}{k}-\frac{2}{k}\left(\underbrace{\left[e^{-2t}\frac{\sin(kt)}{k}\right]_0^\pi}_{=0}-\int_0^\pi(-2e^{-2t})\frac{\sin(kt)}{k}\,dt\right)$$

$$=\frac{1+(-1)^{k+1}e^{-2\pi}}{k}-\frac{4}{k^2}I_k.$$ Tedy $\left(1+\frac{4}{k^2}\right)I_k=\frac{1+(-1)^{k+1}e^{-2\pi}}{k}$, a tedy

$$I_k = \frac{1+(-1)^{k+1}e^{-2\pi}}{k} \cdot \frac{k^2}{k^2+4} =$$

$$= \left((-1)^{k+1}e^{-2\pi}+1\right)\frac{k}{k^2+4}, \quad k \in \mathbb{N}.$$

Odtud $b_k = \frac{2}{\pi}\left((-1)^{k+1}e^{-2\pi}+1\right)\frac{k}{k^2+4}, \quad k \in \mathbb{N}.$

Tedy $$S^f(t) = \frac{2}{\pi}\sum_{k=1}\left((-1)^{k+1}e^{-2\pi}+1\right)\frac{k}{k^2+4}\sin(kt)$$

$$= \frac{2}{\pi}\left(\sum_{k=1}^{\infty}\left(1-e^{-2\pi}\right)\frac{2k}{4k^2+4}\sin(2kt) + \sum_{k=0}^{\infty}\left(e^{-2\pi}+1\right)\frac{2k+1}{(2k+1)^2+4}\sin((2k+1)t)\right.$$

Funkce f je na $[-\pi,\pi]$ po částech monotónní, a tedy $f \in BV([-\pi,\pi])$.

Podle Jordanova-Dirichletova kritéria tedy platí

$$S^f(t) = \frac{f(t+)+f(t-)}{} \quad \text{pro každé } t \in \mathbb{R}.$$

Dosadíme $t=\frac{\pi}{2}$. Potom $\sin(2kt)=0$ $\forall k\in\mathbb{N}$, a tedy

$$s^f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\sum_{k=0}^{\infty}\left(e^{-2\pi}+1\right)\frac{2k+1}{(2k+1)^2+4}\sin\left((2k+1)\frac{\pi}{2}\right).$$

Protože f je spojitá v $\frac{\pi}{2}$ a $\sin\left((2k+1)\frac{\pi}{2}\right)=(-1)^k$, $k\in\mathbb{N}$, plyne odtud

$$e^{-\pi} = \frac{2}{\pi}\sum_{k=0}^{\infty}\left(e^{-2\pi}+1\right)(-1)^k\frac{2k+1}{(2k+1)^2+4}.$$

Tedy $$\sum_{k=0}^{\infty}(-1)^k\frac{2k+1}{(2k+1)^2+4} = \frac{\pi e^{-\pi}}{2\left(e^{-2\pi}+1\right)}.$$