THEORY OF INTERPOLATION 1

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1. Introduction

If not stated otherwise, (R, μ) and (S, ν) will throughout denote σ-finite measure spaces. By $\mathcal{M}(R, \mu)$ (or just $\mathcal{M}(R)$ for short in cases when it is clear which measure is considered) we denote the set of all μ-measurable real-valued functions on R, and by $\mathcal{M}_+(R, \mu)$ the set of all nonnegative functions in $\mathcal{M}(R, \mu)$. For $p \in [1, \infty]$, we define p' by

$$p' = \begin{cases} \infty & \text{if } p = 1, \\ \frac{p}{p-1} & \text{if } p \in (1, \infty), \\ 1 & \text{if } p = \infty. \end{cases}$$

If X and Y are (quasi)-normed spaces, we say that X is *embedded* into Y if there exists a constant C such that for every $x \in X$ one has $\|x\|_Y \le C\|x\|_X$. By $X + Y$ we denote the set of all elements z for which there exists a decomposition $z = x + y$ with $x \in X$ and $y \in Y$. We define the functional $\|\cdot\|_{X+Y} \colon (X + Y) \to [0, \infty]$ by $\|z\|_{X+Y} = \inf_{z=x+y}(\|x\|_X + \|y\|_Y)$.

Theorem 1 (embeddings of Lebesgue spaces)**.** *Let $0 < p, q \le \infty$. Then the embedding*

$$L^q(R, \mu) \hookrightarrow L^p(R, \mu)$$

holds if and only if one of the following conditions hold:

- $p = q$,
- $p < q$ *and* $\mu(R) < \infty$,
- $p > q$ *and there is an* $\varepsilon > 0$ *such that for every measurable* $E \subset R$ *of positive measure one has* $\mu(E) \ge \varepsilon$.

Definition. The *Laplace transform* is defined by the formula

$$\mathcal{L}f(t) = \int_0^\infty f(s)e^{-st}\, ds \quad \text{for } t \in (0, \infty)$$

and every $f \in \mathcal{M}(0, \infty)$ for which the integral makes sense.

Remark. One has

$$\|\mathcal{L}f\|_{L^\infty(0,\infty)} \le \|f\|_{L^1(0,\infty)}.$$

Theorem 2 (Laplace transform on L^2)**.** *For every $f \in L^2(0, \infty)$ one has*

$$\|\mathcal{L}f\|_{L^2(0,\infty)} \le \sqrt{\pi}\|f\|_{L^2(0,\infty)}.$$

The constant is optimal.

Theorem 3 (interpolation principle for Lebesgue spaces)**.** *Let $0 < p < r < q \le \infty$. Assume that $f \in L^p(R, \mu) \cap L^q(R, \mu)$. Let $\theta \in [0, 1]$ and let r be defined by*

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

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Then $f \in L^r(R,\mu)$ *and*

$$\|f\|_r \le \|f\|_p^{1-\theta}\|f\|_q^{\theta}.$$

2. Classical interpolation theorems

2.1. Interpolation of positive operators.

Definition. Let T be an operator defined on simple functions on (R,μ) with values in $\mathcal{M}(S,\nu)$. Let $p,q \in (0,\infty]$. We say that T is of *strong type* (p,q) if there exists a constant M such that

$$\|Tf\|_{L^q(S,\nu)} \le M\|f\|_{L^p(R,\mu)} \quad \text{for every } \mu\text{-simple function } f.$$

The smallest such M is called the *norm* of T and it is denoted by $\|T\|_{L^p\to L^q}$.

Theorem 4 (Riesz's theorem for positive operators)**.** *Let* $1 \le p_0,p_1,q_0,q_1 \le \infty$ *and* $\theta \in [0,1]$. *Let*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let T *be a positive linear operator of the form*

$$Tf(y) = \int_R f(x)A(x,y)d\mu(x) \quad \textit{for } y \in S,$$

where A *is a nonnegative measurable function on* $R\times S$. *Assume that* T *is of strong type* (p_0,q_0) *and, at the same time, of strong type* (p_1,q_1) *with norms* M_0 *and* M_1, *respectively. Then* T *is of strong type* (p,q) *with norm* M_θ *satisfying*

$$M_\theta \le M_0^{1-\theta}M_1^{\theta}.$$

2.2. Riesz's-Thorin's interpolation theorem.

Theorem 5 (Hadamard's three-line theorem)**.** *Let* F *be a bounded continuous function on* $\overline{\Omega}$ *and analytic in* Ω, *where*

$$\Omega = \{z \in \mathbb{C}\colon \operatorname{Re} z \in (0,1)\}.$$

Then the function M_θ, *defined by*

$$M_\theta = \sup\{|F(\theta+iy)|\colon y \in \mathbb{R}\} \quad \textit{for } \theta \in [0,1],$$

satisfies

$$M_\theta \le M_0^{1-\theta}M_1^{\theta} \quad \textit{for } \theta \in [0,1].$$

Theorem 6 (Riesz's-Thorin's interpolation theorem)**.** *Let* $1 \le p_0,p_1,q_0,q_1 \le \infty$ *and let* $\theta \in [0,1]$. *Let* T *be a linear operator which is of strong type* (p_0,q_0) *with norm* M_0 *and, at the same time, of strong type* (p_1,q_1) *with norm* M_1. *Suppose that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T *is of strong type* (p,q) *with norm* M_θ *satisfying*

$$M_\theta \le 2M_0^{1-\theta}M_1^{\theta}.$$

The constant 2 *can be dropped if the function spaces are complex.*

Definition. The *Fourier transform* is defined by the formula

$$\mathcal{F}f(x) = \int_{\mathbb{R}^n} f(y)e^{2\pi ixy}\,dy \quad \text{for } x \in \mathbb{R}^n$$

and every $f \in \mathcal{M}(\mathbb{R}^n)$ for which the integral makes sense.

Theorem 7 (Hausdorff's-Young's theorem). *Assume that $1 \leq p \leq 2$. Then there exists a constant C such that*

$$\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.1}$$

Theorem 8 (Young's convolution theorem). *Let $p, q, r \in [1, \infty]$ and assume that*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Then

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}.$$

2.3. Interpolation of compact operators.

Theorem 9 (interpolation of compact operators). *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and let T be a linear operator which is of strong type (p_0, q_0) and, at the same time, it is compact from $L^{p_1}(R, \mu)$ to $L^{q_1}(S, \nu)$. Let $\theta \in (0, 1]$, $\nu(S) < \infty$, and suppose that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T is compact from $L^p(R, \mu)$ to $L^q(S, \nu)$.

Corollary. *The Hardy operator T, defined by*

$$Tf(t) = \int_0^t f(s)\, ds \quad \textit{for } t \in (0, 1)$$

for every $f \in \mathcal{M}(0, 1)$ for which the integral makes sense, is compact from $L^q(0, 1)$ to $L^\infty(0, 1)$ for every $q \in (1, \infty]$.

2.4. Interpolation of weak-type operators.

Definition. Let $n \in \mathbb{N}$ and $\gamma \in (0, n)$. The *Riesz potential* I_γ is defined by the formula

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\gamma}} \quad \text{for } x \in \mathbb{R}^n$$

and every function $f \in \mathcal{M}(\mathbb{R}^n)$ for which the integral makes sense.

Definition. Let $\delta > 0$. The *dilation operator* τ_δ is defined by the formula

$$\tau_\delta f(x) = f(\delta x) \quad \text{for } x \in \mathbb{R}^n$$

and every function $f \in \mathcal{M}(\mathbb{R}^n)$.

Theorem 10 (weak type estimate for the Riesz potential). *Let $n \in \mathbb{N}$ and $\gamma \in (0, n)$. Then there exists a constant C such that*

$$\sup_{\lambda \in (0,\infty)} \lambda |\{x \in \mathbb{R}^n : |I_\gamma f(x)| > \lambda\}|^{1-\frac{\gamma}{n}} \leq C\|f\|_{L^1(\mathbb{R}^n)}$$

for every $f \in L^1(\mathbb{R}^n)$.

Definition. The *Hardy averaging operator* A is defined by the formula

$$Af(t) = \frac{1}{t}\int_0^t f(s)\, ds \quad \text{for } s \in (0, \infty)$$

and every function $f \in \mathcal{M}(0, \infty)$ for which the integral makes sense.

Remark. We have

$$\sup_{\lambda \in (0,\infty)} \lambda |\{x \in (0, \infty) : |Af(x)| > \lambda\}| \leq \|f\|_{L^1(0,\infty)}$$

for every $f \in L^1(0, \infty)$.

Theorem 11 (interpolation of weak-type operators in the diagonal case)**.** *Let T be a quasilinear operator, that is, T is positively homogeneous and, moreover,*

$$|T(f+g)| \le K(|Tf| + |Tg|)$$

for some positive K and every f, g for which the right-hand side makes sense. Assume that there exists a constant C_∞ such that

$$\|Tf\|_{L^\infty(S,\nu)} \le C_\infty \|f\|_{L^\infty(R,\mu)}$$

for all $f \in L^\infty(R,\mu)$, and, at the same time, there exists a constant C_1 such that

$$\sup_{\lambda\in(0,\infty)} \lambda\nu(\{y \in S\colon |Tf(y)| > \lambda\}) \le C_1 \|f\|_{L^1(R,\mu)}$$

for all $f \in L^1(R,\mu)$. Then for every $p \in (1,\infty]$ there exists a constant C_p such that

$$\|Tf\|_{L^p(S,\nu)} \le C_p \|f\|_{L^p(R,\mu)}$$

for every $f \in L^p(R,\mu)$ and

$$C_p \le 2KC_1^{\frac{1}{p}} C_\infty^{1-\frac{1}{p}} \left(\frac{p}{p-1}\right)^{\frac{1}{p}}.$$

Definition. Let $n \in \mathbb{N}$ and $\gamma \in [0,n)$. The *fractional maximal operator* M_γ is defined by the formula

$$M_\gamma f(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\frac{\gamma}{n}}} \int_Q |f(y)|\,dy \quad \text{for } x \in \mathbb{R}^n$$

and every function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where the supremum is extended over all cubes with sides parallel to coordinate axes. In particular, M_0 is the *Hardy–Littlewood maximal operator*.

Definition. Let $f\colon (R,\mu) \to \mathbb{R}$ be a measurable function. Then the function $f_*\colon [0,\infty) \to [0,\infty]$, defined by

$$f_*(\lambda) = \mu\left(\{x \in R : |f(x)| > \lambda\}\right) \quad \text{for } \lambda \in [0,\infty), \tag{2.2}$$

is called the *distribution function* of f.

Proposition. *Let $f\colon (R,\mu) \to \mathbb{R}$ be a measurable function. Then f_* is nonnegative, nonincreasing and right continuous on $[0,\infty)$.*

Proposition. *Let $f, g \in \mathcal{M}(R,\mu)$, and let $\{f_n\}_{n=1}^\infty$ be a sequence of functions from $\mathcal{M}(R,\mu)$. Let $a \in \mathbb{R}$, $a \neq 0$. Then*

$$|g| \le |f| \ \mu\text{-a.e.} \quad \Rightarrow \quad g_* \le f_*, \tag{2.3}$$

$$(af)_*(\lambda) = f_*\left(\frac{\lambda}{|a|}\right) \quad \textit{for } \lambda \in [0,\infty), \tag{2.4}$$

$$(f+g)_*(\lambda_1+\lambda_2) \le f_*(\lambda_1) + g_*(\lambda_2) \quad \textit{for } \lambda_1, \lambda_2 \ge 0, \tag{2.5}$$

$$|f| \le \liminf_{n\to\infty} |f_n| \ \mu\text{-a.e.} \quad \Rightarrow \quad f_* \le \liminf_{n\to\infty} (f_n)_*, \tag{2.6}$$

in particular,

$$|f_n| \uparrow |f| \ \mu\text{-a.e.} \quad \Rightarrow \quad (f_n)_* \uparrow f_*. \tag{2.7}$$

Definition. Let $f \in \mathcal{M}(R,\mu)$ and $g \in \mathcal{M}(S,\nu)$. We say that f and g are *equimeasurable* if they have the same distribution function, that is, if $f_*(\lambda) = g_*(\lambda)$ for all $\lambda \in [0,\infty)$. We write $f \sim g$.

Definition. Let $f \in \mathcal{M}(R,\mu)$. Then the function $f^*\colon [0,\infty) \to [0,\infty]$ defined by

$$f^*(t) = \inf\{\lambda > 0 : f_*(\lambda) \le t\} \quad \text{for } t \in [0,\infty),$$

is called the *nonincreasing rearrangement* of f.

Definition. Assume that $p, q \in (0, \infty]$. We define the functional $\|\cdot\|_{p,q} \colon \mathcal{M}(R,\mu) \to [0,\infty]$ by

$$\|f\|_{p,q} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)}.$$

In other words, we have

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^\infty \left[t^{\frac{1}{p}} f^*(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup\limits_{0<t<\infty} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases}$$

The collection of all functions $f \in \mathcal{M}(R,\mu)$ such that $\|f\|_{p,q} < \infty$ is called the *Lorentz space* and is denoted by $L^{p,q}(R,\mu)$.

Proposition. *Let $f \in \mathcal{M}(R,\mu)$. Then*

$$f^*(f_*(\lambda)) \le \lambda \quad \text{for } \lambda \in [0,\infty) \quad \text{and} \quad f_*(f^*(t)) \le t \quad \text{for } t \in [0,\infty).$$

Proposition. *Let $f, g \in \mathcal{M}(R,\mu)$. Then*

$$(f+g)^*(s+t) \le f^*(s) + g^*(t) \quad \text{for every } s,t \in [0,\infty).$$

Proposition. *Let $f \in \mathcal{M}(R,\mu)$ and $p \in (0,\infty)$. Then*

$$\int_R |f(x)|^p \, d\mu = \int_0^\infty f^*(t)^p \, dt.$$

Proposition. *Let $f \in \mathcal{M}(R,\mu)$ and let $E \subset R$ be μ-measurable. Then*

$$\int_E |f(x)| \, d\mu \le \int_0^{\mu(E)} f^*(t) \, dt.$$

Theorem 12 (the inequality of Hardy and Littlewood)**.** *For every $f, g \in \mathcal{M}(R,\mu)$, one has*

$$\int_R f(x) g(x) \, d\mu \le \int_0^\infty f^*(t) g^*(t) \, dt.$$

Definition. Let $f \in \mathcal{M}_0(R,\mu)$. Then the function $f^{**} \colon (0,\infty) \to [0,\infty]$ defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds \quad \text{for } t \in [0,\infty),$$

is called the *maximal nonincreasing rearrangement* of f.

Remark. For every $f \in \mathcal{M}_0(R,\mu)$, the function f^{**} is nonincreasing on $(0,\infty)$ and one has $f^*(t) \le f^{**}(t)$ for every $t \in (0,\infty)$.

Theorem 13 (subadditivity of f^{**})**.** *For every $f, g \in \mathcal{M}(R,\mu)$ and every $t \in (0,\infty)$, one has*

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t).$$

Theorem 14 (Hardy's lemma)**.** *Assume that f, g are nonnegative measurable functions on $(0,\infty)$ such that*

$$\int_0^t f(s) \, ds \le \int_0^t g(s) \, ds \quad \text{for every } t \in (0,\infty).$$

Then, for every nonnegative nonincreasing function h on $(0,\infty)$ we have

$$\int_0^\infty f(t) h(t) \, dt \le \int_0^\infty g(t) h(t) \, dt.$$

Theorem 15 (on Lorentz norms)**.** *If $1 \le q \le p \le \infty$, then $\|\cdot\|_{p,q}$ is a norm.*

Theorem 16 (embeddings of Lorentz spaces)**.** *Let $p, q, r \in [0,\infty]$ be such that $q \le r$. Then $L^{p,q} \hookrightarrow L^{p,r}$.*

Definition. Assume that $p, q \in (0,\infty]$. We define the functional $\|\cdot\|_{(p,q)}\colon \mathcal{M}(R,\mu) \to [0,\infty]$ by

$$\|f\|_{(p,q)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t)\|_{L^q(0,\infty)}.$$

Theorem 17 (alternative norm in a Lorentz space)**.** *Assume that $p \in (1,\infty]$ and $q \in [1,\infty]$. Then $\|\cdot\|_{(p,q)}$ is a norm. Moreover, the functionals $\|\cdot\|_{(p,q)}$ and $\|\cdot\|_{p,q}$ are equivalent in the sense that there exists a constant C such that*

$$\|f\|_{p,q} \le \|f\|_{(p,q)} \le C\|f\|_{p,q} \quad \textit{for every } f \in \mathcal{M}(R,\mu).$$

Theorem 18 (weighted Hardy's inequality)**.** *Let $1 < p < \infty$ and $f \in \mathcal{M}_+(0,\infty)$.*

(a) *If $\alpha < p-1$, then*

$$\int_0^\infty \left(\frac{1}{t}\int_0^t f(s)\,ds\right)^p t^\alpha\,dt \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty f(t)^p t^\alpha\,dt.$$

(b) *If $\alpha > p-1$, then*

$$\int_0^\infty \left(\frac{1}{t}\int_t^\infty f(s)\,ds\right)^p t^\alpha\,dt \le \left(\frac{p}{\alpha+1-p}\right)^p \int_0^\infty f(t)^p t^\alpha\,dt.$$

Theorem 19 (Minkowski's integral inequality)**.** *Let (R,μ) and (S,ν) be σ-finite measure spaces. Let $p \in [1,\infty)$ and let $F\colon (R\times S) \to \mathbb{R}$ be measurable with respect to $\mu\times\nu$. Assume that*

$$\int_S \left(\int_R |F(x,y)|^p\,d\mu(x)\right)^{\frac{1}{p}} d\nu(y).$$

Then $\int_S F(x,y)d\nu(y)$ converges for μ-a.e. $x \in R$ and

$$\left(\int_R \left|\int_S F(x,y)d\nu(y)\right|^p d\mu(x)\right)^{\frac{1}{p}} \le \int_S \left(\int_R |F(x,y)|^p\,d\mu(x)\right)^{\frac{1}{p}} d\nu(y).$$

Definition. Let $p \in [1,\infty)$, $q \in [1,\infty]$ and let T be an operator defined on $L^{p,1}(R,\mu)$ and taking values in $\mathcal{M}(S,\nu)$. Then T is said to be of *weak type* (p,q) if it is a bounded operator from $L^{p,1}(R,\mu)$ into $L^{q,\infty}(S,\nu)$, that is, if there exists a constant M such that

$$\|Tf\|_{q,\infty} \le M\|f\|_{p,q} \quad \text{for every } f \in L^{p,1}(R,\mu).$$

The lease such constant M is called the *weak-type* (p,q) *norm* of T. We say that T is of weak type (∞,q) if it is a bounded operator from $L^\infty(R,\mu)$ into $L^{q,\infty}(S,\nu)$.

Theorem 20 (Marcinkiewicz's interpolation theorem)**.** *Let $1 \le p_0 < p_1 < \infty$, $1 \le q_0, q_1 \le \infty$, $q_0 \ne q_1$, $0 < \theta < 1$ and $1 \le r \le \infty$. Let p, q be defined by the formulas*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let T be a quasilinear operator defined on $(L^{p_0,1} + L^{p_1,1})(R,\mu)$ and taking values in $\mathcal{M}(S,\nu)$. Let T be of weak types (p_0,q_0) and (p_1,q_1) with respective weak-type norms M_0 and M_1. Then $T\colon L^{p,r} \to L^{q,r}$. More precisely, there exists a constant C such that

$$\|Tf\|_{q,r} \le \frac{C\max\{M_0,M_1\}}{\theta(1-\theta)}\|f\|_{p,r}.$$

Remarks. (a) Theorem 20 holds also in the case $p_1 = \infty$ provided that the hypothesis "of weak type (p_1,q_1)" is replaced by "of strong type (p_1,q_1)".

(b) If $p_i \le q_i$, $i = 0,1$, then it follows under the hypotheses of Theorem 20 that T is of strong type (p,q).

(c) The assumption $q_0 \neq q_1$ cannot be omitted. For instance, let α be a bounded linear functional on $L^1(0,1)$ and let the operator T be defined on $L^1(0,1)$ by

$$Tf(t) = \alpha(f)\frac{1}{\sqrt{t}} \quad \text{for } t \in (0,1).$$

Then T is of weak type $(1,2)$ and of weak type $(\infty,2)$, but it is not of strong type $(2,2)$.

Example. Assume that $1 \leq p \leq 2$. Then there exists a constant C depending on n and p such that

$$\|\mathcal{F}f\|_{L^{p',p}(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for every } f \in L^p(R^n),$$

where $\mathcal{F}$ denotes the Fourier transform. Note that, thanks to Theorem 15, this is a better estimate than (2.1).

Example. Let $n \in \mathbb{N}$, $\gamma \in (0,n)$ and $p \in (1, \frac{n}{n-\gamma})$. Then there exists a constant C depending on n, p and γ such that

$$\|I_\gamma f\|_{L^{\frac{np}{n-p},p}(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for every } f \in L^p(R^n),$$

where I_γ denotes the Riesz potential.

Example. Let $p \in (1,\infty]$. Then there exists a constant C depending on p such that

$$\|Af\|_{L^p(0,\infty)} \leq C\|f\|_{L^p(0,\infty)} \quad \text{for every } f \in L^p(0,\infty),$$

where A denotes the Hardy averaging operator.

Example. Let $p \in (1,\infty]$. Then there exists a constant C depending on p such that

$$\|\mathcal{L}f\|_{L^{p',p}(0,\infty)} \leq C\|f\|_{L^p(0,\infty)} \quad \text{for every } f \in L^p(0,\infty),$$

where $\mathcal{L}$ denotes the Laplace transform.

Definition. The *Hilbert transform* H is defined by the formula

$$Hf(x) = p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \quad \text{for } x \in \mathbb{R}$$

and every function $f \in \mathcal{M}(\mathbb{R})$ for which the integral makes sense.

Example. Let $p \in (1,\infty)$. Then there exists a constant C depending on p such that

$$\|Hf\|_{L^p(0,\infty)} \leq C\|f\|_{L^p(0,\infty)} \quad \text{for every } f \in L^p(0,\infty),$$

where H denotes the Hilbert transform.

2.5. **Extrapolation and Yano's theorem.**

Theorem 21 (Young's inequality). *Let $a\colon [0,\infty) \to [0,\infty]$ be a nondecreasing left-continuous function such that $a(0) = 0$ and a is not identically equal to zero. Let a^{-1} be the left-continuous inverse of a defined by*

$$a^{-1}(t) = \inf\{s : a(s) \geq t\} \quad \textit{for } t \in [0,\infty).$$

Let A and $\tilde{A}$ be defined by the formulas

$$A(t) = \int_0^t a(s)\,ds, \quad \tilde{A}(t) = \int_0^t a^{-1}(s)\,ds \quad \textit{for } t \in [0,\infty).$$

Then

$$st \leq A(s)\tilde{A}(t) \quad \textit{for every } s,t \in [0,\infty).$$

Theorem 22 (Yano's extrapolation theorem)**.** *Let $\mu(R) < \infty$ and $\nu(S) < \infty$. Let T be an operator defined on $L^1(R,\mu)$ such that for every sequence of functions $\{f_n\}_{n=0}^\infty$ in $\mathcal{M}(R,\mu)$ one has*

$$\left| T\left(\sum_{n=0}^\infty f_n\right)\right| \le \sum_{n=0}^\infty |Tf_n|,$$

and for every function $f \in \mathcal{M}(R,\mu)$ and every $\lambda \in \mathbb{R}$ we have

$$|T(\lambda f)| = |\lambda||Tf|.$$

Assume that for some $C > 0$ and $\alpha > 0$, every $f \in \mathcal{M}(R,\mu)$ and every $p \in (1,2]$ one has

$$\|Tf\|_{L^p(S,\nu)} \le \frac{C}{(p-1)^\alpha}\|f\|_{L^p(R,\mu)}.$$

Then there exist positive constants C_1, C_2 such that for every $f \in \mathcal{M}(R,\mu)$ we have

$$\int_S |Tf(y)|\,d\nu \le C_1 \int_R |f(x)| \log^\alpha(1+|f(x)|)\,d\mu + C_2.$$

Theorem 23 (exponential extrapolation)**.** *Let $\mu(R) < \infty$ and $\nu(S) < \infty$. Let T be a sublinear operator defined on $(L^1+L^\infty)(R,\mu)$ and taking values in $\mathcal{M}(S,\nu)$. Assume that for some $C > 0$ and $\alpha > 0$, every $f \in \mathcal{M}(R,\mu)$ and every $p \in [2,\infty)$ one has*

$$\|Tf\|_{L^p(S,\nu)} \le Cp^\alpha \|f\|_{L^p(R,\mu)}.$$

Then there exist positive constants C_1, C_2 such that for every $f \in L^\infty(R,\mu)$ we have

$$\int_S \exp\left(\left[\frac{|Tf(y)|}{C_1\|f\|_\infty}\right]^{\frac{1}{\alpha}}\right) d\nu \le C_2.$$

2.6. **Orlicz spaces.**

Definition. We say that a function $A\colon [0,\infty) \to [0,\infty]$ is a *Young function* if it is left continuous, convex and non-decreasing on $[0,\infty)$, satisfying $A(0) = 0$, and such that A is not identically equal to zero on $[0,\infty)$.

Remark. A function $A\colon [0,\infty) \to [0,\infty]$ is a Young function if and only if there exists a left-continuous nondecreasing function $a\colon [0,\infty) \to [0,\infty]$ such that $a(0) = 0$, a is not identically equal to zero on $[0,\infty)$ and

$$A(t) = \int_0^t a(s)\,ds \quad \text{for } t \in (0,\infty). \tag{2.8}$$

Proposition. *Let A be a Young function. If $\lambda \in [0,1]$, then*

$$A(\lambda t) \le \lambda A(t) \quad \textit{for every } t \in [0,\infty). \tag{2.9}$$

If $\lambda \in [1,\infty)$, then

$$A(\lambda t) \ge \lambda A(t) \quad \textit{for every } t \in [0,\infty). \tag{2.10}$$

Definition. Let A be a Young function. We define the functional ϱ_A on $\mathcal{M}(R,\mu)$ by

$$\varrho_A(f) = \int_R A(|f|)\,d\mu. \tag{2.11}$$

We define the *Orlicz class* $\mathcal{L}^A = \mathcal{L}^A(R,\mu)$ as the collection

$$\mathcal{L}^A = \{f \in \mathcal{M}(R,\mu) : \varrho_A(f) < \infty\}.$$

Definition. Given a Young function A, the *Orlicz space* $L^A = L^A(R,\mu)$ is the collection of all functions $f \in \mathcal{M}(R,\mu)$ such that

$$\int_R A\left(\frac{|f|}{\lambda}\right) d\mu < \infty$$

for some $\lambda \in (0,\infty)$. The *Luxemburg norm* $\|\cdot\|_A \colon \mathcal{M}(R,\mu) \to [0,\infty]$ is defined by

$$\|f\|_{L^A} = \inf\left\{\lambda > 0 : \int_R A\left(\frac{|f|}{\lambda}\right) d\mu \le 1\right\}.$$

Proposition. *Let A be a Young function.*
 (a) *If $\|f\|_A \le 1$, then $\varrho_A(f) \le \|f\|_A$,*
 (b) *if $\|f\|_A \ge 1$, then $\varrho_A(f) \ge \|f\|_A$,*
 (c) *$\|f\|_A \le \varrho_A(f) + 1$.*

Theorem 24 (relation between norm and modular estimates)**.** *Suppose that A_1, A_2 are Young functions, T is a sublinear operator and there exist positive constants C_1, C_2 such that*

$$\varrho_{A_1}(Tf) \le C_1 + C_2\varrho_{A_2}(f) \quad \textit{for every } f \in \mathcal{M}(R,\mu).$$

Then there exists a constant C_3 such that

$$\|Tf\|_{L^{A_1}(R,\mu)} \le C_3\|f\|_{L^{A_2}(S,\nu)} \quad \textit{for every } f \in \mathcal{M}(R,\mu).$$

3. Real interpolation

3.1. K-method.

Definition. Let X_0 and X_1 be Banach spaces. We say that (X_0, X_1) is a *compatible couple* if there exists a Hausdorff topological linear space $\mathfrak{M}$ such that $X_0 \hookrightarrow \mathfrak{M}$ and $X_1 \hookrightarrow \mathfrak{M}$ (here $\hookrightarrow$ denotes a continuous embedding).

Definition. Let (X_0, X_1) be a compatible couple with the corresponding space $\mathfrak{M}$. We define the *sum* of spaces $X_0 + X_1$ as the collection of all elements $x \in \mathfrak{M}$ which are representable as $x = x_0 + x_1$ with $x_0 \in X_0$ and $x_1 \in X_1$. For each $x \in X_0 + X_1$, set

$$\|x\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1\},$$

where the infimum is extended over all possible representations $x = x_0 + x_1$, $x_0 \in X_0$, $x_1 \in X_1$. For each $x \in X_0 \cap X_1$, set

$$\|x\|_{X_0\cap X_1} = \max\{\|x\|_{X_0}; \|x\|_{X_1}\}.$$

Theorem 25 (sums and intersections of Banach spaces)**.** *Let (X_0, X_1) be a compatible couple. Then $X_0 + X_1$ and $X_0 \cap X_1$ are Banach spaces.*

Definition. Let (X_0, X_1) be a compatible couple. The *K-functional* is defined for each $f \in X_0 + X_1$ and each $t \in (0,\infty)$ by

$$K(f,t;X_0,X_1) = \inf\{\|g\|_{X_0} + t\|h\|_{X_1} : f = g + h\},$$

where the infimum is extended over all possible representations $f = g + h$, $g \in X_0$, $h \in X_1$.

Observation. Obviously,

$$\min\{1,t\}\|f\|_{X_0+X_1} \le K(f,t;X_0,X_1) \le \max\{1,t\}\|f\|_{X_0+X_1}.$$

Consequently, the functionals $f \mapsto K(f,t;X_0,X_1)$ define a family of mutually equivalent norms on $X_0 = X_1$.

Observation. Every $x \in X_0$ has a trivial representation $f = f + 0$, hence

$$K(f,t;X_0,X_1) \le \|f\|_{X_0} \quad \text{for every } t \in (0,\infty).$$

Similarly, every $x \in X_1$ has a trivial representation $f = 0 + f$, hence

$$K(f,t;X_0,X_1) \le t\|f\|_{X_1} \quad \text{for every } t \in (0,\infty).$$

Thus, for every $f \in X_0 \cap X_1$ one has

$$K(f,t;X_0,X_1) \le \min\{\|f\|_{X_0}; t\|f\|_{X_1}\} \quad \text{for every } t \in (0,\infty).$$

Notation. Sometimes we write $K(f,t)$ is place of $K(f,t;X_0,X_1)$ when no confusion can arise.

Proposition. *For each $f \in X_0 + X_1$, $t \mapsto K(f,t;X_0,X_1)$ is a nonnegative nondecreasing concave function such that $t^{-1}K(f,t;X_0,X_1)$ is nonincreasing and*

$$t^{-1}K(f,t;X_0,X_1) = K(f,t^{-1};X_1,X_0).$$

Example. If X_0 and X_1 are either Lebesgue or Lorentz or Orlicz spaces, then one can take

$$\mathfrak{M} = \{\mu\text{-measurable a.e. finite functions on } R\},$$

endowed with metric

$$\varrho(f,g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{1}{\mu(R_n)} \int_{R_n} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} d\mu(x),$$

where $R = \bigcup R_n$, $0 < \mu(R_n) < \infty$ for each $n \in \mathbb{N}$. Then $(\mathfrak{M},\varrho)$ is a complete metric space such that the convergence in ϱ coincides with the convergence in measure on sets of finite measure.

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