PŘÍKLAD 2

$f(x,y,z)=(x,y,z)$, $\Omega=\{[x,y,z]\in\mathbb{R}^3,\ x^2+y^2<1\}$

tok f hranicí $H(\Omega)$

Řešení Ω omezená, otevřená, neprázdná, $\mathcal{H}^2(H(\Omega))<\infty$

Označme

$$H_1=\{[x,y,1];\ x^2+y^2<1\}$$

$$H_2=\{[x,y,z];\ x^2+y^2=z,\ z\in(0,1)\}$$

$$H_3=\{[x,y,1];\ x^2+y^2=1\}$$

Položme $h_1(x,y,z)=z-1$, pak $Dh_1(x,y,z)=(0,0,1)$, $\nu_\Omega^1=(0,0,1)$

$h_2(x,y,z)=x^2+y^2-z$, pak $Dh_2(x,y,z)=(2x,2y,-1)$, $\nu_\Omega^2=\dfrac{(2x,2y,-1)}{\sqrt{4x^2+4y^2+1}}=\dfrac{(2x,2y,-1)}{\sqrt{4z+1}}$

h_1, h_2 jsou rozhraničující $\Rightarrow$ $H_*(\Omega)\supset H_1\cup H_2$

ověření $\mathcal{H}^2(H(\Omega)\setminus H_*(\Omega))=0$

$H(\Omega)\setminus H_*(\Omega) \subseteq H_3$, H_3 je kružnice $\Rightarrow$ lip. obraz intervalu

$\Rightarrow \mathcal{H}^1(H_3) < \infty \Rightarrow \mathcal{H}^2(H_3) = 0$

(1) výpočet pomocí Gaussovy věty

$$\text{TOK} = \int_{H(\Omega)} \langle f, \nu_\Omega \rangle \, d\mathcal{H}^2 \overset{\text{Gauss}}{=} \int_\Omega \operatorname{div} f \, d\lambda^3 \overset{\operatorname{div} f = 3}{=} 3\lambda^3(\Omega).$$

$$\text{Jest } \lambda^3(\Omega) \overset{\text{Fubini}}{=} \int_0^1 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dy\,dz = 2\int_0^1 \int_{-\sqrt{z}}^{\sqrt{z}} \sqrt{z-y^2}\,dy\,dz$$

$$= 2\int_0^1 \int_{-\sqrt{z}}^{\sqrt{z}} \sqrt{z} \cdot \sqrt{1-\frac{y^2}{z}}\,dy\,dz$$

$y = \sqrt{z}\sin t$

$dy = \sqrt{z}\cos t\,dt$

y	$-\sqrt{z}$	$\sqrt{z}$
t	$-\frac{\pi}{2}$	$\frac{\pi}{2}$

$$= 2\int_0^1 \int_{-\pi/2}^{\pi/2} \sqrt{z} \cdot \sqrt{1-\sin^2 t} \cdot \sqrt{z}\cos t\,dt\,dz$$

$$= 2 \int^1 \int^{\pi/2} z\cos^2 t\,dt\,dz = 2\int^1 z\,dz \cdot \int^{\pi/2} \cos^2 t\,dt = 2\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{2}$$

$$= \int_0 \int_{-\pi/2} \ldots \qquad \int_0 \quad \int_{-\pi/2}$$

Tedy $\quad \mathrm{TOK} = \frac{3\pi}{2}$

② výpočet pomocí area formule

$$\mathrm{TOK} = \int_{H(\Omega)} \langle f, \nu_\Omega \rangle \, d\mathcal{H}^2 = \int_{H_1} \langle f, \nu_\Omega^1 \rangle \, d\mathcal{H}^2 + \int_{H_2} \langle f, \nu_\Omega^2 \rangle \, d\mathcal{H}^2$$

(nutno ověřit, že $\mathcal{H}^2(H_3) = 0$, což už máme)

$$= \int_{H_1} z \, d\mathcal{H}^2 + \int_{H_2} \frac{2x^2 + 2y^2 - z}{\sqrt{4z+1}} \, d\mathcal{H}^2$$

$$= \underline{I} + \underline{\overline{II}}$$

<u>parametrisace H_1</u>: $\quad \varphi(r,t) = \begin{pmatrix} r\cos t \\ r\sin t \\ 1 \end{pmatrix}, \quad \varphi'(r,t) = \begin{pmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \\ 0 & 0 \end{pmatrix},$

$$(\mathrm{vol}\,\varphi')^2 = \det\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} = r^2, \qquad \mathrm{vol}\,\varphi' = r$$

předpoklady: $G = (0,1)\times(0,2\pi)$ ot. v $\mathbb{R}^2$, φ prosté, C^1, $\mathrm{vol}\,\varphi' \neq 0$ $\Rightarrow$ regulární

Tedy
$$I_1 \overset{AF}{=} \int_0^1\int_0^{2\pi} 1.r \, dt\, dr + \int_{c_1} z \, d\mathcal{H}^2, \quad \text{kde } c_1 = \{[s,0,1];\ s\in[0,1]\}$$

Potom c_1 je úsečka v $\mathbb{R}^3$, $\mathcal{H}^1(c_1) < \infty \Rightarrow \mathcal{H}^2(c_1) = 0$.

Tedy $I_1 = 2\pi \cdot \frac{1}{2} = \pi$.

<u>parametrisace H_2</u>: $\varphi(z,t) = \begin{pmatrix} \sqrt{z}\cos t \\ \sqrt{z}\sin t \\ z \end{pmatrix}$, $(z,t)\in G := (0,1)\times(0,2\pi)$,

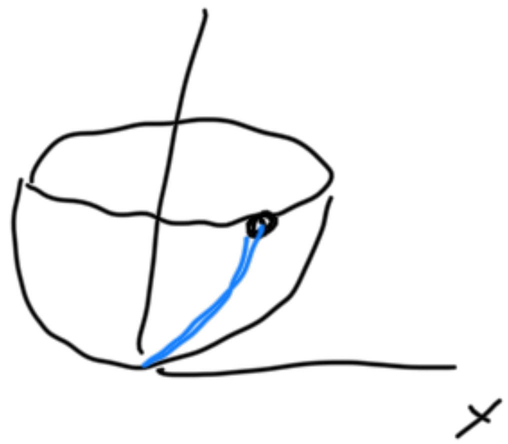
$$\varphi'(z,t) = \begin{pmatrix} \frac{\cos t}{2\sqrt{z}} & -\sqrt{z}\sin t \\ \frac{\sin t}{2\sqrt{z}} & \sqrt{z}\cos t \\ 1 & 0 \end{pmatrix}, \qquad (\mathrm{vol}\,\varphi')^2 = \det\begin{pmatrix} 1+\frac{1}{4z} & 0 \\ 0 & z \end{pmatrix} = z + \frac{1}{4}$$

$\mathrm{vol}\,\varphi' = \sqrt{z+\frac{1}{4}}$, G ot., φ prosté, C^1, $\mathrm{vol}\,\varphi' \neq 0$

$$\text{Tedy } \underline{\mathrm{II}} = \int_{M_2} \frac{2x^2+2y^2-z}{\sqrt{4z+1}}\, d\mathcal{H}^2 \overset{AF}{=} \int_0^1\int_0^{2\pi} \frac{2z-z}{\sqrt{4z+1}}\sqrt{z+\frac{1}{4}}\, dt\, dz + \int_{C_2} \frac{2x^2+2y^2-z}{\sqrt{4z+1}}\, d\mathcal{H}^2,$$

$\mathcal{H}^1(C_2) < \infty$ (lip. obraz intervalu) $\Rightarrow \mathcal{H}^2(C_2) = 0$

$C_2 = \{(0,s,s^2) \in \mathbb{R}^3,\ s \in [0,1]\}$



$$\text{Tedy } \underline{\mathrm{II}} = \int_0^1\int_0^{2\pi} \frac{z}{2}\, dz = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}.$$

$$\text{Celkem } \mathrm{TOK} = \underline{\mathrm{I}} + \underline{\mathrm{II}} = \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$