De Rham complex twisted by the oscillator bundle

Svatopluk Krýsl

Matematický ústav UK

November 3, 2014

C^*-algebras

Definition

A is called a C^*-algebra if

- A is an associative algebra, i.e.,
  - A vector space over $\mathbb{C}$
  - multiplication map $A \times A \to A$ associative: $(ab)c = a(bc)$, distributes the additive and scalar structure: $(a+b)c = ac + bc$, $a(b+c) = ab + ac$, $(ka)b = k(ab) = a(kb)$
- $* : A \to A$ an anti-involution, $(xy)^* = y^* x^*$ and $** = (*^2) = Id_A$
- $\nu : A \to [0, \infty)$ norm
  - $\nu(x+y) \leq \nu(x) + \nu(y)$, $\nu(\lambda x) = |\lambda| \nu(x)$
  - $\nu(x) \geq 0$ and $\nu(x) = 0$ implies $x = 0$

Examples:

Matrices

- V vector space of finite dimension n (over complex numbers)
- $A = \{L : V \to V | L \text{ is a linear map}\} = End(A) = M_n(C)$
- addition of linear maps, multiplication is composition of maps (multiplication of matrices)
- $*A = A^\dagger$
- $\nu(A) = \sup\{|Av|; v \in V, |v| = 1\} = \max\{|Av|; v \in V, |v| = 1\}$

Compact operators

- H a separable Hilbert space, $(,)_H : H \times H \to \mathbb{C}$, $|| = \sqrt{(,)_H}$
- $K(H) = \overline{\{T : H \to H, \dim \operatorname{Im} T < \infty\}}$ - algebra of compact operators
- $|T| = \sup\{\frac{|Tx|_H}{|x|_H}; 0 \neq x \in H\}$
- $*T = T^*$ - operator adjoint (separability)

The difficulty of axioms for endomorphisms

$K(H)$ is a C^*-algebra.

- $K(H)$ is associative (composition of maps is assoc.)
- $* : K(H) \to K(H)$ and $*^2 = \mathrm{Id}_{K(H)}$
- $|\,| : K(H) \to [0, \infty)$ is a norm because $|\,|$ on H is a norm
- $|TT^*| = |T|^2$ (quite difficult, spectras) $|T^*T| \leq |T||T^*|$ (easy)
- $K(H)$ is complete with respect to $|\,|$ (it is so defined)

Further examples

Continuous functions

- X locally compact topological vector space $\Longrightarrow X$ has a one-point compactification (infinity)
- $A = \mathcal{C}_o(X)$ vector space of continuous complex valued functions vanishing in infinity
- $(fg)(x) = f(x)g(x)$, $x \in X$
- $f, g \in A$ implies $fg \in A$
- $|f| = \sup\{|f(x)|; x \in X\}$
- $\overline{f}(x) = \overline{f(x)}$, $x \in X$
- C^*-identity: easy consequence of the properties of sup: $sup(|fg|) \leq sup(|f|)sup(|g|)$, but $|ff^*| = \sup|ff*| = sup|f|^2 = (\sup|f|)^2 = |f|^2$

Convolution algebra on a locally compact group is in general not a C^*-algebra.

Topology of the symplectic group

- $Sp(2n, \mathbb{R})$ non-compact, retractible onto $K = Sp(2n, \mathbb{R}) \cap SO(n) \subseteq Sp(2m, R)$, K is isomorphic to $U(n)$
- $U(n)$ is of homotopy type of $S^1 = \{e^{\imath\phi}; \phi \in [0, 2\pi)\} \subseteq U(n)$.
- $\pi_1(S^1) \simeq \mathbb{Z}$,i.e., S^1 can be entangled only by a spiral with $\mathbb{Z}$ leaves
- Consequently, $Sp(2n, \mathbb{R})$ is also of this type
- 2-folded covering (unbranched) is called the metaplectic group $Mp(2n, \mathbb{R})$
- $\lambda : Mp(2n, R) \to Sp(2n, R)$ the two-fold covering
- A (very nice almost irreducible) faithful unitary representation of $Mp(2n, \mathbb{R})$ exists $\sigma : Mp(2n, R) \to \mathcal{U}(L^2(R^n))$

Definition on elements

Let $\tilde{g} \in Mp(2n, R)$ denotes an element from two-point $\lambda^{-1}(g)$.
Let $A \in M_n(R)$ be symmetric ($A^t = A$) and $B \in GL(n, R)$.

$$g_1 = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}, \qquad (\sigma(\tilde{g_1})f)(x) = e^{-\imath(Ax,x)/2}f(x)$$

$$g_2 = \begin{pmatrix} B & 0 \\ 0 & (B^t)^{-1} \end{pmatrix}, \qquad (\sigma(\tilde{g_2})f)(x) = \sqrt{\det B}f(B^t x)$$

$$g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (\sigma(\tilde{g_3})f)(x) = \pm e^{i\pi n}(\mathcal{F}f)(x),$$

where $\mathcal{F} : L^2(R^n) \to L^2(R^n)$ denotes the Fourier transform, $f \in L^2(R^n)$ and $x \in R^n$.

History

- David Shale (when doing Ph.D. on Quantization of Klein-Gordon fields by Segal, Irving Ezra Segal)
- Irving Ezra Segal: constructive quantum theory (C^*-algebras, representations of locally compact groups, a definition of the state etc.), use of Stone-von Neumann theorem in QP
- André Weil (French number theorist and geometer, member of the Bourbaki group) - representations of some "discrete" Lie groups arising in number theory
- Berezin - infinitesimal level (angeblich, according to S. Gindikin)
- Bertram Kostant - use in geometric quantization (rediscovering via polarization structures)

Sketch reason for the existence

- **Construction**: Schrödinger representation of the Heisenberg group (Heisenberg CCR, H_n), $r : H_n \to \mathcal{U}(L^2(R^n))$
- symplectic twist $Sp(2n, \mathbb{R}) \times H_n \to H_n$ gives rise to other (twisted) Schrödinger representations $r_g : H_n \to \mathcal{U}(L^2(R^n))$
- Stone-von Neumann: All are equivalent (even if twisted) ;-)
- The intertwiners T_g ('realizing' the equivalences) compose in the same way as the elements of the symplectic group modulo signs (elements in $e^{i\phi}$)
- Weil co-cycle computation: it is a true rep of the double-cover of $Sp(2n, R)$, i.e., of the metaplectic group $Mp(2n, R)$

Symplectic manifolds - Phase spaces

(M, ω) a symplectic manifold

1) $\omega \in \Omega^2(M)$ - exterior (anti-symmetric) differential two-form
2) $\omega_m : T_m M \times T_m M \to \mathbb{R}$ non-degenerate for any $m \in M$
3) $\mathrm{d}\omega = 0$ (crucial for the Jacobi identity for the Poisson brackets)

1) and 2) imply dim $T_m M (= \dim M)$ is even

Basic examples:

- $(\mathbb{R}^{2n}[q^1, \ldots, q^n, p_1, \ldots, p_n], \sum_{i=1}^n dq^i \wedge dp_i)$ canonical symplectic space
- $(T^* M, \omega = d\theta_L)$ cotangent spaces

Examples

- $(S^2(r_0), \omega = r_0^2 \sin\theta \mathrm{d}\phi \wedge \mathrm{d}\theta)$, i.e., sphere with the volume form
- no other sphere (except perhaps $S^0 = \{-1, 1\}$), Stokes theorem
- even dimensional tori ($T^{2n} = S^1 \times \ldots \times S^1$, $\omega = \sum_{i=1}^n \mathrm{d}\phi_i \wedge \mathrm{d}\theta_i$)
- Kähler manifolds, many of homogeneous spaces (e.g., G/H where G, H are complex Lie groups), connection to Einstein manifolds (many Einstein manifolds are Kähler or homogeneous spaces)

Metaplectic structures

- M a symplectic manifold
- $\mathcal{P} = \{e = (e_1, \ldots, e_{2n}); e \text{ is a symplectic basis of } (T_m M, \omega_m), m \in M\}$
- bundle of symplectic reperes
- $\mathcal{P}$ is a $Sp(2n, \mathbb{R})$-principal bundle (Sp acts from the right)
- $\mathcal{Q}$ be a two-fold covering of $\mathcal{P}$ metaplectic structure
- $\mathcal{Q} \to M$ defines a bundle over M, a principal $Mp(2n, \mathbb{R})$-bundle
- Mild condition on (M, ω) for the existence of $\mathcal{Q}$
- All cotangent bundles of orientable manifolds

Oscillator bundle and symplectic spinors

- Set $H = L^2(R^n)$
- $\mathcal{H} = \mathcal{Q} \times_\sigma H$ associated bundle, induced bundle, fiber change
- $\mathcal{H} = \mathcal{Q} \times H/ \simeq$
- $(e, f) \simeq (eg, \sigma(g^{-1})f)$
- "From observers to observable quantities"
- metaplectic, symplectic spinor, Kostant's spinor, Segal-Shale-Weil, Weil, oscillator bundle
- An analogue of the spinor bundle (at the algebraic and geometric level)
- one can construct Dirac-type operators on $\Gamma(\mathcal{H})$ (K. Habermann)

Definition of Hilbert and pre-Hilbert A-modules

Definition

Let A be a C^*-algebra and H be a vector space over the complex numbers. We call $(H, (,))$ a pre-Hilbert A-module if

- H is a right A-module – operation $\cdot : H \times A \to H$
- $(,) : H \times H \to A$ is a $\mathbb{C}$-bilinear mapping
- $(f \cdot T + g, h) = T^*(f, h) + (g, h)$
- $(f, g) = (g, f)^*$
- $(f, f) \geq 0$ and $(f, f) = 0$ implies $f = 0$

We say $T \in A$ is non-negative $(T \geq 0)$ if $T = T^*$ and $\mathrm{Spec}(T) \subseteq [0, \infty)$.
$Spec(T) = \{\lambda \in \mathbb{C}; T - \lambda\bar{1}$ is not invertible in $A^0\}$, where $\bar{1} = (0, 1)$ is the unit in $A^0 = A \oplus \mathbb{C}$ (augmentation)

Definition of Hilbert and pre-Hilbert A-modules

Definition

If $(H, (,))$ is a pre-Hilbert A-module we call it Hilbert A-module if it is complete with respect to the norm $|\,|: H \to [0, \infty)$ defined by $f \ni A \mapsto |f| = \sqrt{|(f,f)|_A}$ where $|\,|_A$ is the norm in A.

Trivial example: A a C^*-algebra
Define $\cdot : A \times A \to A$ by $a \cdot b = ab$ and $(a, b) = a^* b$.
Right: $a \cdot (b \cdot c) = a \cdot (bc) = a(bc) = (ab)c = (ab) \cdot c$
Further: $(a \cdot b, c) = (ab, c) = (ab)^* c = (b^* a^*)c = b^*(a, c)$.
$(a, b)^* = (a^* b)^* = b^* a = (b, a)$

Examples of Hilbert A-modules

- For $A = K(H)$, the C^*-algebra of compact operators on a separable Hilbert space $(H, (,)_H)$, $M = H$ is a Hilbert A-module with respect to
  - $f, g, h \in H$ and $T \in K(H)$
  - $f \cdot T := T^*(f) \in H$
  - $(f, g) = f \otimes g^* \in K(H)$ where $(f \otimes g^*)(h) = (g, h)_H f$
- Proof.

Examples of Hilbert A-modules

- $A^n = A \oplus \ldots \oplus A$ is a Hilbert A-module with respect to $a \cdot (a_1, \ldots, a_n) = (aa_1, \ldots, aa_n)$ and the product given by $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = \sum_{i=1}^n a_i^* b_i$
- If M is a Hilbert A-module, then $M^n = M \oplus \ldots \oplus M$ is a Hilbert A-module with respect to $a \cdot (m_1, \ldots, m_n) = (a \cdot m_1, \ldots, a \cdot m_n)$ and the product given by $(m_1, \ldots, m_n) \cdot (m_1', \ldots, m_n') = \sum_{i=1}^n (m_i, m_i')$
- Further generalizes to $\ell^2(M)$ controlled by the convergence in A. Special case $\ell^2(A)$ $(M = A)$

Distinguished features of the $K(H)$-module H

$$H = L^2(R^n)$$

- H is a $Mp(2n, R)$-module and it is a $K(H)$-module
- The $K(H)$-structure make us able to measure the quantities in H
- They do not commute or anti-commute.
- The metaplectic structure makes us able to place H on our manifold (in accordance with the dynamics/geometry)
- On a manifold - on the oscillator bundle - the $K(H)$ and $Mp(2n, R)$-structures are compatible

Banach bundles

- $p : \mathcal{G} \to M$ be a Banach bundle, bundle of Banach spaces with transitions into the homeomorphisms of a Banach space.
- $\mathcal{G}_x := p^{-1}(\{x\})$
- $x \mapsto \mathcal{G}_x$ (family of Banach e.g. Hilbert spaces parametrized by $x \in M$)
- Let $s : M \to \mathcal{G}$ be a section of $\mathcal{G}$, i.e., $p \circ s = Id_M$
- $\Gamma(\mathcal{G}) = \{s : M \to \mathcal{G} | p \circ s = Id_M\}$
- Any family is a section. Any section is family.
- $\Gamma = \Gamma(\mathcal{G})$ is a vector space
- M compact, one can make a completion of Γ $W^{0,2}(\mathcal{G})$
- Defined similarly as the Sobolev spaces but we must know how to integrate Banach valued functions (on a measure space or on a manifold)

C^*-Hilbert bundles

Bundles /Fibrations/Bündeln/Fibré (Champs continus, Dixmier)/Stohy /Snůšky
Not sheaves (= ne svazky). But bundles give rise to sheaves.

- An A-Hilbert bundle is a Banach bundle the fibers of which are homeomorphic to a fixed Hilbert A-module H and the transition functions are into $\mathrm{Aut}_A(H)$
- If $\mathcal{E} \to M$ is an A-Hilbert bundle over a compact M, then $\Gamma(\mathcal{E})$ is a pre-Hilbert A-module.
- completions of $\Gamma(\mathcal{E})$ as in the Banach case, $W^{k,2}(\mathcal{H})$
- These completions form Hilbert A-modules
- $W^{k,2}(\mathcal{H})$ isomorphic to $\ell^2(H)$ via Kasparov stabilization - quite complicated

Avoiding the symplectic and the compactness assumption

- M contact manifold with a Riemannian structure
- take Finsler manifold (some necessary compatibilities)
- The group of projective canonical transformation act on the contact repére bundle (projective - can change the time)
- It is the so-called contact parabolic subgroup $P \subseteq Sp(2n+2, \mathbb{R})$
- It has also "the" Segal-Shale-Weil representation (by inducing)
- Make the association
- You have a Hilbert bundle
- Do the analysis: One define the infinity
- Infinity in the time dimension
- At each time, the universe might be a modeled by a compact manifold and then the analysis above may apply.

De Rham complex tensored by the oscillatory bundle

- (M^{2n}, ω) symplectic manifold
- admitting metaplectic structure
- $\mathcal{H} \to M$
- $\bigoplus_{k=0}^{2n} \bigwedge^k T^*M \to M$
- $\bigwedge^\bullet T^*M \otimes \mathcal{H} \to M$
- Kuiper ('60): $\mathcal{H}$ is globally trivial; trivializing section defines a flat connection ∇
- $d_k^\nabla(\alpha \otimes h) = d\alpha \otimes h + (-1)^k \epsilon^i \wedge \alpha \otimes \nabla_{e_i} h$ where $(e_i)_{i=1,\ldots,2n}$ $(\epsilon^i)_{i=1,\ldots,2n}$ frame and dual coframe
- $d_{k+1} d_k = 0$ since d (de Rham is flat) and ∇ is flat

Cohomology groups are Hausdorff if $A = K(H)$!

Theorem (Krýsl, Ann. Glob.Anal. Geom. 2014): Let M be a compact manifold, A a C^*-algebra, $(\mathcal{E}^k)_{k\in\mathbb{N}_0}$ a sequence of finitely generated projective A-Hilbert bundles over M and $D_k : \Gamma(\mathcal{E}^k) \to \Gamma(\mathcal{E}^{k+1})$, $k \in \mathbb{Z}$, a complex D of differential operators. Suppose that the Laplace operators $\triangle_k$ of D have closed image in the norm topology of $\Gamma(\mathcal{E}^k)$. If D is elliptic, then D is a self-adjoint parametrix possessing complex in $K(H_A^*)$. Moreover, the cohomology groups of D are finitely generated and projective Hilbert A-modules.

Theorem (Krýsl): If A is a C^*-subalgebra of the algebra of compact operators $K(H)$, one may drop the closed image assumption on the Laplacians.

- Fomenko, A., Mishchenko, A., The index of elliptic operators over C^*-algebras, Izv. Akad. Nauk SSSR, Ser. Mat. 43, No. 4, 1979, pp. 831–859, 967.
- Habermann, K., Habermann, L., Introduction to symplectic Dirac Operators, Lect. Notes Math., Springer, 2004.
- Jordan, P., von Neumann, J., Wigner, E., On an algebraic generalization of the quantum mechanical formalism, Ann. of Math. (2) 35 (1934), No. 1, 29–64.
- Krýsl, S., Cohomology of the de Rham complex twisted by the oscillatory representation, Diff. Geom. Appl., Vol. 33 (Supplement), 2014, pp. 290–297.
- Krýsl, S., Hodge theory for elliptic complexes over unital C^*-algebras, Annals Glob. Anal. Geom., Vol. 45(3), 2014, 197–210. DOI 10.1007/s10455-013-9394-9.

- Lance, C., Hilbert C^*-modules. A toolkit for operator algebraists. London Mathematical Society Lecture Note Series, 210, Cambridge University Press, Cambridge, 1995.
- Manuilov, A., Troitsky, A., Hilbert $C^?$-modules. Translations of Mathematical Monographs, 226. American Mathematical Society, Providence, Rhode-Island, 2005.
- Solovyov, Y., Troitsky, E., C^*-algebras and elliptic operators in differential topology. Transl. of Mathem. Monographs, 192, AMS, Providence, Rhode-Island, 2001.
- Wegge-Olsen, N., K-theory and C^*-algebras - a friendly approach, Oxford University Press, Oxford, 1993.
- Weil, A., Sur certains groupes des operateurs unitaires, Acta Math, 111, 1964.