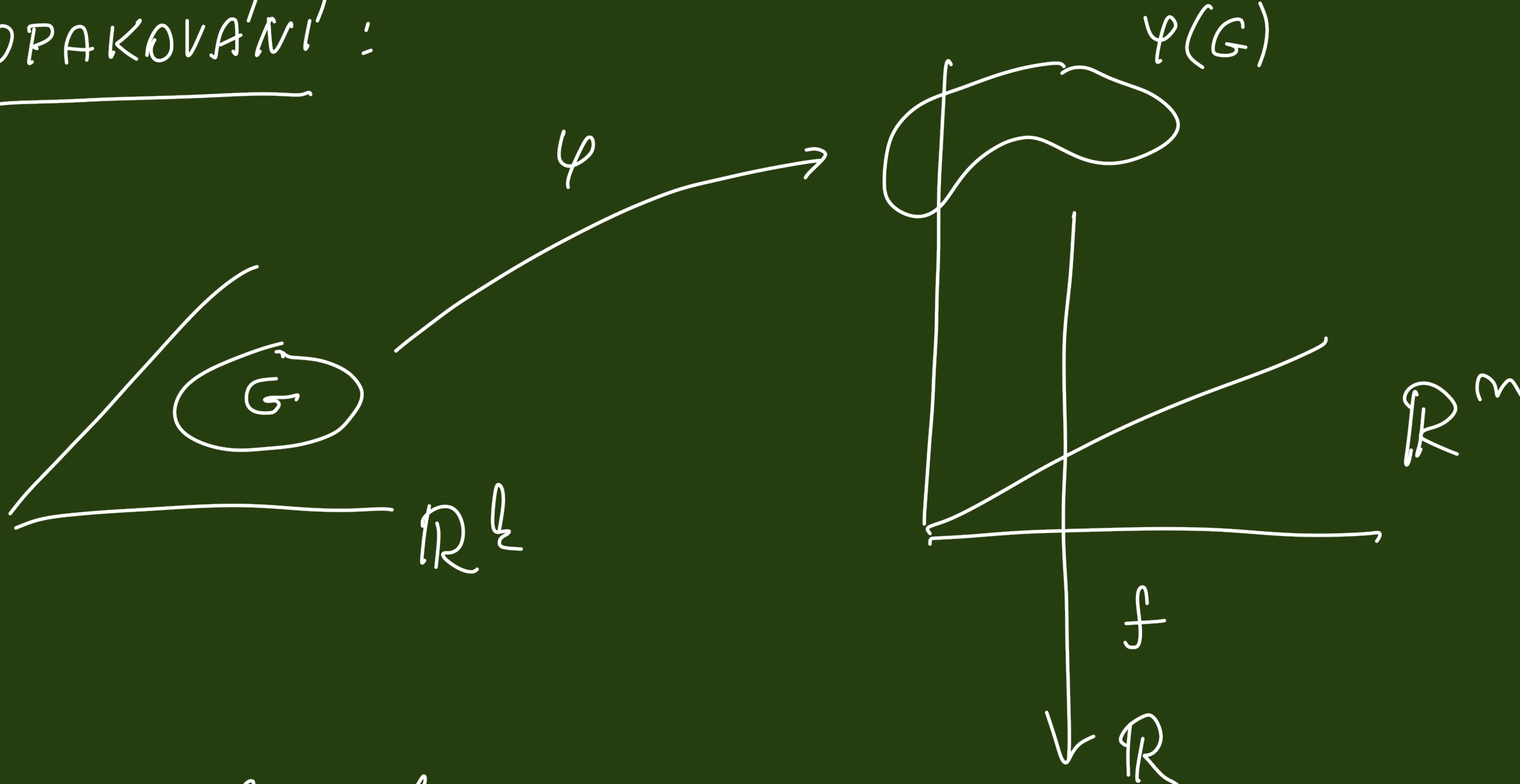
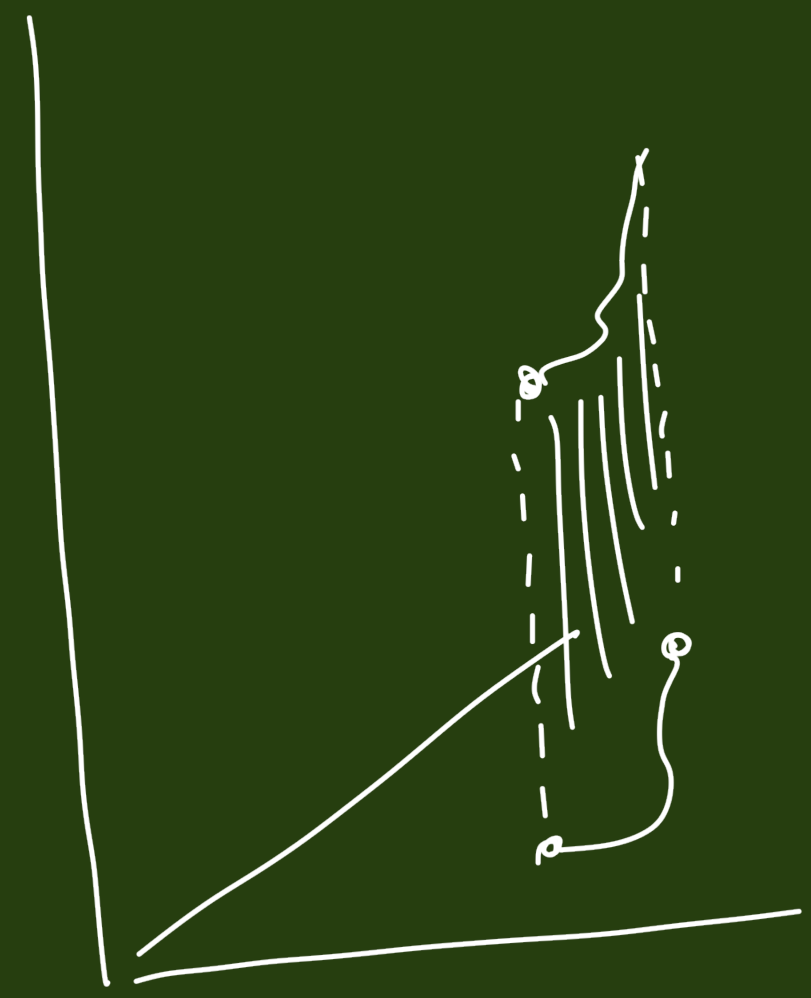
OPAKOVÁNÍ:



Area formule:

$$\int_{\varphi(G)} f \, d\mathcal{H}^k = \int_G (f \circ \varphi) \operatorname{vol} \varphi'(t) \, d\lambda^k$$



POZOROVÁNÍ. Je-li $k=1$, pak

$$\operatorname{vol} \varphi'(t) = \| \varphi'(t) \| ,$$ neboť

$$\varphi'(t)^T \cdot \varphi'(t) = \left(\varphi_1'(t), \dots, \varphi_m'(t) \right) \begin{pmatrix} \varphi_1'(t) \\ \vdots \\ \varphi_m'(t) \end{pmatrix} = \varphi_1'(t)^2 + \dots \varphi_m'(t)^2 ,$$

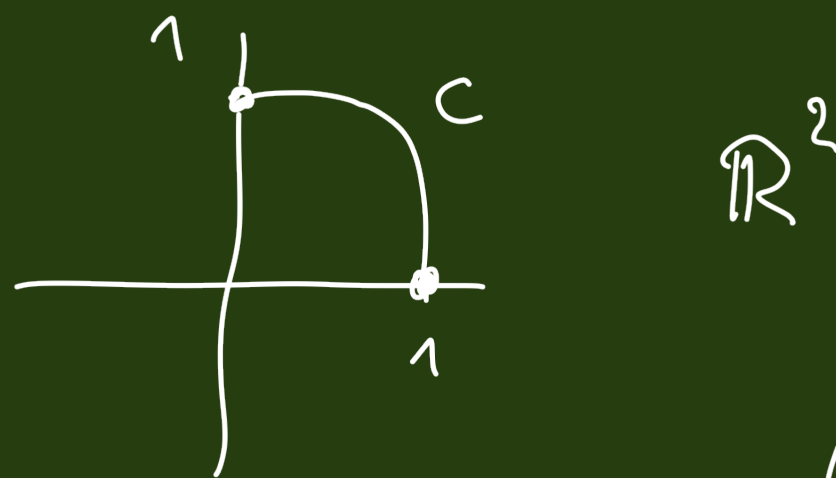
$$\operatorname{vol} \varphi'(t) = \sqrt{\det \varphi'(t)^T \varphi'(t)} = \sqrt{\varphi_1'(t)^2 + \dots + \varphi_m'(t)^2}$$

$$= \| \varphi'(t) \| .$$

Tedy pro $k=1$

$$\int_{\varphi(G)} f \, d\mathcal{H}^1 = \int_G f(\varphi(t)) \cdot \| \varphi'(t) \| \, dt$$ pro f borelovskou.

PŘÍKLAD. Spočtěte $\int_C f \, d\mathcal{H}^1$, kde $f(x,y) = (x+y)^2$ a C je severovýchodní čtvrtina jednotkové kružnice v $\mathbb{R}^2$.



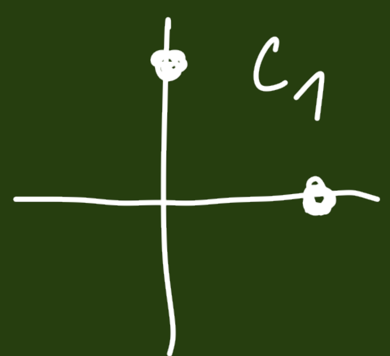
Řešení: Hledáme vhodná G a φ tak, aby G byla otevřená v $\mathbb{R}$ a $\varphi(G) = C$ (nebo alespoň $\varphi(G) = C_2$, kde $C_2 = C \setminus C_1$ a $\mathcal{H}^1(C_1) = 0$).

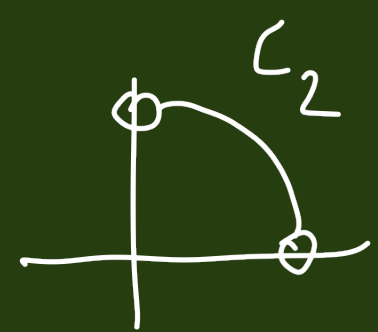
Volíme $G = (0, \frac{\pi}{2})$ a $\varphi(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, $t \in G$.

Potom $\varphi(G) = C_2$, kde $C = C_1 \cup C_2$, $C_2 = C \setminus C_1$,

$C_1 = \{ [1,0], [0,1] \}$.

Potom φ je prosté a $\varphi(G) = C_2$.





$\underline{C_1}$: $\mathcal{H}^1(C_1) = 0$, takže $\int_{C_1} f \, d\mathcal{H}^1 = 0$.

$\underset{\sim}{C_2}$: $\displaystyle\int_{C_2} f \, d\mathcal{H}^1 \overset{AF}{=} \int_G f(\varphi(t)) \, \|\varphi'(t)\| \, dt$

Dosadíme: $G = \left(0, \frac{\pi}{2}\right)$, $\varphi'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$,

$\varphi(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ $\quad \|\varphi'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1 \ \forall t \in G$

(tedy φ je regulární)

$f(x,y) = (x+y)^2$, $\quad x = \cos t$, $y = \sin t$, takže $f(x,y) = (\cos t + \sin t)^2$.

Tedy $\displaystyle\int_{C_2} f \, d\mathcal{H}^1 = \int_0^{\pi/2} (1 + \sin 2t) \, dt = \frac{\pi}{2} + \left[-\frac{\cos 2t}{2} \right]_0^{\pi/2} = \frac{\pi}{2} + 1.$

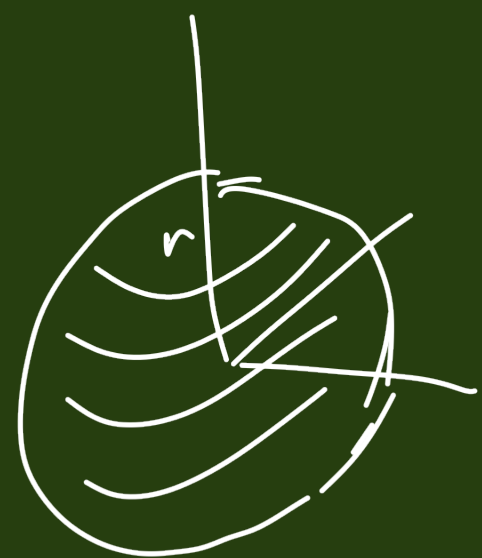
Celkem

$$\int_C f \, d\mathcal{H}^1 = \int_{C_1} f \, d\mathcal{H}^1 + \int_{C_2} f \, d\mathcal{H}^1 = 0 + \frac{\pi}{2} + 1 = \underline{\underline{\frac{\pi}{2} + 1}}.$$

DALŠÍ CÍL: počítat „povrchy těles" v $\mathbb{R}^3$

Dáno $r>0$. Určete povrch sféry

$$S(0,r) = \{x \in \mathbb{R}^3, \ \|x\| = r\}.$$



Mělo by mi vyjít $4\pi r^2$.

OTÁZKY: 1) Nalezneme φ, G tak, aby

$$S(0,r) = \varphi(G) \ ?$$

2) Nalezneme vzorec pro vol $\varphi'(t)$ pro $k \geq 2$?

AD 1) NE, ale skoro (kousek zůstane nepokryt, snad bude $\mathcal{H}^2$-míry 0.

AD 2) Obecně NE, ale ANO pro $k = n-1$.

HÁČEK: Budeme potřebovat novou operaci.

20.2. VEKTOROVÝ SOUČIN

DEFINICE. Nechť $n \in \mathbb{N}$, $n \geq 2$, a $u^1, \dots, u^{n-1} \in \mathbb{R}^n$.

Definujeme vektorový součin $u^1, \dots, u^{n-1}$ předpisem

$$u^1 \times \dots \times u^{n-1} = \left(\det\left(e^i, u^1, \dots, u^{n-1}\right)\right)_{i=1}^n, \quad n \geq 3,$$

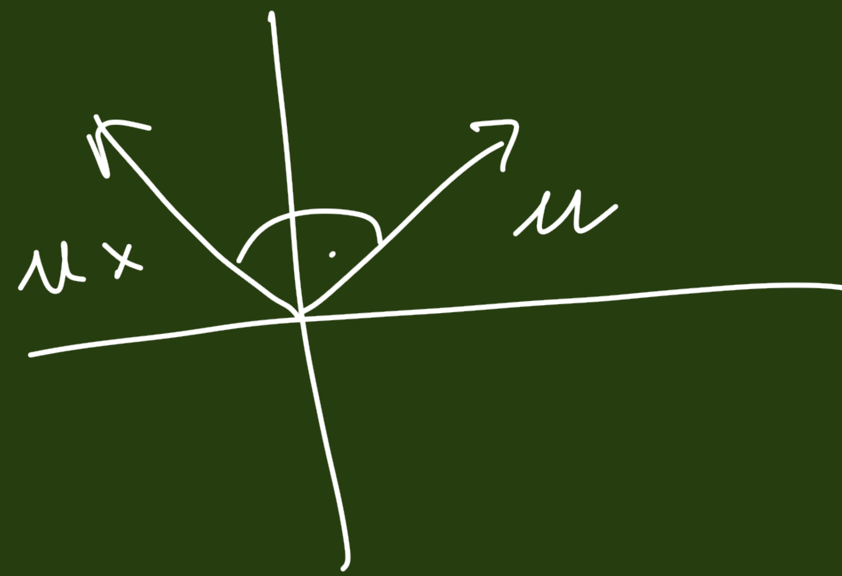
a

$$u^1 \times \quad = \left(u_2^1, -u_1^1\right).$$

POZNÁMKA.

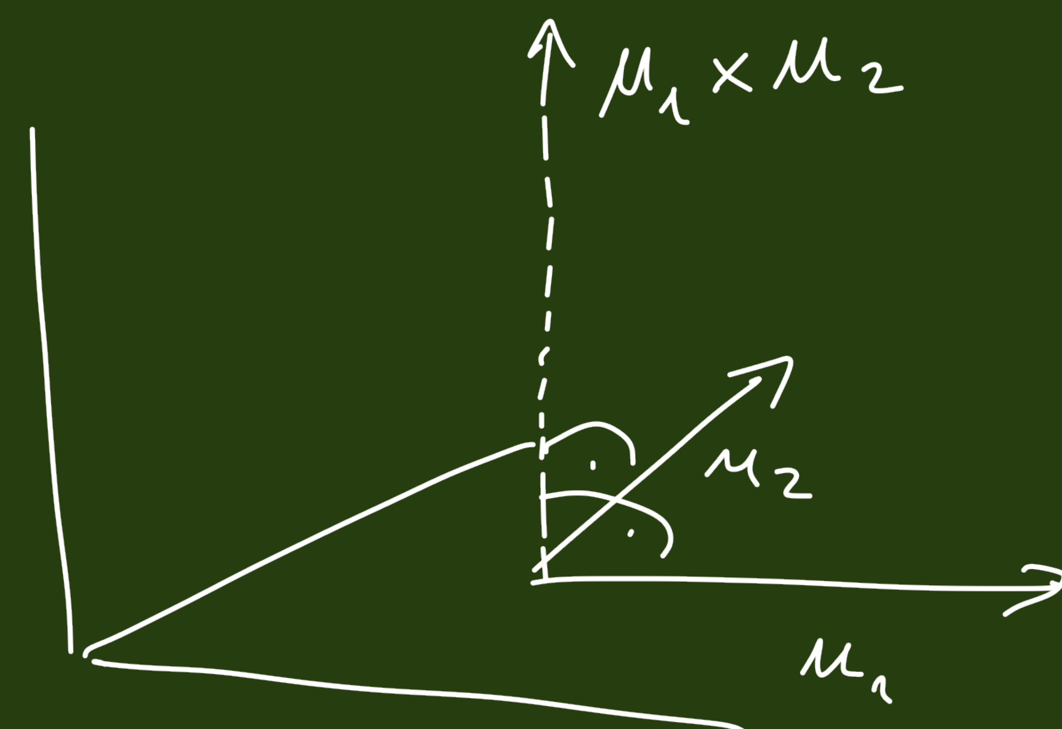
$u^1 \times \dots \times u^{n-1} \in \mathbb{R}^n$

$n = 2$



$u^\times = (u_2, -u_1)$

$n = 3$



Věta 20.12 (vlastnosti vektorového součinu).

Nechť $n \in \mathbb{N}$, $n \geq 2$, a $u^1, \dots, u^{n-1} \in \mathbb{R}^n$.

(a) $\forall v \in \mathbb{R}^n: \quad \langle v, u^1 \times \dots \times u^{n-1} \rangle = \det(v, u^1, \dots, u^{n-1})$,

(b) $u^1, \dots, u^{n-1}$ jsou LZ $\iff u^1 \times \dots \times u^{n-1} = o$,

(c) $\forall i \in \{1, \dots, n-1\}: \quad \langle u^i, u^1 \times \dots \times u^{n-1} \rangle = 0$,

(d) $\operatorname{vol}(u^1, \dots, u^{n-1}) = \| u^1 \times \dots \times u^{n-1} \|$.

PŘIPOMEŇME: $\operatorname{vol} A = \sqrt{\det A^T \cdot A}$

<u>Důkaz.</u> (a) Zvolme $v \in \mathbb{R}^n$. Potom

$$\langle v, u^1 \times \dots \times u^{n-1} \rangle = \sum_{i=1}^{n} v_i \det(e^i, u^1, \dots, u^{n-1})$$

$$= \sum_{i=1}^{n} \det(v_i e^i, u^1, \dots, u^{n-1}) \quad \text{(rozvoj dle 1. sloupce).}$$

$$= \det(v, u^1, \dots, u^{n-1})$$

(b) „$\Rightarrow$" je triviální

„$\Leftarrow$" Nechť $u^1 \times \dots \times u^{n-1} = o$. Potom dle (a) je matice $(v, u^1, \dots, u^{n-1})$ singulární pro každé $v \in \mathbb{R}^n$. Tedy $u^1, \dots, u^{n-1}$ jsou LZ.

(c) Bezprostředně plyne z (a).

(d) Označme $w = u^1 \times \dots \times u^{n-1}$. Jestliže $w = o$, pak dle (b) jsou $u^1, \dots, u^{n-1}$ LZ, a tedy $\mathrm{vol}\,(u^1, \dots, u^{n-1}) = 0$. Předpokládejme, že $w \neq o$. Potom

$$\mathrm{vol}(u^1,\dots,u^{n-1})^2 = \det\left(\left(\langle u^i,u^j\rangle\right)_{i,j=1}^{n-1}\right)$$

$$= \frac{1}{\|w\|^2}\det\begin{pmatrix} \langle w,w\rangle & 0 & \dots & 0 \\ 0 & \langle u^1,u^1\rangle & \dots & \langle u^1,u^{n-1}\rangle \\ \vdots & \vdots & & \vdots \\ 0 & \langle u^{n-1},u^1\rangle & \dots & \langle u^{n-1},u^{n-1}\rangle \end{pmatrix}$$

$$\overset{(c)}{=} \frac{1}{\|w\|^2}\det\left((w,u^1,\dots,u^{n-1})^T\cdot(w,u^1,\dots,u^{n-1})\right)$$

$$= \frac{1}{\|w\|^2}\det(w,u^1,\dots,u^{n-1})^2 \overset{(a)}{=} \frac{1}{\|w\|^2}\langle w,u^1\times\dots\times u^{n-1}\rangle^2$$

$$= \frac{\langle w,w\rangle^2}{\|w\|^2} = \frac{\|w\|^4}{\|w\|^2} = \|w\|^2. \qquad \square$$