Elliptic complexes over C^*-algebras of compact operators

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Abstract

For a C^*-algebra A of compact operators and a compact manifold M, we prove that the Hodge theory holds for A-elliptic complexes of pseudodifferential operators acting on smooth sections of finitely generated projective A-Hilbert bundles over M. For these C^*-algebras, we get also a topological isomorphism between the cohomology groups of an A-elliptic complex and the space of harmonic elements. Consequently, the cohomology groups appear to be finitely generated projective C^*-Hilbert modules and especially, Banach spaces. We prove as well, that if the Hodge theory holds for a complex in the category of Hilbert A-modules and continuous adjointable Hilbert A-module homomorphisms, the complex is self-adjoint parametrix possessing.

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1 Introduction

This paper is a continuation of papers [13] and [15], devoted to the Hodge theory for Hilbert and pre-Hilbert C^*-modules and to an application of this theory to A-elliptic complexes of operators acting on sections of specific C^*-Hilbert bundles over compact manifolds.

Let A be a C^*-algebra and M be a compact manifold. In [15], the Hodge theory is proved to hold for an arbitrary A-elliptic complex of operators acting on smooth sections of finitely generated projective A-Hilbert bundles over M if the images of the extensions to the Sobolev spaces of the Laplacians of the complex are closed. One of the main results achieved in this paper is that one

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can omit the assumption on the images if A is a C^*-algebra of compact operators and still get the claim of the theorem.

We define what it means that *the Hodge theory holds* for a complex in an additive and dagger category and study this concept in a detail in categories of pre-Hilbert and Hilbert modules and continuous adjointable A-equivariant maps. These categories constitute a special class of the so-called R-module categories that are in addition, equipped with an involution on the morphisms spaces. We say that the *Hodge theory holds* for a complex $d^\bullet = (U^i, d_i : U^i \to U^{i+1})_{i\in\mathbb{Z}}$ in an additive and dagger category $\mathfrak{C}$ or that $d^\bullet$ is of *Hodge type* if for each $i \in \mathbb{Z}$, we have

$$U^i = \mathrm{Im}\ d_{i-1} \oplus \mathrm{Im}\, d_i^* \oplus \mathrm{Ker}\,\Delta_i,$$

where $\Delta_i = d_i^* d_i + d_{i-1}d_{i-1}^*$, and d_i^* and d_{i-1}^* are the adjoints of d_i and d_{i-1}, respectively. The operators Δ_i, $i \in \mathbb{Z}$, are called the *Laplace operators* of $d^\bullet$. The term "dagger category" is explained in the paper.

For a C^*-algebra A, we consider the category PH_A^* of right pre-Hilbert A-modules and continuous adjointable A-equivariant maps. The full subcategory of PH_A^*, the object of which are right Hilbert A-modules is denoted by H_A^* and it is called the category of Hilbert A-modules. See Kaplansky [11], Paschke [22], Lance [16] and Manuilov, Troitsky [18] for information on (pre-)Hilbert modules. Recall that each object in PH_A^* inherits a norm derived from the A-product defined on it. The categories PH_A^* and H_A^* are additive and dagger with respect to the orthogonal direct sum and an involution defined by the A-product.

In Krýsl [15], the so-called self-adjoint parametrix possessing complexes in PH_A^* are introduced. According to results in that paper, any self-adjoint parametrix possessing complex in PH_A^* is of Hodge type and its cohomology groups are pre-Hilbert A-modules isomorphic to the kernels of the Laplace operators as pre-Hilbert A-modules. Especially, the cohomology groups are normed spaces. In the present paper, we prove the opposite implication in the category H_A^*, i.e., that if the Hodge theory holds for a complex in H_A^*, the complex is self-adjoint parametrix possessing. Thus, in H_A^* the condition of being self-adjoint parametrix possessing characterizes the Hodge type complexes.

Let us recall that the Hodge theory is well known to hold for elliptic complexes of pseudodifferential operators acting on smooth sections of finite rank vector bundles over compact manifolds. Classical examples are deRham and Dolbeault complexes over compact manifolds. See, e.g., Palais [21] or Wells [28]. Fomenko, Mishchenko prove in [8], that the continuous extensions of an A-elliptic operator to the Sobolev section spaces are A-Fredholm. In [2], Bakić and Guljaš prove that any A-Fredholm endomorphism $F : U \to U$ in H_A^* has closed image if A is a C^*-algebra of compact operators. By a simple transforming, we generalize this result to the case of an A-Fredholm morphism $F : U \to V$ acting between Hilbert A-modules U and V. In particular, we prove that the image of F is closed. For C^*-algebras of compact operators, we further derive a transfer theorem which roughly speaking, enables us to deduce certain properties of pre-Hilbert A-module maps from the appropriate properties of their extensions. We specify the properties and the maps in the theorem formulation. Applying the mentioned theorem generalizing the result of Bakić and Guljaš, we get that the images of the extended Laplace operators of an A-elliptic complex are closed. The transfer theorem enables us to prove that in this case, the Laplace operators themselves have closed images, they are self-adjoint parametrix possessing, and consequently, that the complex is of Hodge type.

The motivation for our research comes from quantum field theories which aim to include constraints – especially, from the Becchi, Rouet, Stora and Tyutin or simply BRST quantization. See Henneaux, Teitelboim [9], Horuzhy, Voronin [10], Carchedi, Roytenberg [5] and the references there. Let us explain the connection in a more detail. In the BRST quantization, one constructs complexes whose cohomology groups represent state spaces of a given physical system. Because the state spaces in quantum theories are usually formed by infinite dimensional vector spaces, the co-cycle spaces for the cohomology groups have to be infinite dimensional as well. It is agreed that the state spaces shall be equipped with a topology because of the testing of the theory by measurements. Since the measurements do not give the precise value of a measured observable (a result of a measurement is always a value together with an error estimate), the state spaces shall have good a good behavior of limits of converging sequences. Especially, it is desirable that the limit of a converging sequence is unique. It is well known that the uniqueness of limits in a topological space forces the space to be T1. However, the T1 separation axiom in a topological vector space implies that the topological vector space is already Hausdorff. (For it, see, e.g., Theorem 1.12 in Rudin [23].) The quotient of a topological vector space is non-Hausdorff in the quotient topology if and only if the space by which one divides is not closed. If we insist that the state spaces are cohomology groups, we shall be able to assure that the spaces of co-boundaries are closed. For a explanation of the requirements on a physical theory considered above, we refer to Ludwig [17] and to a still appealing paper of von Neumann [20]. We hope that our work can be relevant for physics at least in the case when a particular BRST complex appears to be self-adjoint parametrix possessing in the categories PH^*_A or H^*_A for an arbitrary C^*-algebra A, or an A-elliptic complex in finitely generated projective A-Hilbert bundles over a compact manifold if A is a C^*-algebra of compact operators. A further inspiring topic from physics is the parallel transport in Hilbert bundles considered in a connection with quantum theory. See, e.g., Drechsler, Tuckey [7].

Let us notice that in Troitsky [26], indices of A-elliptic complexes are investigated. In that paper the operators are, quite naturally, allowed to be changed by an A-compact perturbation in order the index is an element of the appropriate K-group. See also Schick [24]. If the reader is interested in a possible application of the Hodge theory for A-elliptic complexes, we refer to Krýsl [12].

In the second chapter, we give a definition of the Hodge type complex, recall definitions of a pre-Hilbert and a Hilbert C^*-module, and give several examples of them. We prove that complexes in the category of Hilbert spaces and continuous maps are of Hodge type if the images of their Laplace operators are closed (Lemma 1). Further, we recall the definition of a self-adjoint parametrix complex in PH^*_A and some of its properties including the fact that they are of Hodge type (Theorem 2). We prove that if a complex in H^*_A is of Hodge type, it is already self-adjoint parametrix possessing (Theorem 3). At the end of the second section, we give examples of complexes the cohomology groups of which are not Hausdorff spaces. In the third chapter, we summarize the result of Bakić and Guljaš (Theorem 4), give the mentioned generalization of it (Corollary 5), and prove the transfer theorem (Theorem 6). In the fourth section, basic facts on differential operators acting on sections of A-Hilbert bundles over compact manifolds are recalled. In this chapter, the theorem on properties of A-elliptic complexes in finitely generated projective A-Hilbert bundles over compact manifolds is proved (Theorem 9).

Preamble: All manifolds and bundles are assumed to be smooth. The base manifolds of bundles are assumed to be finite dimensional. When an index of a labeled object exceeds its allowed range, the object is set to be zero. We do not suppose the Hilbert spaces to be separable.

2 Self-adjoint maps and complexes possessing a parametrix

Let us recall that a category $\mathfrak{C}$ is called a *dagger category* if there is a contravariant functor $* : \mathfrak{C} \to \mathfrak{C}$ which is the identity on the objects and satisfies the following property. For any objects U, V and W and any morphisms $F : U \to V$ and $G : V \to W$, we have $*F : V \to U$, and the relations $*\mathrm{Id}_U = \mathrm{Id}_U$ and $*(*F) = F$ hold. The functor $*$ is called the involution or the dagger. The morphism $*F$ is denoted by F^*, and it is called the adjoint of F. See Burgin [4] or Brinkmann, Puppe [3].

Let us give some examples of categories which are additive and dagger.

Example 1:

1) The category of finite dimensional inner product spaces over $\mathbb{R}$ or $\mathbb{C}$ and linear maps is an example of an additive and a dagger category. The addition (product) of objects is given by the orthogonal sum and the addition of morphism is the standard addition of linear maps. The involution is defined as the adjoint of maps with respect to the inner products. The existence of the adjoint to any linear map is based on the Gram-Schmid process which guarantees the existence of an orthonormal basis. The matrix of the adjoint of a morphism with respect to orthonormal bases in the domain and target spaces is given by taking the transpose or the transpose and the complex conjugate of the matrix of the original map. Such an adjoint operation on morphisms is easily proved to be unambiguous.

2) The category of Hilbert spaces and continuous maps equipped with the addition of objects and maps and the involution given as in item 1 is an

example of an additive and dagger category. For the existence of the adjoints, see Meise, Vogt [19]. The proof is based on the Riesz representation theorem for Hilbert spaces.

Definition 1: Let $\mathfrak{C}$ be an additive and dagger category. We say that the *Hodge theory holds* for a complex $d^\bullet = (U^i, d_i : U^i \to U^{i+1})_{i\in\mathbb{Z}}$ in $\mathfrak{C}$ or that $d^\bullet$ is of *Hodge type* if for each $i \in \mathbb{Z}$, we have

$$U^i = \mathrm{Im}\ d_{i-1} \oplus \mathrm{Im}\, d_i^* \oplus \mathrm{Ker}\, \Delta_i$$

where $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$, and d_i^* and d_{i-1}^* are the adjoints of d_i and d_{i-1}, respectively. We call the morphism Δ_i the i^{th} *Laplace operator* of $d^\bullet$, $i \in \mathbb{Z}$. We say that the Hodge theory holds for a subset $\mathfrak{K} \subseteq \mathcal{K}(\mathfrak{C})$ of complexes in $\mathfrak{C}$ if it holds for each element $d^\bullet \in \mathfrak{K}$.

Remark 1:

1) In Definition 1, we demand no compatibility of the involution with the additive structure. However, in the categories of pre-Hilbert and Hilbert A-modules that we will consider mostly, the relations $(F+G)^* = F^* + G^*$ and $(zF)^* = z^* F^*$ are satisfied for each objects U, V, morphisms $G, F : U \to V$, and complex number $z \in \mathbb{C}$.

2) The existence of the Laplace operators of $d^\bullet$ is guaranteed by the definitions of the additive and of the dagger category. If the dagger structure is compatible with the additive structure in the sense of item 1, we see that the Laplace operators are self-adjoint, i.e., $\Delta_i^* = \Delta_i$, $i \in \mathbb{Z}$.

Lemma 1: Let $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}}$ be a complex in the category of Hilbert spaces and continuous maps. If the images of the Laplace operators of $d^\bullet$ are closed, the Hodge theory holds for $d^\bullet$.

Proof. On the level of symbols, we do not distinguish the dependence of the inner products on the Hilbert spaces and denote them by $(,)$. It is easy to realize that $\mathrm{Ker}\, \Delta_i = \mathrm{Ker}\, d_{i-1}^* \cap \mathrm{Ker}\, d_i$. Namely, the inclusion $\mathrm{Ker}\, \Delta_i \supseteq \mathrm{Ker}\, d_i \cap \mathrm{Ker}\, d_{i-1}^*$ is immediate due to the definition of Δ_i, and the opposite one can be seen as follows. For any $u \in \mathrm{Ker}\, \Delta_i$, we have $0 = (\Delta_i u, u) = (d_i^* d_i u + d_{i-1} d_{i-1}^* u, u) = (d_i u, d_i u) + (d_{i-1}^* u, d_{i-1}^* u)$. Since inner products are positive definite, we have $d_i u = 0$ and $d_{i-1}^* u = 0$. Because we assume the image of Δ_i to be closed, taking the orthogonal complement of $\mathrm{Ker}\, \Delta_i = \mathrm{Ker}\, d_i \cap \mathrm{Ker}\, d_{i-1}^*$, we get $(\mathrm{Ker}\, d_{i-1}^*)^\perp \subseteq (\mathrm{Ker}\, \Delta_i)^\perp = \overline{\mathrm{Im}\, \Delta_i} = \mathrm{Im}\, \Delta_i$ and $(\mathrm{Ker}\, d_i)^\perp \subseteq (\mathrm{Ker}\, \Delta_i)^\perp = \overline{\mathrm{Im}\, \Delta_i} = \mathrm{Im}\, \Delta_i$. Summing-up,

$$(\mathrm{Ker}\, d_{i-1}^*)^\perp + (\mathrm{Ker}\, d_i)^\perp \subseteq \mathrm{Im}\, \Delta_i.$$

Further, it is immediate to see that $\mathrm{Im}\, d_{i-1} \subseteq (\mathrm{Ker}\, d_{i-1}^*)^\perp$ and $\mathrm{Im}\, d_i^* \subseteq (\mathrm{Ker}\, d_i)^\perp$. Indeed, for any $u \in \mathrm{Im}\, d_{i-1}$ there exists an element $u' \in U^{i-1}$ such that $u = d_{i-1} u'$. For each $v \in \mathrm{Ker}\, d_{i-1}^*$, we have $(u, v) = (d_{i-1} u', v) = (u', d_{i-1}^* v) = 0$. Thus, the inclusion follows. The other inclusion can be seen similarly. Using the result of the previous paragraph, we obtain

$$\mathrm{Im}\, d_{i-1} + \mathrm{Im}\, d_i^* \subseteq (\mathrm{Ker}\, d_{i-1}^*)^\perp + (\mathrm{Ker}\, d_i)^\perp \subseteq \mathrm{Im}\, \Delta_i. \tag{1}$$

We prove that the sum $\operatorname{Im} d_{i-1} + \operatorname{Im} d_i^*$ is direct. For it, we take $u = d_{i-1}u'$ and $v = d_i^* v'$ for $u' \in U^{i-1}$ and $v' \in U^{i+1}$, and compute $(u, v) = (d_{i-1}u', d_i^* v') = (d_i d_{i-1} u', v') = 0$ which holds since $d^\bullet$ is a complex. Therefore, we have $\operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1} \subseteq \operatorname{Im} \Delta_i$. The inclusion $\operatorname{Im} \Delta_i \subseteq \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$ is immediate. Thus, we conclude that $\operatorname{Im} \Delta_i = \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$.

Since for each $i \in \mathbb{Z}$, Δ_i is self-adjoint and its image is closed, we have $U^i = \operatorname{Im} \Delta_i \oplus \operatorname{Ker} \Delta_i$. Substituting the equation for $\operatorname{Im} \Delta_i$ found at end of the previous paragraph, we get $U^i = \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1} \oplus \operatorname{Ker} \Delta_i$ proving that the Hodge theory holds for $d^\bullet$. □

Remark 2:
By Lemma 1, the Hodge theory holds for any complex in the category $\mathfrak{C} = V_{\mathrm{fin}}$ of finite dimensional inner product spaces over real or complex numbers and linear maps since any linear subspace of a finite dimensional vector space is closed. However, it is possible to prove that the Hodge theory holds for $\mathfrak{K} = \mathcal{K}(\mathfrak{C})$ in a simpler way than in the general case of Hilbert spaces. The relation $\operatorname{Ker} \Delta_i = \operatorname{Ker} d_i \cap \operatorname{Ker} d_{i-1}^*$ is proved in the same way as in the proof of Lemma 1. Since for any $A, B \subseteq U^i$, the equation $(A \cap B)^\perp = A^\perp + B^\perp$ holds, we have $(\operatorname{Ker} d_i \cap \operatorname{Ker} d_{i-1}^*)^\perp = (\operatorname{Ker} d_i)^\perp + (\operatorname{Ker} d_{i-1}^*)^\perp$. Due to the finite dimension, we can write $(\operatorname{Ker} d_i)^\perp = \operatorname{Im} d_i^*$ and $(\operatorname{Ker} d_{i-1}^*)^\perp = \operatorname{Im} d_{i-1}$, and thus $(\operatorname{Ker} \Delta_i)^\perp = (\operatorname{Ker} d_{i-1} \cap \operatorname{Ker} d_i^*)^\perp = \operatorname{Im} d_{i-1} + \operatorname{Im} d_i^*$. The sum is direct as follows from $0 = (d_i d_{i-1} u, v) = (d_{i-1} u, d_i^* v)$, $u \in U^{i-1}$, $v \in U^{i+1}$. Substituting $(\operatorname{Ker} \Delta_i)^\perp = \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$ into $U_i = \operatorname{Ker} \Delta_i \oplus (\operatorname{Ker} \Delta_i)^\perp$, we get $U^i = \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1} \oplus \operatorname{Ker} \Delta_i$. (Let is notice that in Lemma 1, we proved that the images of d_i and d_{i-1}^* are closed.)

Next we define the pre-Hilbert and Hilbert modules over C^*-algebras. Our reference for C^*-algebras is Dixmier [6].

Definition 2: For a C^*-algebra A, a *pre-Hilbert A-module* is a complex vector space U, which is a right A-module (the action is denoted by a dot) and which is moreover equipped with a map $(,) : U \times U \to A$ such that for each $z \in \mathbb{C}$, $a \in A$ and $u, v, w \in U$ the following relations hold

1) $(u, zv + w) = z(u, v) + (u, w)$

1) $(u, v \cdot a) = (u, v)a$

2) $(u, v) = (v, u)^*$

3) $(u, u) \geq 0$ and

4) $(u, u) = 0$ implies $u = 0$

where z^* denotes the complex conjugate of the element $z \in \mathbb{C}$. A pre-Hilbert A-module $(U, (,))$ is called a *Hilbert A-module* if U is a Banach space with respect to the norm $U \ni u \mapsto |u| = \sqrt{|(u, u)|_A} \in [0, +\infty)$. The map $(,) : U \times U \to A$ is called the *A-product*.

Note that if A is the algebra of complex numbers, Definition 2 coincides with the one of a pre-Hilbert and of a Hilbert space, respectively.

Morphisms of pre-Hilbert A-modules $(U,(,))$ and $(V,(,)_V)$ are assumed to be continuous, A-linear and adjointable maps. Recall that a map $L : U \to V$ is called A-linear if the equivariance condition $L(u) \cdot a = L(u \cdot a)$ holds for any $a \in A$ and $u \in U$. An adjoint $L^* : V \to U$ of a pre-Hilbert A-module morphism $L : U \to V$ is a map which satisfies $(Lu, v)_V = (u, L^*v)_U$ for any $u \in U$ and $v \in V$. It is known that the adjoint need not exist in general, and that if it exists, it is unique and a pre-Hilbert A-module homomorphism, i.e., continuous and A-linear. Morphisms of Hilbert A-modules have to be morphisms of these modules considered as pre-Hilbert A-modules. The category the objects of which are pre-Hilbert A-modules and the morphisms of which are continuous, A-linear and adjointable maps will be denoted by PH_A^*. The category H_A^* of Hilbert A-modules is defined to be the full subcategory of PH_A^* the object of which are Hilbert A-modules. If we drop the condition on the adjointability of morphisms, we denote the resulting categories by PH_A and H_A. By an *isomorphism* $F : U \to V$ in PH_A^* or H_A^*, we mean a morphism which is right and left invertible by a morphism in PH_A^* or H_A^*, respectively. In particular, we demand an isomorphism in these categories neither to preserve the appropriate A-products nor the induced norms.

Submodules of a (pre-)Hilbert A-module have to be (pre-)Hilbert A-modules with respect to the restrictions both of the algebraic and of the norm structure. In particular, they are closed in the super-module. Further, if U is a submodule of the (pre-)Hilbert A-module V, we can construct the space $U^\perp = \{v \in V, (v, u) = 0 \text{ for all } u \in U\}$ which is a (pre-)Hilbert A-module. Further, U is called orthogonally complemented in V if $V = U \oplus U^\perp$. There are Hilbert A-submodules which are not orthogonally complemented. (See Lance [16].) For the convenience of the reader, we give several examples of Hilbert A-modules and an example of a pre-Hilbert A-module. For further examples, see Solovyov, Troitsky [25], Manuilov, Troitsky [18], Lance [16], and Wegge-Olsen [27].

Example 2:

1) Let H be a Hilbert space with the inner product denoted by $(,)_H$. The action of the C^*-algebra $A = B(H)$ of bounded linear operators on H is by evaluation on the adjoint, i.e., $h \cdot a = a^*(h)$ for any $a \in B(H)$ and $h \in H$. The $B(H)$-product is defined by $(u, v) = u \otimes v^*$, where $(u \otimes v^*)w = (v, w)_H u$ for $u, v, w \in H$. In this case, the product takes values in the C^*-algebra $K(H)$ of compact operators on H. In fact, the A-product maps into the algebra of finite rank operators.

2) For a locally compact topological space X, consider the C^*-algebra $A = \mathcal{C}_0(X)$ of continuous functions vanishing at infinity with the product given by the point-wise multiplication, with the complex conjugation as the involution, and with the classical supremum norm $|\,|_A : \mathcal{C}_0(X) \to [0, +\infty)$

$$|f|_A = \sup\{|f(x)|, \, x \in X\}$$

where $f \in A$. For U, we take the C^*-algebra $\mathcal{C}_0(X)$ itself with the module structure given by the point-wise multiplication, i.e., $(f \cdot g)(x) = f(x)g(x)$, $f \in U$, $g \in A$ and $x \in X$. The A-product is defined by $(f, g) = \overline{f}g$. Note that this is a particular example of a Hilbert A-module with $U = A$, right action $a \cdot b = ab$ for $a \in U = A$ and $b \in A$, and A-product $(a, b) = a^*b$, $a, b \in U$.

3) If U is a Hilbert A-module, the orthogonal direct sums of a finite number of copies of U form a Hilbert A-module in a natural way. One can also construct the space $\ell^2(U)$, i.e., the space consisting of sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in U$, $n \in \mathbb{N}$, for which the series $\sum_{i=1}^{\infty}(a_i, a_i)$ converges in A. The A-product is given by $((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) = \sum_{i=1}^{\infty}(a_i, b_i)$, where $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \ell^2(U)$. See Manuilov, Troitsky [18].

4) Let A be a C^*-algebra. For a compact manifold M^n, pick a Riemannian metric g and choose a volume element $|\mathrm{vol}_g| \in \Gamma(M, |\bigwedge^n T^*M|)$. Then for any A-Hilbert bundle $\mathcal{E} \to M$ with fiber a Hilbert A-module E, one defines a *pre-Hilbert A-module* $\Gamma(M, \mathcal{E})$ of smooth sections of $\mathcal{E} \to M$ by setting $(s \cdot a)_m = s_m \cdot a$ for $a \in A$, $s \in \Gamma(M, \mathcal{E})$, and $m \in M$. One sets

$$(s', s) = \int_{m\in M} (s'_m, s_m)_m |\mathrm{vol}_g|_m$$

where $s, s' \in \Gamma(M, \mathcal{E})$, $(,)_m$ denotes the A-product in fiber $\mathcal{E}_m$, and $m \in M$. Taking the completion of $\Gamma(M, \mathcal{E})$ with respect to the norm associated to the A-product $(,)$ (Definition 2), we get the Hilbert A-module $(W^0(M, \mathcal{E}), (,)_0)$. Further Hilbert A-modules $(W^t(M, \mathcal{E}), (,)_t)$, $t \in \mathbb{N}_0$ are derived from the space $\Gamma(M, \mathcal{E})$ by mimicking the construction of Sobolev spaces defined for finite rank bundles. See Wells [28] for the finite rank case and Solovyov, Troitsky [25] for the case of A-Hilbert bundles.

Let us turn our attention to the so-called self-adjoint parametrix possessing morphisms in the category $\mathfrak{C} = PH^*_A$.

Definition 3: A pre-Hilbert A-module endomorphism $F : U \to U$ is called *self-adjoint parametrix possessing* if F is self-adjoint, i.e., $F^* = F$, and there exist a pre-Hilbert A-module homomorphism $G : U \to U$ and a self-adjoint pre-Hilbert A-module homomorphism $P : U \to U$ such that

$$\begin{aligned} 1_U &= GF + P \\ 1_U &= FG + P \\ FP &= 0. \end{aligned}$$

Remark 3:

1) The map G from Definition 3 is called a *parametrix* or a *Green operator* and the first two equations in this definition are called the *parametrix equations*.

2) Composing the first parametrix equation from the right with P and using the third equation, we get that $P^2 = P$.

3) If $F : U \to U$ is a self-adjoint parametrix possessing morphism in PH_A^*, then $U = \operatorname{Ker} F \oplus \operatorname{Im} F$ (see Theorem 6 in Krýsl [15]). In particular, the image of F is closed. Note that we do not assume that U is complete.

4) A morphism in H_A^* is self-adjoint parametrix possessing if its image is closed. Indeed, the Mishchenko theorem (Theorem 3.2 on pp. 22 in Lance [16]) enables us to write for a self-adjoint morphism $F : U \to U$ with closed image, the orthogonal decomposition $U = \operatorname{Ker} F \oplus \operatorname{Im} F$. Then we can define the projection onto $\operatorname{Ker} F$ along $\operatorname{Im} F$. It is easy to see that the projection is self-adjoint. Inverting F on its image and defining it by zero on the kernel of F, we get a map G which satisfies the parametrix equations and it is continuous due to the open map theorem. Thus, in H_A^* a self-adjoint map F is self-adjoint parametrix possessing if and only if its image is closed.

Let us notice that if $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}}$ is a co-chain complex in the category PH_A^*, the i^{th} Laplace operator $\Delta_i = d_{i-1}d_{i-1}^* + d_i^* d_i$ is self-adjoint, $i \in \mathbb{Z}$.

Definition 4: A co-chain complex $d^\bullet \in \mathcal{K}(PH_A^*)$ is called *self-adjoint parametrix possessing* if all of its Laplace operators are self-adjoint parametrix possessing maps.

Remark 4:

1) Since $\Delta_{i+1}d_i = (d_{i+1}^* d_{i+1} + d_i d_i^*)d_i = d_i d_i^* d_i = d_i d_i^* d_i + d_i d_{i-1} d_{i-1}^* = d_i(d_i^* d_i + d_{i-1}d_{i-1}^*) = d_i\Delta_i$, the Laplace operators are co-chain endomorphisms of $d^\bullet$. Similarly, one derives that the Laplace operators are chain endomorphisms of the chain complex $(U^i, d_i^* : U^{i+1} \to U^i)_{i\in\mathbb{Z}}$ "dual" to $d^\bullet$.

2) Let us assume that the Laplace operators Δ_i of a complex $d^\bullet$ in PH_A^* satisfy equations $\Delta_i G_i + P_i = G_i \Delta_i + P_i = 1_{U_i}$ and that the identity $\Delta_i P_i = 0$ holds. Notice that we do not suppose that the idempotent P_i is self-adjoint. Still, we can prove that the Green operators G_i satisfy $G_{i+1}d_i = d_i G_i$, i.e., that they are co-chain endomorphisms of the complex $d^\bullet$ we consider. For it, see Theorem 3, Krýsl [13].

3) In the following picture, facts from the previous two items are summarized in a diagrammatic way.

$$
\begin{array}{ccccc}
U_{i-1} & \underset{d_{i-1}^*}{\overset{d_{i-1}}{\rightleftarrows}} & U_i & \underset{d_i^*}{\overset{d_i}{\rightleftarrows}} & U_{i+1} \\
\Delta_{i-1} \Big\uparrow\Big\downarrow G_{i-1} & & \Delta_i \Big\uparrow\Big\downarrow G_i & & \Delta_i \Big\uparrow\Big\downarrow G_{i+1} \\
U_{i-1} & \underset{d_{i-1}^*}{\overset{d_{i-1}}{\rightleftarrows}} & U_i & \underset{d_i^*}{\overset{d_i}{\rightleftarrows}} & U_{i+1}
\end{array}
$$

Let us consider the cohomology groups $H^i(d^\bullet) = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$ of a complex $d^\bullet \in \mathcal{K}(PH_A^*)$, $i \in \mathbb{Z}$. If $\operatorname{Im} d_{i-1}$ is orthogonally complementable in $\operatorname{Ker} d_i$, then one can define an A-product in $H^i(d^\bullet)$ by $([u],[v])_{H^i(d^\bullet)} = (p_i u, p_i v)$, where $u, v \in U^i$ and p_i is the projection along $\operatorname{Im} d_{i-1}$ onto the orthogonal complement $(\operatorname{Im} d_{i-1})^\perp$ in $\operatorname{Ker} d_i$. Let us call this A-product the *canonical quotient product*. For information on A-products on quotients in PH_A^*, see [15].

In the next theorem, we collect results on self-adjoint parametrix complexes from [15].

Theorem 2: Let A be a C^*-algebra. If $d^\bullet = (U^i, d_i)_{i \in \mathbb{Z}} \in \mathcal{K}(PH_A^*)$ is self-adjoint parametrix possessing complex, then for any $i \in \mathbb{Z}$,

1) $U^i = \operatorname{Ker} \Delta_i \oplus \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1}$, i.e., $d^\bullet$ is a Hodge type complex

2) $\operatorname{Ker} d_i = \operatorname{Ker} \Delta_i \oplus \operatorname{Im} d_{i-1}$

3) $\operatorname{Ker} d_i^* = \operatorname{Ker} \Delta_{i+1} \oplus \operatorname{Im} d_{i+1}^*$

4) $\operatorname{Im} \Delta_i = \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1}$

5) $H^i(d^\bullet)$ is a pre-Hilbert A-module with respect to the canonical quotient product $(,)_{H^i(d^\bullet)}$

6) The spaces $\operatorname{Ker} \Delta_i$ and $H^i(d^\bullet)$ are isomorphic as pre-Hilbert A-modules. Moreover, if $d^\bullet$ is self-adjoint parametrix possessing complex in $\mathcal{K}(H_A^*)$, then $H^i(d^\bullet)$ is an A-Hilbert module and $\operatorname{Ker} \Delta_i \simeq H^i(d^\bullet)$ are isomorphic as A-Hilbert modules.

Proof. See Theorem 11 in Krýsl [15] for item 1; Theorem 13 in [15] for items 2 and 3; Remark 12 (1) in [15] for item 4; and Corollary 14 in [15] for items 5 and 6. □

Next we prove that in the category $\mathfrak{C} = H_A^*$, the property of a complex to be self-adjoint parametrix possessing **characterizes** the complexes of Hodge type.

Theorem 3: If the Hodge theory holds for a complex $d^\bullet \in \mathcal{K}(H_A^*)$, then $d^\bullet$ is self-adjoint parametrix possessing.

Proof. Because the Hodge theory holds for $d^\bullet$, we have the decomposition of U^i into Hilbert A-modules

$$U^i = \operatorname{Ker}\Delta_i \oplus \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$$

$i \in \mathbb{Z}$. In particular, the ranges of d_{i-1} and d_i^* are closed topological vector spaces. It is easy to verify that

$$\operatorname{Ker} d_i^* d_i = \operatorname{Ker} d_i \qquad\qquad \operatorname{Ker} d_{i-1}^* = \operatorname{Ker} d_{i-1} d_{i-1}^*.$$

For $i \in \mathbb{Z}$ and $u \in U^i$, we have

$$(\Delta_i u, \Delta_i u) = (d_i^* d_i u, d_i^* d_i u) + (d_{i-1} d_{i-1}^* u, d_{i-1} d_{i-1}^* u)$$

since $(d_i^* d_i u, d_{i-1} d_{i-1}^* u) = (d_i u, d_i d_{i-1} d_{i-1}^* u) = 0$ for any $u \in U^i$. Due to the definition of the Laplace operator and the positive definiteness of the A-Hilbert product, we have $\operatorname{Ker}\Delta_i = \operatorname{Ker} d_i \cap \operatorname{Ker} d_{i-1}^*$. For $u \in (\operatorname{Ker}\Delta_i)^\perp = (\operatorname{Ker} d_i)^\perp + (\operatorname{Ker} d_{i-1}^*)^\perp$, there exist $u_1 \in (\operatorname{Ker} d_i)^\perp = \operatorname{Im} d_i^*$ and $u_2 \in (\operatorname{Ker} d_{i-1}^*)^\perp = \operatorname{Im} d_{i-1}$ such that $u = u_1 + u_2$. Consequently, $(\Delta_i u, \Delta_i u) =$

$$\begin{aligned}
&= (d_i^* d_i(u_1+u_2), d_i^* d_i(u_1+u_2)) + (d_{i-1} d_{i-1}^*(u_1+u_2), d_{i-1} d_{i-1}^*(u_1+u_2)) \\
&= (d_i^* d_i u_1, d_i^* d_i u_1) + (d_i^* d_i u_2, d_i^* d_i u_2) + (d_i^* d_i u_1, d_i^* d_i u_2) + (d_i^* d_i u_2, d_i^* d_i u_1) \\
&\quad + (d_{i-1} d_{i-1}^* u_1, d_{i-1} d_{i-1}^* u_1) + (d_{i-1} d_{i-1}^* u_2, d_{i-1} d_{i-1}^* u_2) \\
&\quad + (d_{i-1} d_{i-1}^* u_1, d_{i-1} d_{i-1}^* u_2) + (d_{i-1} d_{i-1}^* u_2, d_{i-1} d_{i-1}^* u_1) \\
&= (d_i^* d_i u_1, d_i^* d_i u_1) + (d_{i-1} d_{i-1}^* u_2, d_{i-1} d_{i-1}^* u_2)
\end{aligned}$$

since $(d_i^* d_i u_2, d_i^* d_i u_2) = (d_i^* d_i u_1, d_i^* d_i u_2) = 0$ due to $u_2 \in \operatorname{Im} d_{i-1}$, and $(d_{i-1} d_{i-1}^* u_1, d_{i-1} d_{i-1}^* u_1) = (d_{i-1} d_{i-1}^* u_1, d_{i-1} d_{i-1}^* u_2) = 0$ due to $u_1 \in \operatorname{Im} d_i^*$. Since both summands on the right-hand side of

$$(\Delta_i u, \Delta_i u) = (d_i^* d_i u_1, d_i^* d_i u_1) + (d_{i-1} d_{i-1}^* u_2, d_{i-1} d_{i-1}^* u_2)$$

are non-negative, we obtain $(\Delta_i u, \Delta_i u) \geq (d_i^* d_i u_1, d_i^* d_i u_1)$ and $(\Delta_i u, \Delta_i u) \geq (d_{i-1} d_{i-1}^* u_2, d_{i-1} d_{i-1}^* u_2)$. Consequently

$$|\Delta_i u| \geq |d_i^* d_i u_1| \tag{2}$$

$$|\Delta_i u| \geq |d_{i-1} d_{i-1}^* u_2| \tag{3}$$

(See paragraph 1.6.9 on pp. 18 in Dixmier [6].) Notice that $d_i^* d_i$ and $d_{i-1}^* d_{i-1}$ is injective on $(\operatorname{Ker} d_i^* d_i)^\perp = (\operatorname{Ker} d_i)^\perp$ and $(\operatorname{Ker} d_{i-1} d_{i-1}^*)^\perp =$ $= (\operatorname{Ker} d_{i-1}^*)^\perp$, respectively, and zero on the complements of these spaces. Due to an equivalent characterization of closed image maps on Banach spaces, there are positive real numbers α, β such that $|d_i^* d_i u_1| \geq \alpha |u_1|$ and $|d_{i-1} d_{i-1}^* u_2| \geq \beta |u_2|$ (see, e.g., Abramovich, Aliprantis [1]). Substituting these inequalities into (2) and (3) and adding the resulting inequalities, we see that $2|\Delta_i u| \geq \alpha |u_1| + \beta |u_2|$. Thus $|\Delta_i u| \geq \frac{1}{2}\min\{\alpha,\beta\}(|u_1| + |u_2|) \geq \frac{1}{2}\min\{\alpha,\beta\}|u_1 + u_2| = \frac{1}{2}\min\{\alpha,\beta\}|u|$ by the triangle identity. Due to the characterization of closed image maps again, we get that the image of Δ_i is closed. This implies that $d^\bullet$ is self-adjoint parametrix possessing using Remark 3 item 4.

□

Remark 5:

1) Let $\mathfrak{C} = H^*_{\mathbb{C}}$ be the category of Hilbert spaces and continuous maps. Let us consider such complexes in $\mathfrak{C}$ whose differentials are Fredholm maps. Especially their images and the ones of their adjoint maps are closed. It is easy to prove that the Laplacians are Fredholm as well. Namely, due to the fact that the images of d_i and d_i^*, $i \in \mathbb{Z}$, are closed we can use the inequalities in the proof of Theorem 3 and conclude that the Laplacian has closed image by the characterization of closed image maps as used above. By Lemma 1, the complex is necessarily of Hodge type and we have the decomposition $\operatorname{Im}\Delta_i = \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$ (e.g., by Theorem 2). In particular, the cokernel of the Laplacian is a finite dimensional space. The kernel of Δ_i is finite dimensional due to the finite dimensionality of $\operatorname{Ker} d_i$ and due to $\operatorname{Ker}\Delta_i = \operatorname{Ker} d_i \cap \operatorname{Ker} d^*_{i-1}$ which follows from the definition of the Laplacian operator.

2) From Theorem 2, Theorem 3 and Remark 3 item 4, we get that a complex in H^*_A is of Hodge type if and only if the images of its Laplace operators are closed if and only if it is self-adjoint parametrix possessing.

Example 3:

1) For a compact manifold M of positive dimension, let us consider the Sobolev spaces $W^{k,l}(M)$ for k, l non-negative integers. For $l = 2$, these space are complex Hilbert spaces. Due to the Rellich-Kondrachov embedding theorem and the fact that the dimension of $W^{k,2}(M)$ is infinite, the canonical embedding $i : W^{k,2}(M) \hookrightarrow W^{l,2}(M)$ has a non-closed image for $k > l$. We take $d^\bullet =$

$$0 \longrightarrow W^{k,2}(M) \xrightarrow{\ i\ } W^{l,2}(M) \longrightarrow 0\ .$$

Labeling the first element in the complex by zero, the second cohomology $H^2(d^\bullet) = \operatorname{Ker} 0/\operatorname{Im} i = W^{l,2}(M)/i(W^{k,2}(M))$ is non-Hausdorff in the quotient topology. The complex is not self-adjoint parametrix possessing due to Theorem 2 item 5. Consequently, it is not of Hodge type (Theorem 3).

2) This example shows a simpler construction of a complex in $\mathcal{K}(H^*_{\mathbb{C}})$ which is not of Hodge type. Without any reference to a manifold, we can define mapping $i : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by setting $i(e_n) = e_n/n$, where $(e_n)_{n=1}^{+\infty}$ denotes the canonical orthonormal system of $\ell^2(\mathbb{Z})$. It is easy to check that i is continuous. Further, the set $i(\ell^2(\mathbb{Z}))$ is not closed. For it, the sequence $(1, 1/2, 1/3, \dots) \in \ell^2(\mathbb{Z})$ is not in the image. Indeed, the preimage of this

element had to be the sequence $(1, 1, 1, \ldots)$ which is not in $\ell^2(\mathbb{Z})$. On the other hand, $(1, 1/2, 1/3, \ldots)$ lies in the closure of $i(\ell^2(\mathbb{Z}))$ since it is the limit of the sequence $i((1, 0 \ldots)), i((1, 1, 0, \ldots)), i((1, 1, 1, 0 \ldots)), \ldots$. The complex $0 \to \ell^2(\mathbb{Z}) \xrightarrow{i} \ell^2(\mathbb{Z}) \to 0$ is not of Hodge type and it is not self-adjoint parametrix possessing by similar reasons as given in the example above.

3 C^*-Fredholm operators over C^*-algebras of compact operators

In this section, we focus on complexes over C^*-algebras of compact operators, and study C^*-Fredholm maps acting between Hilbert modules over such algebras. For the convenience of the reader, let us recall some necessary notions.

Definition 5: Let $(U, (,)_U)$ and $(V, (,)_V)$ be Hilbert A-modules.

1) For any $u \in U$ and $v \in V$, the operator $F_{u,v} : U \to V$ defined by $U \ni u' \mapsto F_{u,v}(u') = v \cdot (u, u')$ is called an *elementary operator*. A morphism $F : U \to V$ in H_A^* is called of *A-finite rank* if it can be written as a finite sum of A-linear combinations of the elementary operators.

2) The set $K_A(U, V)$ of *A-compact* operators on U is defined to be the closure of the vector space of the A-finite rank morphisms in the operator norm in $\mathrm{Hom}_{H_A^*}(U, V)$, induced by the norms $| \, |_U$ and $| \, |_V$.

3) We call $F \in \mathrm{Hom}_{H_A^*}(U, V)$ *A-Fredholm* if there exist Hilbert A-module homomorphisms $G_V : V \to U$ and $G_U : U \to V$ and A-compact homomorphisms $P_U : U \to U$ and $P_V : V \to V$ such that

$$G_U F = 1_U + P_U$$

$$F G_V = 1_V + P_V$$

i.e., if F is left and right invertible modulo A-compact operators.

Remark 5:

1) *Equivalent definition of A-compact operators.* The A-finite rank operators are easily seen to be adjointable. Suppose for a moment that we define the "A-compact" operators as such morphisms in the category H_A of Hilbert A-modules and continuous A-module homomorphisms that lie in the operator norm closure in $\mathrm{Hom}_{H_A}(U, V) \supseteq \mathrm{Hom}_{H_A^*}(U, V)$ of the A-finite rank operators. One can prove that these operators are adjointable and that their set coincides with class of the A-compact operators defined above (Definition 5). For it see, e.g., Corollary 15.2.10 in Wegge-Olsen [27].

2) *A-compact vs. compact.* It is well known that in general, the notion of an A-compact operator does not coincide with the notion of a compact operator in a Banach space. Indeed, let us consider an infinite dimensional *unital* C^*-algebra A ($1 \in A$), and take $U = A$ with the right action given by the multiplication in A and the A-product $(a, b) = a^* b$, $a, b \in A$. Then the identity $1_U : U \to U$ is A-compact since it is equal to $F_{1,1}$. But it is not a compact operator in the classical sense since U is infinite dimensional.

Example 4:

1) *A-Fredholm operator with non-closed image.* Let us consider the space $X = [0, 1] \subseteq \mathbb{R}$, the C^*-algebra $A = \mathcal{C}([0, 1])$ and the tautological Hilbert A-module $U = A = \mathcal{C}([0, 1])$ (second paragraph of Example 2). We give a simple proof of the fact that there exists an endomorphism on U which is A-Fredholm but the image of which is not closed. Let us take an arbitrary map $T \in \mathrm{End}_{H_A^*}(U)$. Writing $f = 1 \cdot f$, we have $T(1 \cdot f) = T(1) \cdot f = T(1)f$. Thus, T can be written as the elementary operator $F_{f_0,1}$ where $f_0 = T(1)$. Since T is arbitrary, $K_A(U, U) = \mathrm{End}_{H_A^*}(U)$. Consequently, any endomorphism $T \in \mathrm{End}_{H_A^*}(U)$ is A-Fredholm since $T 1_U = 1_U T = 1_U + (T - 1_U)$ Let us consider operator $Ff = xf$, $f \in U$. This operator satisfies $F = F^*$, and it is clearly a morphism of the Hilbert A-module U. It is immediate to realize that $\mathrm{Ker}\, F = 0$. Suppose that the image of $F = F^*$ is closed. Using Theorem 3.2 in Lance [16], we obtain $\mathcal{C}([0, 1]) = \mathrm{Im}\, F^* \oplus \mathrm{Ker}\, F = \mathrm{Im}\, F$. Since the constant function $1 \notin \mathrm{Im}\, F$, we get a contradiction. Therefore $\mathrm{Im}\, F$ is not closed although F is an A-Fredholm operator as shown above. Let us recall that the image of a Fredholm operator on a Banach space, in the classical sense, is closed.

2) *Hilbert space over its compact operators.* Let us note that if $U = H$ and $A = K(H)$ with the action and the A-product as in Example 2 item 1, we have $F_{u,v} = (v, u)$ for any $u, v \in H$. Especially, $F_{u,v}$ are rank one operators. Thus, their finite A-linear combinations are finite rank operators on H and their closure is $K(H)$ itself, i.e., $K_{K(H)}(H) = K(H)$.

Remark 6: Let us remark that the definition of an A-Fredholm operator on pp. 841 in Mishchenko, Fomenko [8] is different from the definition of an A-Fredholm operator given in item 3 of Definition 5 of our paper. However, an A-Fredholm operator in the sense of Fomenko and Mishchenko is necessarily invertible modulo an A-compact operator (see Theorem 2.4 in Fomenko, Mishchenko [8]), i.e., it is an A-Fredholm operator in our sense.

Definition 6: A C^*-algebra is called a *C^*-algebra of compact operators* if it is a C^*-subalgebra of the C^*-algebra of compact operators $K(H)$ on a Hilbert space H.

If A is a C^*-algebra of compact operators, an analogue of an orthonormal system in a Hilbert space is introduced for the case of Hilbert A-modules in the paper of Bakić, Guljaš [2]. For a fixed Hilbert A-module, the cardinality of any of its orthonormal systems does not depend on the choice of such a system. We denote the cardinality of an orthonormal system of a Hilbert A-module U over a C^*-algebra A of compact operators by $\dim_A U$. Let us note that in particular, an orthonormal system forms a set of generators of the module as follows from the definition in [2].

Theorem 4: Let A be a C^*-algebra of compact operators, U be a Hilbert A-module, and $F \in \mathrm{End}_{H_A^*}(U)$. Then F is A-Fredholm, if and only if its image is closed and $\dim_A \mathrm{Ker}\, F$ and $\dim_A (\mathrm{Im}\, F)^\perp$ are finite.

Proof. Bakić, Guljaš [2], pp. 268. □

Corollary 5: Let A be a C^*-algebra of compact operators, U and V be Hilbert A-modules, and $F \in \mathrm{Hom}_{H_A^*}(U, V)$. Then F is an A-Fredholm operator, if and only if its image is closed and $\dim_A \mathrm{Ker}\, F$ and $\dim_A (\mathrm{Im}\, F)^\perp$ are finite.

Proof. Let $F : U \to V$ be an A-Fredholm operator and G_U, P_U and G_V, P_V be the corresponding left and right inverses and projections, respectively, i.e., $G_U F = 1_U + P_U$ and $F G_V = 1_V + P_V$.

Let us consider the element $\mathfrak{F} = \left(\begin{smallmatrix} 0 & F^* \\ F & 0 \end{smallmatrix}\right) \in \mathrm{End}_{H_A^*}(U \oplus V)$. For this element, we can write

$$\begin{pmatrix} 0 & G_U \\ G_V^* & 0 \end{pmatrix} \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix} = \begin{pmatrix} 1_U + P_U & 0 \\ 0 & 1_V + P_V \end{pmatrix} = \begin{pmatrix} 1_U & 0 \\ 0 & 1_V \end{pmatrix} + \begin{pmatrix} P_U & 0 \\ 0 & P_V \end{pmatrix}$$

Since the last written matrix is an A-compact operator in $\mathrm{End}_{H_A^*}(U \oplus V)$, $\mathfrak{F}$ is left invertible modulo an A-compact operator on $U \oplus V$. The right invertibility is proved in a similar way. Summing-up, $\mathfrak{F}$ is A-Fredholm. According to Theorem 4, $\mathfrak{F}$ has closed image. This implies that F has closed image as well due to the orthogonality of the modules U and V in $U \oplus V$. Let us denote the orthogonal projections of $U \oplus V$ onto U and V by proj_U and proj_V, respectively. Due to Theorem 4, $\dim_A(\mathrm{Ker}\, \mathfrak{F})$ and $\dim_A(\mathrm{Im}\, \mathfrak{F})^\perp$ are finite. Since $\mathrm{Ker}\, F = \mathrm{proj}_U(\mathrm{Ker}\, \mathfrak{F})$ and $(\mathrm{Im}\, F)^\perp = \mathrm{Ker}\, F^* = \mathrm{proj}_V(\mathrm{Ker}\, \mathfrak{F}^*) = \mathrm{proj}_V(\mathrm{Im}\, \mathfrak{F})^\perp$, the finiteness of $\dim_A \mathrm{Ker}\, F$ and $\dim_A(\mathrm{Im}\, F)^\perp$ follows from the finiteness of $\dim_A(\mathrm{Ker}\, \mathfrak{F})$ and $\dim_A(\mathrm{Im}\, \mathfrak{F})^\perp$.

On the other hand, if $\dim_A \mathrm{Im}\, F$ and $\dim_A(\mathrm{Im}\, F)^\perp$ are finite and the image of F is closed, we deduce the same for $\mathfrak{F}$ using the orthogonality of the direct sum $U \oplus V$ and the fact that the image of F^* is closed as well (see Theorem 3.2 in Lance [16]). Since $\mathfrak{F}$ satisfies the assumptions of Theorem 4, we have that $\mathfrak{F}$ is A-Fredholm. Consequently, there exists a map $\mathfrak{G} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{End}_{H_A^*}(U \oplus V)$ such that $\mathfrak{F}\mathfrak{G} = 1_{U \oplus V} + P_{U \oplus V}$ for an A-compact operator $P_{U \oplus V}$ in $U \oplus V$. Expanding this equation, we get $FD = 1_V + \mathrm{proj}_V P_{U \oplus V}{}_{|V}$. It is immediate to realize that $\mathrm{proj}_V P_{U \oplus V}{}_{|V}$ is an A-compact operator in V. Thus, F is right invertible modulo an A-compact operator in V. Similarly, one proceeds in the case of the left inverse. Summing-up, F is an A-Fredholm morphism. □

In the next theorem, we study how certain properties of the continuous extensions of pre-Hilbert module morphisms transfer to properties of the original map.

Theorem 6: Let A be a C^*-algebra of compact operators, $(V,(,)_V)$ and $(W,(,)_W)$ be Hilbert A-modules, and $(U,(,)_U)$ be a pre-Hilbert A-module which is a vector subspace of V and W such that the norms $||_W$ and $||_U$ coincide on U and $||_V$ restricted to U dominates $||_U$. Suppose that $D \in \mathrm{End}_{PH_A^*}(U)$ is a self-adjoint morphism having a continuous adjointable extension $\widetilde{D} : V \to W$ such that

i) $\widetilde{D}$ is A-Fredholm,

ii) $\widetilde{D}^{-1}(U), \widetilde{D}^{*-1}(U) \subseteq U$ and

iii) $\mathrm{Ker}\, \widetilde{D}$ and $\mathrm{Ker}\, \widetilde{D}^*$ are subsets of U.

Then D is a self-adjoint parametrix possessing operator in U.

Proof. We construct the parametrix and the projection.

1) Using assumption (i), $\widetilde{D}$ has closed image by Corollary 5. By Theorem 3.2 in Lance [16], the image of $\widetilde{D}^* : W \to V$ is closed as well, and the following decompositions

$$V = \mathrm{Ker}\, \widetilde{D} \oplus \mathrm{Im}\, \widetilde{D}^*,$$

$$W = \mathrm{Ker}\, \widetilde{D}^* \oplus \mathrm{Im}\, \widetilde{D}$$

hold. Restricting $\widetilde{D}$ to the Hilbert A-module $\mathrm{Im}\, \widetilde{D}^*$, we obtain a continuous bijective Hilbert A-module homomorphism $\mathrm{Im}\, \widetilde{D}^* \to \mathrm{Im}\, \widetilde{D}$.

Let us set

$$\widetilde{G}(x) = \begin{cases} (\widetilde{D}_{|\mathrm{Im}\, \widetilde{D}^*})^{-1}(x) & x \in \mathrm{Im}\, \widetilde{D} \\ 0 & x \in \mathrm{Ker}\, \widetilde{D}^*. \end{cases}$$

The operator $\widetilde{G} : W \to V$ is continuous by the open map theorem. Due to its construction, $\widetilde{G}$ is a morphism in the category H_A. Because of the adjointability of $\widetilde{D}$, and the definition of $\widetilde{G}$, $\widetilde{G}$ is adjointable as well. Summing-up, $\widetilde{G} \in \mathrm{Hom}_{H_A^*}(W, V)$. Note that $\widetilde{G} : W \to \mathrm{Im}\, \widetilde{D}^*$.

2) It is easy to see that the decomposition $V = \mathrm{Ker}\, \widetilde{D} \oplus \mathrm{Im}\, \widetilde{D}^*$ restricts to U in the sense that $U = \mathrm{Ker}\, D \oplus (\mathrm{Im}\, \widetilde{D}^* \cap U)$. Indeed, let $u \in U$. Then $u \in V$ and thus $u = v_1 + v_2$ for $v_1 \in \mathrm{Ker}\, \widetilde{D}$ and $v_2 \in \mathrm{Im}\, \widetilde{D}^*$. Since $\mathrm{Ker}\, \widetilde{D} \subseteq U$ (assumption (iii)) and $\mathrm{Ker}\, D \subseteq \mathrm{Ker}\, \widetilde{D}$, we have $\mathrm{Ker}\, \widetilde{D} = \mathrm{Ker}\, D$. Similarly, one proves that $\mathrm{Ker}\, \widetilde{D}^* = \mathrm{Ker}\, D$. In particular, $v_1 \in \mathrm{Ker}\, D$. Since U is a vector space, $v_2 = u - v_1$ and $u, v_1 \in U$, v_2 is an element of U as well. Thus, $U \subseteq \mathrm{Ker}\, D \oplus (\mathrm{Im}\, \widetilde{D}^* \cap U)$. Since $\mathrm{Ker}\, D, \mathrm{Im}\, \widetilde{D}^* \cap U \subseteq U$, the announced decomposition holds.

3) Further, we have $\operatorname{Im}\widetilde{D}^* \cap U = \operatorname{Im} D$. Indeed, if $u \in U$ and $u = \widetilde{D}^* w$ for an element $w \in W$ then $w \in U$ due to item (ii) and consequently, $u = \widetilde{D}^* w = D^* w = Dw$ that implies $\operatorname{Im}\widetilde{D}^* \cap U \subseteq \operatorname{Im} D$. The opposite inclusion is immediate. (Similarly, one may prove that $\operatorname{Im}\widetilde{D} \cap U = \operatorname{Im} D$.) Putting this result together with the conclusion of item 2 of this proof, we obtain $U = \operatorname{Ker} D \oplus \operatorname{Im} D$.

4) It is easy to realize that $\widetilde{G}_{|U}$ is into U. Namely, if $v = \widetilde{G}u$ for an element $u \in U \subseteq V$, we may write it as $u = u_1 + u_2$ for $u_1 \in \operatorname{Ker}\widetilde{D}$ and $u_2 \in \operatorname{Im}\widetilde{D}^*$ according to the decomposition of V above. Since $u_2 = u - u_1$ and $u_1 \in U$ (due to (iii)), we see that u_2 is an element of U as well. Consequently, $v = \widetilde{G}_{|U} u = \widetilde{G} u_1 + \widetilde{G} u_2 = \widetilde{D}^{-1}_{|\operatorname{Im}\widetilde{D}^*} u_2$. Since $\widetilde{D}^{-1}(U) \subseteq U$ (item (ii)), we obtain that $v \in U$ proving that $\widetilde{G}_{|U}$ is into U. Let us set $G = \widetilde{G}_{|U}$. Due to the assumptions on the norms and the continuity of $\widetilde{G} : (W, |\,|_W) \to (V, |\,|_V)$, it is easy to see that $G : U \to U$ is continuous as well.

5) Defining P to be the projection of U onto $\operatorname{Ker} D$ along the $\operatorname{Im} D$, we get a self-adjoint projection on the pre-Hilbert module U due to the decomposition $U = \operatorname{Ker} D \oplus \operatorname{Im} D$ derived in item 2 of this proof. The relations $DP = 0$ and $1_U = GD + P = DG + P$ are then easily verified using the relation $\operatorname{Ker}\widetilde{D}^* = \operatorname{Ker} D$.

□

Remark 9: In the preceding theorem, specific properties are generalized which are well known to hold for self-adjoint elliptic operators acting on smooth sections of vector bundles over compact manifolds. For instance, assumption (ii) is the smooth regularity and (iii) expresses the fact that differential operators are of finite order. See, e.g., Palais [21] or Wells [28].

4 Complexes of pseudodifferential operators in C^*-Hilbert bundles

For a definition of a C^*-Hilbert bundle, bundle atlae and differential structures of bundles, see Krýsl [14], [15] or Mishchenko, Fomenko [8]. For definitions of the other notions used in the next two paragraphs, we refer to Solovyov, Troitsky [25]. Let us recall that for an A-pseudodifferential operator $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F})$ acting between smooth sections of A-Hilbert bundles $\mathcal{E}$ and $\mathcal{F}$ over a manifold M, we have the order $\operatorname{ord}(D) \in \mathbb{Z}$ of D and the symbol map $\sigma(D) : \pi^*(\mathcal{E}) \to \pi^*(\mathcal{F})$ of D at our disposal. Here, the map $\pi : T^*M \to M$ denotes the projection of the cotangent bundle. Moreover, if M is compact, then for A-Hilbert bundles $\mathcal{E} \to M$ and $\mathcal{F} \to M$, an A-pseudodifferential operator $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F})$, and an integer $t \geq \operatorname{ord}(D)$, we can form

1) the so called Sobolev type completions $(W^t(M, \mathcal{E}), (\,,\,)_t)$ of $(\Gamma(M, \mathcal{E}), (\,,\,))$

2) the adjoint $D^*: \Gamma(M,\mathcal{F}) \to \Gamma(M,\mathcal{E})$ of D and

3) the continuous extensions $D_t: W^t(M,\mathcal{E}) \to W^{t-\mathrm{ord}(D)}(M,\mathcal{F})$ of D.

Smooth sections $(\Gamma(M,\mathcal{G}),(,))$ of an A-Hilbert bundle $\mathcal{G} \to M$ form a pre-Hilbert A-module and spaces $(W^t(\mathcal{G}),(,)_t)$ are Hilbert A-modules. See Example 2 item 4 for a definition of the A-product $(,)$ on the space of smooth sections. The adjoint D^* of an A-pseudodifferential operator D is considered with respect to the A-products $(,)$ on the pre-Hilbert A-modules of smooth sections of the appropriate bundles. Operators D and D^* are pre-Hilbert A-module morphisms, extensions D_t are Hilbert A-module morphisms, and the symbol map $\sigma(D)$ is a morphism of A-Hilbert bundles.

The A-ellipticity is defined similarly as the ellipticity of differential operators in bundles with finite dimensional fibers over $\mathbb{R}$ or $\mathbb{C}$. We use the following definition, the first part of which is contained in Solovyov, Troitsky [25].

Definition 7: Let $D: \Gamma(M,\mathcal{E}) \to \Gamma(M,\mathcal{F})$ be an A-pseudodifferential operator. We say that D is A-*elliptic* if $\sigma(D)(\xi,-): \mathcal{E} \to \mathcal{F}$ is an isomorphism of A-Hilbert bundles for any non-zero $\xi \in T^*M$. Let $(p_i: \mathcal{E}^i \to M)_{i\in\mathbb{Z}}$ be a sequence of A-Hilbert bundles and $(\Gamma(M,\mathcal{E}^i), d_i: \Gamma(M,\mathcal{E}^i) \to \Gamma(M,\mathcal{E}^{i+1}))_{i\in\mathbb{Z}}$ be a complex of A-pseudodifferential operators. We say that $d^\bullet$ is A-*elliptic* if and only if the complex of symbol maps $(\mathcal{E}^i, \sigma(d_i)(\xi,-))_{i\in\mathbb{Z}}$ is exact for each non-zero $\xi \in T^*M$.

Remark 8: One can show that the Laplace operators $\Delta_i = d_{i-1}d_{i-1}^* + d_i^* d_i$, $i \in \mathbb{Z}$, of an A-elliptic complex are A-elliptic operators in the sense of Definition 7. For a proof in the C^*-case, see Lemma 9 in Krýsl [13]. Let us notice that the assumption on unitality of A is inessential in the proof of the Lemma 9 in [13].

Recall that an A-Hilbert bundle $\mathcal{G} \to M$ is called *finitely generated projective* if its fibers are finitely generated and projective Hilbert A-modules. See Manuilov, Troitsky [18]. Let us recall a theorem of Fomenko and Mishchenko on a relation of the A-ellipticity and the A-Fredholm property.

Theorem 7: Let A be a C^*-algebra, M a compact manifold, $\mathcal{E} \to M$ a finitely generated projective A-Hilbert bundle over M, and $D: \Gamma(M,\mathcal{E}) \to \Gamma(M,\mathcal{E})$ an A-elliptic operator. Then the continuous extension

$$D_t: W^t(M,\mathcal{E}) \to W^{t-\mathrm{ord}(D)}(M,\mathcal{E})$$

is an A-Fredholm morphism for any $t \geq \mathrm{ord}(D)$.

Proof. See Fomenko, Mishchenko [8] and Remark 6. □

Corollary 8: Under the assumptions of Theorem 7, $\mathrm{Ker}\, D_t = \mathrm{Ker}\, D$ for any $t \geq \mathrm{ord}(D)$. If moreover D is self-adjoint, then also $\mathrm{Ker}\, {D_t}^* = \mathrm{Ker}\, D$ for any $t \geq \mathrm{ord}(D)$.

Proof. See Theorem 7 in Krýsl [13] for the first claim, and the formula (5) in [13] for the second one. □

Let us notice that the first assertion in Corollary 8 appears as Theorem 3.1.145 on pp. 101 in Solovyov, Troitsky [25]. Now, we state the theorem saying that the Hodge theory holds for A-elliptic complexes of operators acting on sections of finitely generated projective C^*-Hilbert bundles over compact manifolds if A is a C^*-algebra of compact operators.

Theorem 9: Let A be a C^*-algebra of compact operators, M be a compact manifold, $(p_i : \mathcal{E}^i \to M)_{i\in\mathbb{Z}}$ be a sequence of finitely generated projective A-Hilbert bundles over M and $d^\bullet = (\Gamma(M,\mathcal{E}^i), d_i : \Gamma(M,\mathcal{E}^i) \to \Gamma(M,\mathcal{E}^{i+1}))_{i\in\mathbb{Z}}$ be a complex of A-pseudodifferential operators. If $d^\bullet$ is A-elliptic, then for each $i \in \mathbb{Z}$

1) $d^\bullet$ is of Hodge type, i.e., $\Gamma(M,\mathcal{E}^i) = \operatorname{Ker}\Delta_i \oplus \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1}$

2) $\operatorname{Ker} d_i = \operatorname{Ker}\Delta_i \oplus \operatorname{Im} d_{i-1}$

3) $\operatorname{Ker} d_i^* = \operatorname{Ker}\Delta_{i+1} \oplus \operatorname{Im} d_{i+1}^*$

4) $\operatorname{Im}\Delta_i = \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$

5) The cohomology group $H^i(d^\bullet)$ is a finitely generated projective A-Hilbert module isomorphic to the A-Hilbert module $\operatorname{Ker}\Delta_i$.

Proof. Since $d^\bullet$ is an A-elliptic complex, the associated Laplace operators are A-elliptic operators (Remark 8). The Laplace operators are self-adjoint according to their definition. According to Theorem 7, the extensions $(\Delta_i)_t$ are A-Fredholm for any $t \geq \operatorname{ord}(\Delta_i)$.

Let us set $D = \Delta_i$, $U = \Gamma(M,\mathcal{E}^i)$, $V = W^{\operatorname{ord}(\Delta_i)}(M,\mathcal{E}^i)$ and $W = W^0(M,\mathcal{E}^i)$ considered with the appropriate A-products. Then U is a vector subspace of $V \cap W$, and the restriction of $(,)_W$ to $U \times U$ coincides with $(,)_U$. Since Δ_i is A-elliptic, $\operatorname{Ker}\Delta_i = \operatorname{Ker}(\Delta_i)_t = \operatorname{Ker}(\Delta_i)_t^*$ due to Corollary 8. Because the operator D is of finite order, $\widetilde{D}^{-1}(\Gamma(M,\mathcal{E}^i))$, $\widetilde{D}^{*-1}(\Gamma(M,\mathcal{E}^i)) \subseteq \Gamma(M,\mathcal{E}^i)$. The norm on $U = \Gamma(M,\mathcal{E}^i)$ coincides with the norm on $W = W^0(M,\mathcal{E}^i)$ restricted to U and the norm $|\,|_U$ on U is dominated by the norm $|\,|_V$ on $V = W^{\operatorname{ord}(D)}(M,\mathcal{E}^i)$ restricted to U. Thus, the assumptions on the norms in Theorem 6 are satisfied and we may conclude, that Δ_i is a self-adjoint parametrix possessing morphism, and thus, $d^\bullet$ is self-adjoint parametrix possessing as well.

The assertions in items 1–4 follow from the corresponding assertions of Theorem 2. Using Theorem 2 item 5, $H^i(d^\bullet) \simeq \operatorname{Ker}\Delta_i$. As already mentioned, $\operatorname{Ker}\Delta_i \simeq \operatorname{Ker}(\Delta_i)_t$. Since $(\Delta_i)_t$ is A-Fredholm and A is a C^*-algebra of compact operators, $\dim_A \operatorname{Ker}(\Delta_i)_t$ is finite due to Corollary 5. It follows that the kernel of Δ_i is finitely generated.

Since the image of $(\Delta_i)_t$ is closed (Corollary 5), we have $W^t(M,\mathcal{E}^i) = \operatorname{Ker}(\Delta_i)_t \oplus \operatorname{Im}(\Delta_i)_t^*$ due to the Mishchenko theorem (Theorem 3.2 in Lance

[16]). Consequently, $\mathrm{Ker}\,\Delta_i = \mathrm{Ker}\,(\Delta_i)_t$ is a projective A-Hilbert module by Theorem 1.3 in Fomenko, Mishchenko [8]. □

Remark 10: Let us notice that if the assumptions of Theorem 9 are satisfied, the cohomology groups share the properties of their fibers in the sense that they are finitely generated projective A-Hilbert modules.

In the proof of Theorem 9, we could have shown the cohomology groups to be finitely generated and projective in a shorter way using Theorem 7 with an alternative definition of an A-Fredholm operator given in [8] since also Theorem 7 as appears in [8] uses the alternative notion of an A-Fredholm map.

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