$\boxed{A2}$ $$\int \frac{\cos^2 x}{\cos^2 x + 2\sin^2 x}\, dx$$

Řešení: pro $k \in \mathbb{Z}$ def.

$$\varphi(t) = \operatorname{arctg} t + k\pi, \quad t \in \mathbb{R},$$

potom $\varphi((-\infty,\infty)) = \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right) =: I_k$,

$$\varphi'(t) = \frac{1}{1+t^2} \quad (\text{tedy } \forall t \in \mathbb{R}\ \exists \text{ vl. a nenulová}),$$

$$\varphi^{-1}(x) = \operatorname{tg}(x - k\pi) = \operatorname{tg} x, \quad x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right).$$

Označ $f(x) = \dfrac{\cos^2 x}{\cos^2 x + 2\sin^2 x}$, $x \in \mathbb{R}$. Zvolme $k \in \mathbb{Z}$.

Potom $\cos^2 x = \dfrac{1}{1+\operatorname{tg}^2 x}$, $\sin^2 x = \dfrac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x}$

pro $x \in I_k$, a tedy

$$f(x) = \frac{\frac{1}{1+\operatorname{tg}^2 x}}{\frac{1}{1+\operatorname{tg}^2 x} + \frac{2\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x}} = \frac{1}{1 + 2\operatorname{tg}^2 x}, \quad x \in I_k,$$

takže $f(\varphi(t)) = \dfrac{1}{1+2t^2}$. Jest

$$\int (f \circ \varphi)(t)\, \varphi'(t)\, dt = \int \frac{1}{(1+2t^2)(1+t^2)}\, dt.$$

Položme
$$\frac{1}{(1+2t^2)(1+t^2)} = \frac{At+B}{1+2t^2} + \frac{Ct+D}{1+t^2},$$
pak
$$1 = (At+B)(1+t^2) + (Ct+D)(1+2t^2),$$
takže

„t^3“ $\quad A+2C=0$

„t^2“ $\quad B+2D=0$

„t“ $\quad A+C=0$

„1“ $\quad B+D=1.$

Odtud plyne $A=C=0$, $B=2$, $D=-1$. Tedy
$$\int (f\circ\varphi)(t)\,\varphi'(t)\,dt = \int \frac{2}{1+2t^2}\,dt - \int \frac{dt}{1+t^2}$$
$$\overset{c}{=} \sqrt{2}\,\operatorname{arctg}(\sqrt{2}\,t) - \operatorname{arctg} t, \quad t\in\mathbb{R}.$$
Dle VOS2 tedy platí
$$\int f(x)\,dx \overset{c}{=} \sqrt{2}\,\operatorname{arctg}\left(\sqrt{2}\cdot\operatorname{tg}(x-k\pi)\right) - (x-k\pi)$$
$$= \sqrt{2}\,\operatorname{arctg}(\sqrt{2}\,\operatorname{tg}x) - x + k\pi, \quad x\in I_k.$$

Definujme $F_k : I_k \to \mathbb{R}$ předpisem

$$F_k(x) = \sqrt{2}\,\mathrm{arctg}\left(\sqrt{2}\,\mathrm{tg}\,x\right) - x + k\pi .$$

Potom

$$\lim_{x \to -\frac{\pi}{2}+k\pi+} F_k(x) = -\sqrt{2}\,\frac{\pi}{2} + \frac{\pi}{2} - k\pi + k\pi = \frac{\pi}{2}(1-\sqrt{2}),$$

$$\lim_{x \to \frac{\pi}{2}+k\pi-} F_k(x) = \sqrt{2}\,\frac{\pi}{2} - \frac{\pi}{2} - k\pi + k\pi = \frac{\pi}{2}(\sqrt{2}-1).$$

Položme $F_0(x) = \sqrt{2}\,\mathrm{arctg}\left(\sqrt{2}\,\mathrm{tg}\,x\right) - x + k\pi + k\pi(\sqrt{2}-1)$

$$= \sqrt{2}\,\mathrm{arctg}\left(\sqrt{2}\,\mathrm{tg}\,x\right) - x + \sqrt{2}\,k\pi, \quad x \in I_k,$$

a navíc $F_0(x) = (\sqrt{2}-1)\left(k\pi + \frac{\pi}{2}\right), \quad x = \frac{\pi}{2} + k\pi,$

$k \in \mathbb{Z}.$

Potom $\displaystyle\int \frac{\cos^2 x}{\cos^2 x + 2\sin^2 x}\,dx \stackrel{c}{=} F_0(x), \quad x \in \mathbb{R}.$

Hodnocení:

- VOS .. 1
- rozklad na parc. zlomky -- 1
- integrace zlomků -- 2
- řešení -- 4