Classification of $\mathfrak{p}$-homomorphisms between higher symplectic spinors

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Abstract

In this article, $\mathfrak{p}$-homomorphisms from a first jet prolongation of the so called higher symplectic spinor module into an arbitrary higher symplectic spinor module are classified. This is the first step in the classification of first order invariant differential operators acting between fields with values in higher symplectic spinors over projective contact manifolds. An "infinitesimal version" of the prospective classification is established by a procedure described in this article. MSC: 17B10, 17B20, 53D10.

1 Introduction

The projective contact geometry is one of the two exceptional parabolic geometries for which one needs some additional data together with the filtration of the tangent bundle to construct the corresponding Cartan bundle. This is not the only exception present in this article, the further comes from the representation theory of symplectic Lie algebras. To develop a theory of invariant differential operators, one needs to specify irreducible representations in which appropriate fields take their values. In the projective contact case, important operators which were studied (see, e.g., Kostant [15], Habermann [11], Klein [14] for the so called symplectic Dirac operator and Kadlčáková [13] for the symplectic twistor operator) are acting between fields with values in some infinite dimensional representations. In this article, it is shown that these representations (although infinite dimensional) can be handled for the seeks of the theory of first order invariant differential operators in a similar way as finite dimensional ones, see, e.g., Čap, Slovák, Souček [5, 6, 7]

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and Slovák, Souček [18] for a treatment of theory of invariant differential operators for finite dimensional irreducible representations.

In the first three subsections of the first section (Introduction), some introductory notions of and preliminary facts on parabolic subgroups, parabolic geometries, first order invariant operators (1.1.), contact gradings and projective contact geometry (1.2.) and higher symplectic spinor modules (1.3.) are summarized. In the first part of the second section, a condition on a $\mathfrak{g}_0$-homomorphism to be a $\mathfrak{p}$-homomorphism is derived (2.1.). The main result (description of the set $\mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{V}, \mathbb{W})$, theorem 6) is in the second part of the second section, i.e., (2.2.). The crucial fact needed in this case is a decomposition of a tensor product of the defining representation of the symplectic Lie algebra and a harmonic (= higher symplectic spinor) module (theorem 2, section (1.3.)).

Let us remark that some special cases of operators we are dealing with were recently introduced in the so called covariant quatization in the superstring theory and in the theory of Dirac-Kähler fields, see a paper of Green, Hull [10] for a treatment of geometric quantization and a paper of Reuter [17] for the so called Dirac-Kähler fileds and metaplectic Dirac operators, in which some notions generalized here, are introduced in a setting of physics.

1.1 Parabolic subgroups, parabolic geometries and invariant differential operators

Let G be a semisimple finite dimensional Lie group and let us denote by P an arbitrary parabolic subgroup of G. Let us denote the appropriate Lie subalgebras by $\mathfrak{g}$ and $\mathfrak{p}$, respectively. The parabolic subalgebra possesses the so called Lévi decomposition (direct sum on the level of Lie algebras)

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$$

into a reductive (the Lévi factor) and a nilpotent part, respectively.[1] The first decomposes as $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}$, where $\mathfrak{g}_0^{ss}$ and $\mathfrak{z}$ are the semisimple part and the center of the reductive component $\mathfrak{g}_0$, respectively.

Now let us briefly remark that there is a connection between parabolic subalgebras and the so called $|k|$-gradings ($k \in \mathbb{N}$) of a simple Lie algebra. A $|k|$-grading of a Lie algebra $\mathfrak{g}$ is a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k,$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for $i, j \in \{-k, \ldots, k\}$, where $\mathfrak{g}_j = \{0\}$ for $|j| > k$. Further, let us denote by $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ and $\mathfrak{g}_+ := \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$,

[1] Moreover, $\mathfrak{g}_+$ is a $\mathfrak{g}_0$-module.

the so called negative and positive part, respectively. Notice that to the $|k|$-grading, there is a unique element $E \in \mathfrak{g}$, for which $[E, X] = jX$ holds for $X \in \mathfrak{g}_j$, $j \in \{-k, \ldots, k\}$, a so called *grading element*. Suppose that such a $|k|$-graded simple Lie algebra is given. Then there exists a choice $(\mathfrak{h}, \Phi^+)$ of a Cartan subalgebra and a system of positive roots of ($\mathfrak{h}$ on) $\mathfrak{g}$, such that $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_+$ is a standard parabolic subalgebra with respect to $(\mathfrak{h}, \Phi^+)$. On the other hand, choose a pair $(\mathfrak{h}, \Phi^+)$. Given any parabolic subalgebra, which is standard for the choice of $(\mathfrak{h}, \Phi^+)$, one can form a $|k|$-grading in such a way that this procedure is inverse to that one described above (for the fixed choice $(\mathfrak{h}, \Phi^+)$).

The last data of an algebraic character we need are two representations $(\rho, \mathbb{V})$ and $(\sigma, \mathbb{W})$ of the parabolic group P.

Now, let us come to a geometrical part of our introduction. Let us consider a P-principle fiber bundle $p : \mathcal{G} \to M$ being equipped with a Cartan connection, i.e., a differential $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which is a P-equivariant absolute parallelism reproducing the generators of fundamental vector fields of $p : \mathcal{G} \to M$. The above data form the so called parabolic geometry, and for our purposes it will be referred to as $(\mathcal{G}, p, M, G, P, \omega)$. Let us consider the absolutely invariant derivative ∇^ω canonically associated to the parabolic geometry $(\mathcal{G}, p, M, G, P, \omega)$ and to the representation $(\rho, \mathbb{V})$. This is an operation which transforms a P-equivariant $\mathbb{V}$-valued function $s : \mathcal{G} \to \mathbb{V}$ (which corresponds to a section of $\mathcal{G} \times_\rho \mathbb{V}$) into a smooth function $\nabla^\omega s : \mathcal{G} \to \mathfrak{g}_-^* \otimes \mathbb{V}$. The latter being defined by

$$\nabla^\omega s(u)(X) := \mathcal{L}_{\omega_u^{-1}(X)} s,$$

where $X \in \mathfrak{g}_-, u \in \mathcal{G}$ and $\mathcal{L}$ denotes the Lie derivative. Let us consider two associated vector bundles $VM := \mathcal{G} \times_\rho \mathbb{V}$ and $WM := \mathcal{G} \times_\sigma \mathbb{W}$. The vector space

$$J^1\mathbb{V} := \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V})$$

is called the first jet prolongation of $\mathbb{V}$. This vector space $J^1\mathbb{V}$ comes up with a natural action of P, making it into a P-module in such a way, that $(s, \nabla^\omega s) : \mathcal{G} \to J^1\mathbb{V}$ is a P-equivariant function, see again Slovák, Souček [18]. The restriction of the infinitesimal version of this action to the subalgebra $\mathfrak{g}_+$ is expressed by the following formula, which will be used in the section 2:

$$\begin{aligned} &Z.(v_0, Y \otimes v_1) = \\ &(\rho(Z)v_0, Y \otimes \rho(Z)v_1 + [Z, Y] \otimes v_1 + \sum_\alpha \eta^\alpha \otimes \rho([Z, \xi_\alpha]_{\mathfrak{p}})v_0), \end{aligned} \qquad (1)$$

where $v_0, v_1 \in \mathbb{V}$, $Z \in \mathfrak{g}_i$, $i > 0$, $Y \in \mathfrak{g}_+$, and where $\{\eta^\alpha\}_\alpha$, $\{\xi_\alpha\}_\alpha$ are some bases of $\mathfrak{g}_-$ and $\mathfrak{g}_+$, respectively, which are dual to each other with respect to the Killing form B of the algebra $\mathfrak{g}$. We notice that $\mathfrak{g}_-^* \simeq \mathfrak{g}_+$.

In the realm of parabolic geometries, each P-module homomorphism $\Phi : J^1\mathbb{V} \to \mathbb{W}$ gives rise to a first order invariant differential operator $D : \Gamma(M, VM) \to \Gamma(M, WM)$, which is defined by the formula

$$Ds(u) = \Phi(s(u), \nabla^\omega s(u))$$

for $u \in \mathcal{G}$ and $s \in \Gamma(M, VM)$ considered as a P-equivariant $\mathbb{V}$-valued function on $\mathcal{G}$, i.e., $s \in \mathcal{C}^\infty(\mathcal{G}, \mathbb{V})^P \simeq \Gamma(M, VM)$. So the first step in understanding the first order differential operators is the description of the set $\mathrm{Hom}_P(J^1\mathbb{V}, \mathbb{W})$. We shall characterize the "infinitesimal" version of it, i.e., we give a description of the set $\mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{V}, \mathbb{W})$ in the section 2.2. and this will be the main result of this article.

1.2 Projective contact geometry

Let $\mathfrak{g}$ be a real or complex $|k|$-graded semisimple Lie algebra. We call this grading contact, if it is a depth two grading, i.e.,

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

satisfying the following two properties

(1) $[,] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is nondegenerate and

(2) $\dim \mathfrak{g}_{-2} = 1$.

A classification of complex and real contact graded simple Lie algebras can be found in Yamaguchi [19]. Let us remark that up to an isomorphism each complex simple Lie algebra possesses a unique contact grading. This fact is no more true for real simple Lie algebras, see again Yamaguchi [19] where a list of all real forms of complex simple Lie algebras, which possess no contact grading, together with an information on gradings of simple real Lie algebras which are not real forms of complex simple Lie algebras are presented.

Let us consider a real simple Lie algebra $\mathfrak{g} = \mathfrak{sp}(2k, \mathbb{R})$ and denote the defining symplectic form of $\mathfrak{g}$ by J. Suppose a symplectic basis of $(\mathbb{R}^{2k}, J)$ is given. With respect to this basis, the matrix of J is given in a block form as follows

$$J = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix},$$

where the (i,j)-entry of K equals $\delta_{i,n+1-j}$. With respect to the symplectic basis, the contact grading is given by the following block splitting

$$A = \left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right),$$

where

(1) $\mathfrak{g}_0 = \mathfrak{sp}(2k-2,\mathbb{R}) \oplus \mathbb{R}E$, where E is the grading element,

(2) $\mathfrak{g}_{-1} = \mathbb{R}^{2k-2}$,

(3) $\mathfrak{g}_{-2} = \mathbb{R}$.

Let $G = Sp(2k,\mathbb{R})$. Consider the tautological action of G on the vector space $\mathbb{R}^{2k}$. This action factors to an action on the open rays of $\mathbb{R}^{2k}$. Let us denote the isotropy subgroup of G of an open ray in $\mathbb{R}^{2k}$ by P. Obviously, P is a parabolic subgroup of the group G. It can be proved (see Krýsl [16]) that P is the subgroup of G associated to the contact grading of $\mathfrak{g}$ given above.
Definition: Let $p : \mathcal{G} \to M$ be a P-principal fiber bundle. A parabolic geometry $(\mathcal{G}, p, M, G, P, \omega)$ of type (G,P) on the manifold M with G, and P described above will be called *projective contact geometry*.

For a comprehensive treatment on contact geometries, see Blair [1]; for some connections of projective contact geometries (given by certain data associated to the filtered tangent bundle) and Cartan geometries see Fox [9], or Krýsl [16].

1.3 Higher symplectic modules

Let us describe an important class of simple modules over simple Lie algebras. Let $\mathfrak{g}$ be an arbitrary simple complex Lie algebra and $\mathfrak{h}$ its Cartan subalgebra. An $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module $\mathbb{V}$ is called a *module with bounded multiplicities*, if there is an element $k \in \mathbb{N}_0$, such that for each weight $\nu \in \mathfrak{h}^*$ the dimension of the weight space $\mathbb{V}_(\nu)$ is smaller or equal to k. The minimal k with this property is called *degree* of $\mathbb{V}$. A module with bounded multiplicities is called *completely pointed* provided its degree is 1.

Now, we will focus our attention at representations of the symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}(2k,\mathbb{C})$. Symplectic spinor representations were introduced by Bertram Kostant (see Kostant [15]), when he was seeking for an analogy of the "orthogonal" Dirac operator, which was then studied by many authors, e.g., Habermann [11], Klein [14]. Choose a Cartan subalgebra $\mathfrak{h}$ of the symplectic Lie algebra $\mathfrak{g}$ and a system of positive roots Φ^+ of ($\mathfrak{h}$ on) $\mathfrak{g}$. Denote the basis of fundamental weights of $\mathfrak{g}$ wr. to $(\mathfrak{h}, \Phi^+)$ by $\{\varpi_i\}_{i=1}^k$. Let $L(\nu)$ be the irreducible highest weight module with the highest weight ν. The irreducible highest weight modules $\mathbb{S}_+ := L(-\frac{1}{2}\varpi_k)$ and $\mathbb{S}_- := L(\varpi_{k-1} - \frac{3}{2}\varpi_k)$ are called *basic spinor modules.* These spinor modules are often referred to as *metaplectic representations*, because it is possible to lift them to representations of the double cover of the symplectic group, which is usually called metaplectic.

One can prove (see Britten, Lemire [3], Britten, Hooper, Lemire [2]), that an infinite dimensional $\mathfrak{sp}(2k,\mathbb{C})$-module $\mathbb{V}$ is completely pointed iff either $\mathbb{V} = \mathbb{S}_+$ or $\mathbb{V} = \mathbb{S}_-$. Moreover, Britten, Hooper and Lemire [2] proved that the tensor product $L(\nu) \otimes \mathbb{S}_\pm$ for ν an integral dominant weight for $\mathfrak{g}$ is completely reducible and decomposes into a finite direct sum of irreducible highest weight $\mathfrak{g}$-modules.

Definition: We call a $\mathfrak{sp}(2k,\mathbb{C})$-module $\mathbb{V}$ *higher symplectic spinor* or *harmonic module*, if there is an integral dominant weight ν such that $\mathbb{V}$ is an irreducible summand in $L(\nu) \otimes \mathbb{S}_\pm$.

Further let us define a set

$$\mathbb{A} := \{\sum_{i=1}^{n} \lambda_i \varpi_i; \lambda_i \geq 0, \lambda_i \in \mathbb{Z},\ i = 1, \ldots, n-1;\ \lambda_n \in \mathbb{Z}+\frac{1}{2}, \lambda_{n-1}+2\lambda_n+3 > 0\}.$$

The following fact is known.

Theorem 1: The following are equivalent

1. $\mathbb{V}$ a is harmonic module,
2. the highest weight of $\mathbb{V}$ is in $\mathbb{A}$,
3. $\mathbb{V}$ is an infinite dimensional $\mathfrak{sp}(2k,\mathbb{C})$-module with bounded multiplicities.

Proof. See Britten, Lemire [3].

There is an important theorem on the decomposition of a tensor product of a harmonic module and the defining representation, which will be used in the study of the $\mathfrak{p}$-homomorphisms between the first jet prolongation of a harmonic module and an arbitrary higher symplectic module.

Theorem 2: Let $\lambda \in \mathbb{A}$. Then

$$L(\lambda) \otimes L(\varpi_1) = \bigoplus_{\mu \in \mathbb{A}_\lambda} L(\mu),$$

where $\mathbb{A}_\lambda = \{\lambda+\nu; \nu \in \Pi(\varpi_1)\} \cap \mathbb{A}$ and $\Pi(\varpi_1)$ is the saturated set of weights of the representation $L(\varpi_1)$. [2]
Proof. See Krýsl [16].

The preceding theorem says, how a tensor product of the defining representation and a harmonic module decomposes. The most important information is the fact that the tensor product is completely reducible and multiplicity free. The proof uses some facts on decomposition of a tensor product of a finite dimensional and infinite dimensional highest weight modules into irreducible modules and a the structure of the set $\mathbb{A}$ (conjugacy with respect to the affine action of the Weyl group). In the next section, we shall see, how this decomposition helps us to define all "infinitesimal versions" of invariant differential operators of the first order. Let us also notice, that the last theorem can be generalized in such a way that it is possible to construct invariant differential operators of higher order, see Krýsl [16].

2 First order invariant differential operators for parabolic geometries and irreducible highest weight modules

The aim of this section is to classify the $\mathfrak{p}$-homomorphisms between $J^1\mathbb{V}$ and $\mathbb{W}$, where $\mathbb{V}$ and $\mathbb{W}$ are higher symplectic spinor modules. At first we will consider arbitrary irreducible highest weight modules and then we restrict our attention at the case of higher symplectic spinor ones.

2.1 The general case: values in irreducible highest weight modules

For explanation of notions used in the first paragraph of this subsection, see subsection (1.1). Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra, $\mathfrak{p}$ the associated parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{g}_+$ the positive and $\mathfrak{g}_-$ the negative part of $\mathfrak{g}$. Let $(\rho, \mathbb{V})$, $(\sigma, \mathbb{W})$ be $\mathfrak{p}$-modules. Further, let $J^1\mathbb{V}$ be the first jet prolongation of the $\mathfrak{p}$-module $\mathbb{V}$ associated to the $|k|$-graded Lie algebra $\mathfrak{g}$. The induced action of $\mathfrak{g}_+$ on $J^1\mathbb{V}$ is given by the equation (1):

$$Z.(v_0, Y \otimes v_1) = (\rho(Z)v_0, Y \otimes \rho(Z)v_1 + [Z,Y] \otimes v_1 + \sum_\alpha \eta^\alpha \otimes \rho([Z,\xi_\alpha]_\mathfrak{p})v_0),$$

[2] One can check that $\Pi(\varpi_1) = \{\pm\epsilon_i; i = 1, \ldots, n\}$, in the notation in which $\varpi_i = \sum_{j=1}^{i} \epsilon_j$.

where $v_0, v_1 \in \mathbb{V}$, $Z \in \mathfrak{g}_i$, $i > 0$, $Y \in \mathfrak{g}_+$ and $\{\eta^\alpha\}_\alpha$, $\{\xi_\alpha\}_\alpha$ as before, see section 1.1. or Slovák, Souček [18].

It is a well known fact that $\mathfrak{g}_{-i}^* \simeq \mathfrak{g}_i$ for $i = 1, \ldots, k$ as $\mathfrak{g}_0$-modules. Let $\mathfrak{g}_+^2$ denote the space $[\mathfrak{g}_+, \mathfrak{g}_+] = \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k$. We call the space $J_R^1\mathbb{V} = \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V})/(\{0\} \oplus (\mathfrak{g}_+^2 \otimes \mathbb{V})) \simeq \mathbb{V} \oplus (\mathfrak{g}_{-1}^* \otimes \mathbb{V}) \simeq \mathbb{V} \oplus (\mathfrak{g}_1 \otimes \mathbb{V})$ the *space of restricted jets.* This space carries the structure of $\mathfrak{p}$-module inherited by factorization. Let us denote the bases of $\mathfrak{g}_{\pm 1}$ by $\{\eta^{\alpha'}\}_{\alpha'}, \{\xi_{\alpha'}\}_{\alpha'}$.

Theorem 3: Let $\mathbb{V}, \mathbb{W}$ be irreducible $\mathfrak{p}$-modules with the trivial action of $\mathfrak{g}_+$. Let $\Psi : J^1\mathbb{V} \to \mathbb{W}$ be a $\mathfrak{g}_0$-module homomorphism. Then Ψ is a $\mathfrak{p}$-module homomorphism if and only if Ψ factors through the restricted jets $J_R^1\mathbb{V}$ and for all $Z \in \mathfrak{g}_1$ and $v_0 \in \mathbb{V}$

$$\Psi(\sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}].v_0) = 0.$$

Proof. See Krýsl [16]. $\square$

Let us notice that the proof of the theorem 3 is closely parallel to that of an analogous fact for the finite dimensional case which is given in Slovák, Souček [18].

For further purposes let as define an endomorphism

$$\Phi : \mathfrak{g}_1 \otimes \mathbb{V} \to \mathfrak{g}_1 \otimes \mathbb{V}, \quad \Phi(Z \otimes s) := \sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}].s$$

for $Z \in \mathfrak{g}_1, s \in \mathbb{V}$.

Thus we can reformulate the last theorem as

Corollary 1: Let $\mathbb{V}, \mathbb{W}$ be irreducible $\mathfrak{p}$-modules with the action of $\mathfrak{g}_+$ being trivial. Let $\Psi : J^1\mathbb{V} \to \mathbb{W}$ be a $\mathfrak{g}_0$-module homomorphism. Then Ψ is a $\mathfrak{p}$-module homomorphism if and only if Ψ factors through the restricted jets $J_R^1\mathbb{V}$ and $\Psi_{|Im(\Phi)} = 0$.

In this part, we would like to compute the mapping Φ with help of the universal Casimir element.

First, let us present a well known theorem on the action of the universal Casimir element on a highest weight module over a corresponding simple Lie algebra.

Theorem 4: Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra and $\lambda \in \mathfrak{h}^*$. The action of the universal Casimir element c on a highest weight module of the highest weight λ is by a scalar

$$(\lambda + \delta, \lambda + \delta) - (\delta, \delta) = (\lambda + 2\delta, \lambda)$$

where δ is the sum of fundamental weights of $\mathfrak{g}$.

Proof. See Humphreys [12], pp. 143. $\square$

Second, let us make some assumptions on the Lie algebra $\mathfrak{g}$. Suppose that the reductive subalgebra $\mathfrak{g}_0$ (the Lévi factor) of $\mathfrak{p}$ has a one dimensional center. This center is necessarily generated by the grading element E of the $|k|$-graded Lie algebra $\mathfrak{g}$. Thus we can decompose $\mathfrak{g} = \mathfrak{g}_0^{ss} \oplus \mathbb{C}E$ where the part $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ denotes the semisimple part of $\mathfrak{g}_0$, see the section 1.1. The Killing form B of the Lie algebra $\mathfrak{g}$ when restricted to $\mathfrak{g}_0$ is nondegenerate too, see Yamaguchi [19]. Let us normalize the Killing form B by the condition $B(E,E) = 1$ and denote this resulting nondegenerate invariant form on $\mathfrak{g}_0$ by $(,) : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{C}$. It is easy to compute that the decomposition $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathbb{C}E$ is an orthogonal decomposition. Indeed, take an arbitrary $X \in \mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ in the form $X = [U,V]$ for a $U, V \in \mathfrak{g}_0$ and compute $(E, X) = (E, [U,V]) = ([E,U],V) = 0$ because E is the grading element and $U \in \mathfrak{g}_0$.

Let us denote the basis of $\mathfrak{g}_0^{ss}$ by $\{Y_a\}_{a=1}^k$ and the dual basis with respect to $(,)$ by $\{Y_a'\}_{a=1}^k$. Sometimes we will denote the element E by Y_{k+1}.

Now, we derive the following

Lemma 1: Let $\mathbb{V}$ be a representation of a semisimple $|k|$-graded Lie algebra $\mathfrak{g}$ then

$$\Phi(Z \otimes s) = \sum_{a=1}^{k} Y_a'.Z \otimes Y_a.s$$

for each $Z \in \mathfrak{g}_1$ and $s \in \mathbb{V}$.

Proof. See Krýsl [16]. $\square$

Let us remark that the proof does not differ in a formal sense from the proof of a corresponding statement for finite dimensional representations presented in Slovák, Souček [18].

Now, we make some assumptions on the representations we shall consider. We will consider that $\mathbb{V}$ is an irreducible highest weight $\mathfrak{g}$-module with the highest weight λ, if considered as a $\mathfrak{g}_0^{ss}$-module. Further, we assume that $\mathfrak{g}_1 \otimes \mathbb{V}$ decomposes (as a $\mathfrak{g}_0^{ss}$-module) into a finite direct sum of irreducible $\mathfrak{g}_0^{ss}$-modules without multiplicities and denote by π_μ the projection $\pi_\mu : \mathfrak{g}_1 \otimes \mathbb{V} \to \mathbb{V}_\mu$ where $\mathbb{V}_\mu$ is the representation with highest weight μ which occurs in the decomposition of the completely reducible tensor product $\mathfrak{g}_1 \otimes \mathbb{V}$. Let us suppose that the representation of the center $\mathbb{C}E$ of $\mathfrak{g}_0$ is given by $E.v := wv$ for each $v \in \mathbb{V}$ and a $w \in \mathbb{C}$. So we are given a representation of the whole $\mathfrak{g}_0$ which is characterized by the tuple (λ, w). The complex number w is often called *a generalized conformal weight*. Finally, we assume that $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0^{ss}$-module with the highest weight α. In order to compute the mapping Φ let us evaluate the following expression $\sum_{a=1}^{k+1}(Y_a'Y_a).(Z \otimes s)$ for $s \in \mathbb{V}$ and $Z \in \mathfrak{g}_1$.

$$\sum_{a=1}^{k+1}(Y_a'Y_a).(Z\otimes s) = \sum_{a=1}^{k+1}(Y_a'Y_a).Z\otimes s + Z\otimes\sum_{a=1}^{k+1}(Y_a'Y_a).s + 2\Phi(Z\otimes s), \quad (2)$$

where we have used the lemma 1 above. Now, we would like to compute the first two terms of the last written equation using the universal Casimir element of $\mathfrak{g}_0^{ss}$, see the theorem 4.

$$\sum_{a=1}^{k+1}(Y_a'Y_a).Z\otimes s = (\alpha,\alpha+2\delta)Z\otimes s + Z\otimes s \quad (3)$$

$$Z\otimes\sum_{a=1}^{k+1}(Y_a'Y_a).s = (\lambda,\lambda+2\delta)Z\otimes s + w^2 Z\otimes s \quad (4)$$

Let us compute the L.H.S. of 2

$$\sum_{a=1}^{k+1}(Y_a'Y_a).(Z\otimes s) = \sum_{\mu}(\mu,\mu+2\delta)\pi_\mu(Z\otimes s) + \sum_{\mu}\pi_\mu[Z\otimes s + 2wZ\otimes s + w^2 Z\otimes s] \quad (5)$$

Substituting equations 3, 4 and 5 into the equation 2 we obtain

$$\sum_{\mu}(\mu,\mu+2\delta)\pi_\mu(Z\otimes s) + 2\sum_{\mu}w\pi_\mu(Z\otimes s) + \sum_{\mu}w^2\pi_\mu(Z\otimes s) + Z\otimes s =$$
$$= 2\Phi(Z\otimes s) + (\alpha,\alpha+2\delta)Z\otimes s + Z\otimes s + (\lambda,\lambda+2\delta)Z\otimes s + w^2 Z\otimes s.$$

As a result we obtain

$$\Phi(Z\otimes s) = \sum_{\mu}(w - c^{\mu}_{\lambda\alpha})\pi_\mu(Z\otimes s),$$

where

$$c^{\mu}_{\lambda\alpha} = \frac{1}{2}[(\lambda,\lambda+2\delta) + (\alpha,\alpha+2\delta) - (\mu,\mu+2\delta)].$$

We state our result as a theorem formulating explicitly the assumptions we have made.

Theorem 5: Let $\mathfrak{g}$ be a $|k|$-graded simple Lie algebra such that the subalgebra $\mathfrak{g}_0$ has an one dimensional center $\mathbb{C}E$. Let $\mathbb{V}$ be an irreducible $\mathfrak{g}_0$-module with the highest weight λ, if considered as a $\mathfrak{g}_0^{ss}$-module. Let the grading element E acts by the complex number $w \in \mathbb{C}$ (generalized conformal weight). Further, let $\mathfrak{g}_1$ be an irreducible $\mathfrak{g}_0^{ss}$-module with the highest weight α. Assume that the tensor product $\mathfrak{g}_1 \otimes \mathbb{V}$ decomposes into a finite direct sum of irreducible $\mathfrak{g}_0^{ss}$-modules and has no multiplicities then

$$\Phi(Z \otimes s) = \sum_{\mu} (w - c^{\mu}_{\lambda\alpha}) \pi_{\mu}(Z \otimes s),$$

where

$$c^{\mu}_{\lambda\alpha} = \frac{1}{2}[(\lambda, \lambda + 2\delta) + (\alpha, \alpha + 2\delta) - (\mu, \mu + 2\delta)].$$

Proof. See the analysis above this theorem. $\square$

Due to the corollary 1, we have the following

Corollary 2: In the setting of the preceding theorem, let $\widetilde{\pi}_{\mu}$ be the trivial extension of π_{μ} to $J^1_R\mathbb{V} = \mathbb{V} \oplus (\mathfrak{g}_1 \otimes \mathbb{V})$ and suppose that the multiplicity of the irreducible representation of the highest weight μ in the preceding direct sum is 1. Then $\widetilde{\pi}_{\mu}$ is a $\mathfrak{p}$-homomorphism if and only if

$$c^{\mu}_{\lambda\alpha} = w.$$

Proof. Due to the corollary 1 it is sufficient to show that $\widetilde{\pi}_{\mu}$ factors through the restricted jets and vanishes on the image of Φ. The first is clear from the definition and the second is a consequence of of the theorem 5. The opposite implication is easy, too. Look at the formula for Φ and use the fact that π_{μ} is onto $\mathbb{V}_{\mu}$. $\square$

2.2 The special case: values in higher symplectic spinor modules

Using the corollary 2 we can prove the following statement, which is the main result of the paper. In the next theorem, an infinitesimal version of the prospective classification of first order invariant differential operators acting between sections of to higher symplectic spinor representations associated vector bundles over projective contact geometries is presented.

Theorem 6: Let us consider the complex simple Lie algebra $\mathfrak{sp}(2k, \mathbb{C})$ together with its contact grading, introduced in the section 1.2., let $\mathbb{V}, \mathbb{W}$ be harmonic (= higher symplectic spinor) modules over $\mathfrak{g}_0$ with the highest weight (λ, w) and (μ, w') respectively and let the action of $\mathfrak{g}_+$ be trivial in both cases. Then the vector space

$$\mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{V}, \mathbb{W}) = \begin{cases} \mathbb{C}\widetilde{\pi}_\mu & \mu \in \mathbb{A}_\lambda \quad w = c^\mu_{\lambda\varpi_1} \\ \{0\}, & \text{in other cases.} \end{cases}$$

Proof. Let us start with the second part of the statement, i.e., $\mu \notin \mathbb{A}_\lambda$ or $w \neq c^\mu_{\lambda\varpi_1}$. Let us take an element $T \in \mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{V}, \mathbb{W})$. Then $T \in \mathrm{Hom}_{\mathfrak{g}_0^{ss}}(J^1\mathbb{V}, \mathbb{W})$. Because T is a $\mathfrak{p}$-homomorphism, we know due to the corollary 1, that $T \in \mathrm{Hom}_{\mathfrak{g}_0^{ss}}(J^1_R\mathbb{V}, \mathbb{W})$. If $w \neq c^\mu_{\lambda\varpi_1}$, then we know that T is not a $\mathfrak{p}$-homomorphism, see corollary 1 and theorem 5. Now suppose that $c^\mu_{\lambda\varpi_1} = w$, i.e., $\mu \notin \mathbb{A}_\lambda$. Due to the theorem 2, we know that $J^1_R\mathbb{V} = L(\varpi_1) \otimes L(\lambda)$ decomposes into a finite direct sum of irreducible $\mathfrak{g}_0^{ss}$-modules, i.e., we can write

$$\mathrm{Hom}_{\mathfrak{g}_0^{ss}}(J^1_R\mathbb{V}, \mathbb{W}) = \bigoplus_{\nu \in \mathbb{A}_\lambda} \mathrm{Hom}_{\mathfrak{g}_0^{ss}}(L(\nu), L(\mu)).$$

Due to the theorem 2.6.5 and 2.6.6 in Dixmier [8], we know that each member of the direct sum is zero under our assumption, $\mu \notin \mathbb{A}_\lambda$. Thus $T = 0$.

Now, consider the case $\mu \in \mathbb{A}_\lambda$ and $w = c^\mu_{\lambda\alpha}$ and take a $T \in \mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{V}, \mathbb{W})$. Again as in the previous case, this implies $T \in \mathrm{Hom}_{\mathfrak{g}_0^{ss}}(J^1_R\mathbb{V}, \mathbb{W})$ and $w = c^\mu_{\lambda\alpha}$. Decomposing the tensor product $J^1_R\mathbb{V} = L(\varpi_1) \otimes L(\lambda)$, and substituting this decomposition into the $\mathrm{Hom}_{\mathfrak{g}_0^{ss}}(J^1_R\mathbb{V}, \mathbb{W})$ we obtain a direct sum

$$\bigoplus_{\nu \in \mathbb{A}_\lambda} \mathrm{Hom}_{\mathfrak{g}_0^{ss}}(L(\nu), L(\mu)).$$

According to our assumption, we know that the direct sum simplifies into a space isomorphic to $\mathbb{C}$, using the above cited theorems of Dixmier once more and having the structure of $\mathbb{A}_\lambda$ in mind. Thus we know that $\mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{V}, \mathbb{W}) \subseteq \mathrm{Hom}_{\mathfrak{g}_0^{ss}}(J^1_R\mathbb{V}, \mathbb{W}) \simeq \mathbb{C}$. To obtain an equality in the previous inclusion, consider the vector space of $\mathfrak{g}_0^{ss}$-homomorphisms $\{k\widetilde{\pi}_\mu; k \in \mathbb{C}\}$, which is clearly one dimensional, because we suppose $\mu \in \mathbb{A}_\lambda$. The elements of this vector space are clearly $\mathfrak{g}_0^{ss}$-homomorphisms, for which the mapping Φ vanishes, if $w = c^\mu_{\lambda\alpha}$, and they factorize through the restricted jets, because they are the trivial extensions from the restricted jets to the the whole module of jets, i.e., they are $\mathfrak{p}$-homomorphisms. $\square$

In the last theorem, algebraic homomorphisms are classified. In the future, we would like to use it in a non-infinitesimal version, i.e., for groups. This can be done by introducing a kind of globalization and by characterizing the Harish-Chandra class of the basic spinor (= metaplectic) modules and in this way also of the higher spinor modules consequently.

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