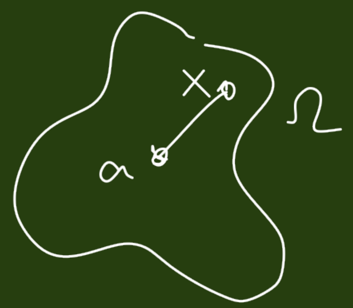
Důkaz V 20.25 (c) BÚNO $a=0$, tj. $\forall x \in \Omega\ \forall t \in [0,1] : tx \in \Omega$



Položme: $u(x) = \int_{\varphi_x} f \cdot d\varphi_x$, $\varphi_x(t) = tx$, $t \in [0,1]$.

CHCEME: $Du = f$

Jest

$$\frac{\partial u}{\partial x_i}(x) = \frac{\partial}{\partial x_i} \int_0^1 \langle f(tx), x \rangle \, dt = \frac{\partial}{\partial x_i} \int_0^1 \sum_{j=1}^n f_j(tx)\, x_j \, dt$$

$$= \int_0^1 \sum_{j=1}^n \frac{\partial}{\partial x_i} \left(f_j(tx)\, x_j \right) dt = \int_0^1 \left(\sum_{j=1}^n \frac{\partial f_j}{\partial x_i}(tx)\, t x_j + f_i(tx) \right) dt,$$

$$f_i(x) = \int_0^1 \frac{d}{dt} \left(t f_i(tx) \right) dt \overset{\text{ŘP}}{=} \int_0^1 \left(f_i(tx) + t \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(tx) \cdot x_j \right) dt,$$

takže $\forall i \in \{1, \dots, n\} : \frac{\partial u}{\partial x_i} = f_i$, tedy $Du = f$. $\square$

PŘÍKLAD $\mathbb{R}^2 \setminus \{0\}$ .. není hvězdicovitá

$$f(x) = \left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2} \right),$$

pak $\operatorname{curl} f = 0$, ale f není potenciální

POZNÁMKY – různá značení

- proměnné v $\mathbb{R}^n$, $x \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$

$[x, y, z] \in \mathbb{R}^3$, $[x, y] \in \mathbb{R}^2$

$$\int_c g\,ds, \quad \int_c f\cdot dc \quad \left(\int_a^b g(c(t))\,\|c'(t)\|\,dt, \quad \int_a^b \langle f(c(t)), c'(t)\rangle\,dt\right)$$

$$\int_\Phi g\,dS, \quad \int_\Phi f\cdot d\Phi \quad \left(\int_G g(\Phi(t))\,\mathrm{vol}\,\Phi'(t)\,dt, \quad \int_G \left\langle f(\Phi(t)), \frac{\partial\Phi}{\partial t_1}(t)\times\cdots\times\frac{\partial\Phi}{\partial t_{n-1}}\right\rangle dt\right)$$

Jiná značení: $\int \vec{f}\,d\vec{s}$, $\int \mathbf{f}\,\mathbf{ds}$, $\oint_c$, $\int_{(c)}\dots$

$$\int_{\mathbb{R}^n} f\,d\mathcal{H}^k, \quad \int_{\mathbb{R}^k} f\,d\lambda^k,$$

integrace vektorových funkcí

možnost zápisu:

$$f = (f_1, f_2, f_3)$$

$$\int f_1\,dx + f_2\,dy + f_3\,dz$$

POZOR $\int f_2\,dy + f_1\,dx$

3.30 – 3.35

plošný: $\int f_1 \, dy\,dz + f_2 \, dz\,dx + f_3 \, dx\,dy$

$$f = (f_1, f_2, f_3).$$

PŘÍKLADY. .. pole s konstantní rotací v $\mathbb{R}^2$

$F = (0, x)$, $\operatorname{curl} F = 1$

$F = (-y, 0)$, — " —

$F = (-y, x)$, $\operatorname{curl} F = 2$

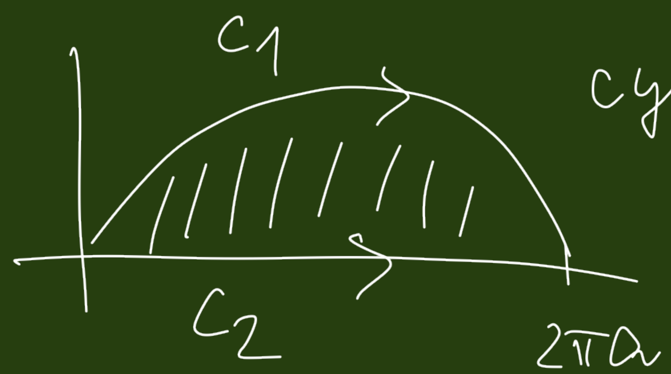
} lze využít pro výpočet obsahu (pomocí Greenovy věty) obsah elipsuhu: Ω = elipsuh, $c = \partial(\Omega)$ = elipsa

$$c(t) = \begin{pmatrix} a\cos t \\ b \sin t \end{pmatrix}, \quad t \in [0, 2\pi], \quad c'(t) = \begin{pmatrix} -a \sin t \\ b \cos t \end{pmatrix},$$

! $\underline{\underline{F = (-y, x)}}$, $\operatorname{curl} F = 2$, takže

$$\mathcal{H}^2(\Omega) \overset{\text{GREEN}}{=} \frac{1}{2} \int_c F \cdot dc = \frac{1}{2} \int_0^{2\pi} (-b \sin t, a \cos t) \begin{pmatrix} -a \sin t \\ b \cos t \end{pmatrix} dt$$

$$= \frac{ab}{2} \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = \underline{\underline{\pi a b}}.$$



cykloida

$$\varphi(t) = \begin{pmatrix} a(t-\sin t) \\ a(1-\cos t) \end{pmatrix}, \quad t \in [0, 2\pi]$$

$$c_2(t) = (t, 0), \quad c_2'(t) = (1, 0), \quad F = (0, x), \quad \operatorname{curl} F = 1$$

$$\mathcal{H}^2(\Omega) \overset{GR}{=} \int_c F \cdot dc = \underbrace{\int_0^{2\pi a} (0, x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt}_{=0 \text{ (přes } c_2)} = \int_0^{2\pi} (0, a(t-\sin t)) \begin{pmatrix} a(1-\cos t) \\ a\sin t \end{pmatrix} dt$$

$$\varphi'(t) = \begin{pmatrix} a(1-\cos t) \\ a \sin t \end{pmatrix}$$

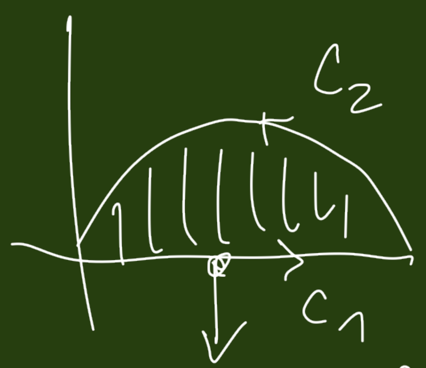
$$= -\int_0^{2\pi} a^2 (t \sin t - \sin^2 t)\, dt = a^2 \underbrace{\int_0^{2\pi} \sin^2 t\, dt}_{\pi} - a^2 \int_0^{2\pi} t \sin t\, dt$$

$$= \underline{3\pi a^2}$$

PŘÍKLAD (3.41) $\displaystyle\int_c \underbrace{e^x(1-\cos y)}_{f_1}\,dx \underbrace{-e^x(y-\sin y)}_{f_2}\,dy$

$f = (f_1, f_2)$

$c = H(\Omega)$, $\Omega = \{[x,y]\in\mathbb{R}^2,\ 0<x<\pi,\ 0<y<\sin x\}$



Řešení: parametrisace c

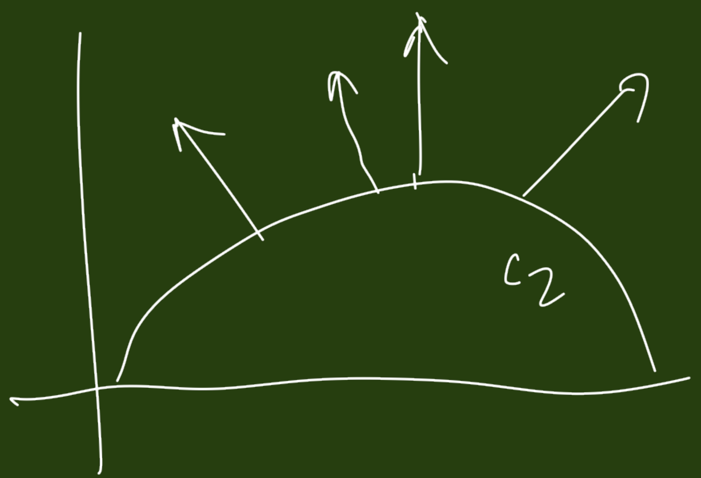
$c = \begin{cases} c_1(t) \\ c_2(t) \end{cases}$

$c_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$, $t\in[0,\pi]$, $c_1'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$c_2(t) = \begin{pmatrix} 2\pi - t \\ -\sin t \end{pmatrix}$, $t\in[\pi, 2\pi]$, $c_2'(t) = \begin{pmatrix} -1 \\ -\cos t \end{pmatrix}$

- c je PČR, jednoduchá a uzavřená,
- test orientace $\exists\, t_0:\ \det\left(\nu_\Omega(c(t_0)),\ c'(t_0)\right)$

na c_1: $t_0\in(0,\pi)$: $h(x,y) = -y$, $Dh(x,y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\det \ldots = \dfrac{1}{\|Dh\|}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 > 0$



$$c_2(t) = \begin{pmatrix} 2\pi - t \\ -\sin t \end{pmatrix}, \quad c_2'(t) = \begin{pmatrix} -1 \\ -\cos t \end{pmatrix}, \quad t \in (\pi, 2\pi),$$

$$h(x,y) = y - \sin x, \quad \nabla h(x,y) = \begin{pmatrix} -\cos x \\ 1 \end{pmatrix},$$

$t \in (\pi, 2\pi)$:
$$\nu_\Omega = \frac{\nabla h}{\|\nabla h\|}, \quad \det(\nu_\Omega, c') = \frac{1}{\|\nabla h\|} \det \begin{pmatrix} -\cos(2\pi - t) & -1 \\ 1 & -\cos t \end{pmatrix}$$

$$= \frac{1}{\|\nabla h\|} \left(\cos t \cdot \cos(2\pi - t) + 1 \right) > 0,$$

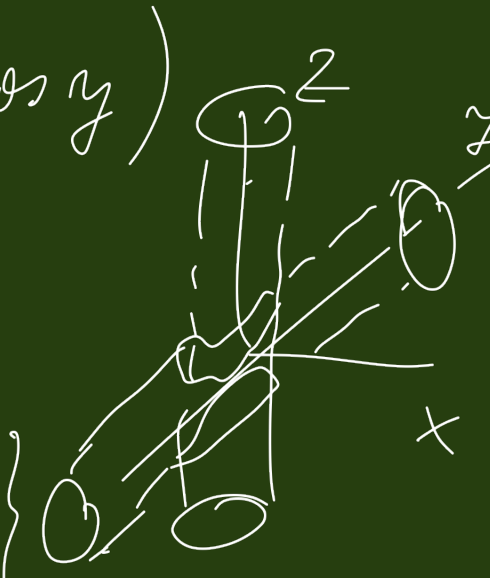
ORIENTACE OK

- $f \in C^1(\mathbb{R}^2)$, $\operatorname{curl} f = -e^x(y - \sin y) + e^x \sin y = -y e^x$ (SNAD)

$$f = \left(e^x (1 - \cos y), \ -e^x (y - \sin y) \right)$$

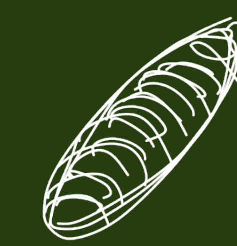
Tedy
$$\int_c f \cdot dc \overset{G-S}{=} \int_\Omega -y e^x \, dx^2 \overset{Fub}{=} \int_0^\pi \int_0^{\sin x} -y e^x \, dy \, dx = \frac{-(e^\pi - 1)}{5}$$

PŘÍKLAD: $I = \int_c f \cdot dc$, $f = \left(-y,\ x - z^2 \sin y,\ z^3 + 2z\cos y\right)$



$c = \left\{ [x,y,z] \in \mathbb{R}^3,\ x^2+y^2=1,\ x^2+z^2=1,\ z \geq 0 \right\}$.

Řešení: $c = H(\Omega)$, $\Omega = \left\{ [x,y,z] \in \mathbb{R}^3,\ x^2+y^2<1,\ z=\sqrt{1-x^2} \right\}$



- parametrisace Ω: „případ graf" $z = g(x,y)$

pak lze parametrisovat: $\varphi(x,y) = \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix}$, pak

$$\frac{\partial \varphi}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial g}{\partial x} \end{pmatrix},\quad \frac{\partial \varphi}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial g}{\partial y} \end{pmatrix},\quad \frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} = \begin{pmatrix} -\frac{\partial g}{\partial x} \\ -\frac{\partial g}{\partial y} \\ 1 \end{pmatrix}$$

orientace: průmět c do xy-roviny je kladně orientován

$\Rightarrow$ orientace OK, neboť ν má 3. složku kladnou

$$f = \left(-y,\ x - z^2 \sin y,\ z^3 + 2z\cos y\right)$$

$$\operatorname{curl} f = (0, 0, 2), \quad G = \{[x,y] \in \mathbb{R}^2,\ x^2 + y^2 < 1\}.$$

Tedy $$I \overset{\text{STOKES}}{=} \int_G (0,0,2) \begin{pmatrix} \frac{x}{\sqrt{1-x^2}} \\ 0 \\ 1 \end{pmatrix} d\mathcal{H}^2 = 2\,\mathcal{H}^2(G) = \underline{2\pi}.$$

21. ČÍSELNÉ ŘADY II

21.1. PŘEROVNÁVÁNÍ ŘAD

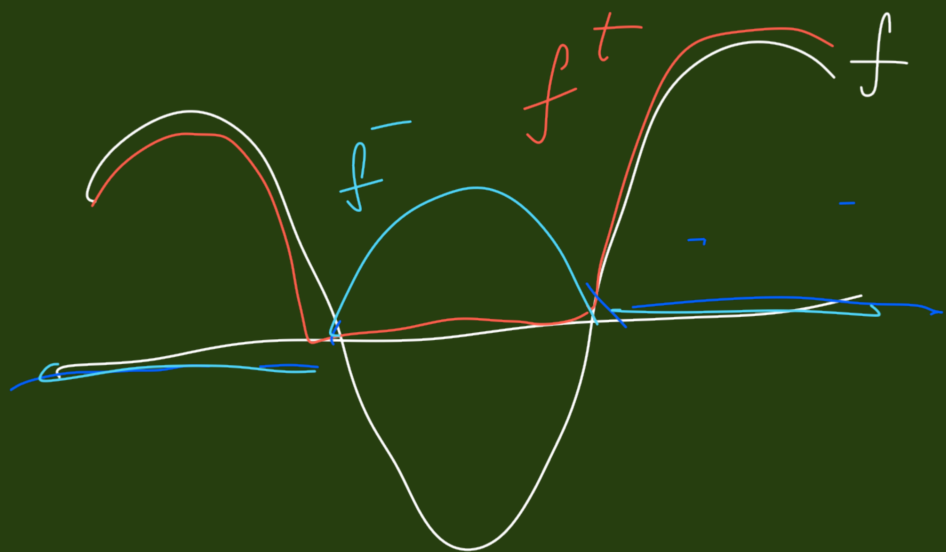
OTÁZKA: $\sum_{n=1}^{\infty} a_n$, $a_n \in \mathbb{R}$

jestliže známe součet $\sum_{n=1}^{\infty} a_n$, co lze říci

o součtu $\sum_{n=1}^{\infty} a_{\pi(n)}$, kde $\pi: \mathbb{N} \to \mathbb{N}$ je bijekce ?

$$a_1 + a_2 + a_3 + \dots = S \overset{?}{\Longrightarrow} a_3 + a_{17} + a_{26} + \dots = ??$$

DEFINICE. Nechť $a \in \mathbb{R}$. Kladnou částí a rozumíme číslo $a^+ = \max\{a, 0\}$, zápornou částí a rozumíme číslo $a^- = \max\{-a, 0\}$. Je-li M množina a $f : M \to \mathbb{R}$, pak značíme $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$.



POZNÁMKA. Platí $(\forall a \in \mathbb{R})$

- $0 \le a^+, a^-$, $\quad a^+, a^- \le |a|$
- $a = a^+ - a^-$
- $|a| = a^+ + a^-$.