Hodge theory for parametrix possessing complexes and a class of elliptic complexes

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Algebra of compact operators

- H a separable Hilbert space, $(,)_H : H \times H \to \mathbb{C}$, $|| = \sqrt{(,)_H}$
- $K(H) = \overline{\{T : H \to H, \dim \operatorname{Im} T < \infty\}}$ – algebra of compact operators
- $|T| = \sup\{\frac{|Tx|_H}{|x|_H}; 0 \neq x \in H\}$
- $*T = T^*$ – operator adjoint (separability)

- $K(H)$ is a C^*-algebra –
  - $K(H)$ is associative
  - $* : K(H) \to K(H)$ and $*^2 = \mathrm{Id}_{K(H)}$
  - $|| : K(H) \to [0, \infty)$ is a norm and $|TT^*| = |T|^2$ (C^*-identity)
  - addition, multiplication and scalar multiplication are continuous (this is a consequence of triangle + C^*-identity)
  - $K(H)$ is complete with respect to $||$ (it is so defined)

H as a Hilbert $K(H)$-module with an additional structure

- Multiplication/left-action
  $\cdot : K(H) \times H \to H$, $T \cdot f = T(f)$, $T \in K(H)$, $f \in H$
- 'Scalar product' into $K(H)$, $K(H)$-product
  $(,) : H \times H \to K(H)$, $(f, g) = f^* \otimes g$, where
  $(f^* \otimes g)(h) = (f, h)_H g$ for each $h \in H$
- Note: $f^* \otimes g$ is finite rank (rank 1), especially compact

Definition of Hilbert and pre-Hilbert A-modules

Definition

Let A be a C^*-algebra and H be a vector space over the complex numbers. We call $(H, (,))$ a pre-Hilbert A-module if

H is a left A-module – operation $\cdot : A \times H \to H$

$(,) : H \times H \to A$ is a $\mathbb{C}$-bilinear mapping

$(T \cdot f + g, h) = T^*(f, h) + (g, h)$

$(f, g) = (g, f)^*$

$(f, f) \geq 0$ and $(f, f) = 0$ implies $f = 0$

We say $T \in A$ is non-negative ($T \geq 0$) if $T = T^*$ and $\mathrm{Spec}(T) \subseteq [0, \infty)$.
$Spec(T) = \{\lambda \in \mathbb{C}; T - \lambda\overline{1}$ is not invertible in $A^0\}$, where $\overline{1} = (0, 1)$ is the unit in $A^0 = A \oplus \mathbb{C}$ (augmentation)

Definition of Hilbert and pre-Hilbert A-modules

Definition

If $(H, (,))$ is a pre-Hilbert A-module we call it Hilbert A-module if it is complete with respect to the norm $|\,| : H \to [0, \infty)$ defined by $f \ni A \mapsto |f| = \sqrt{|(f,f)|_A}$ where $|\,|_A$ is the norm in A.

- Closed submodules need to have neither orthogonal nor only topological complements
- Continuous homomorphisms need not be adjointable
- $F : H_1 \to H_2$ is called adjointable if there is a map $F' : H_2 \to H_1$ such that $(Ff, g) = (f, F'g)$ for each $f \in H_1$ and $g \in H_2$. (F' unique, continuous, A-equivariant.)

Examples of Hilbert A-modules

- X locally compact topological vector space, $A = \mathcal{C}_0(X)$ (continuous complex valued functions vanishing at infinity), $(*f)(x) = \overline{f(x)}$, $x \in X$, $|f| = \sup\{|f(x)|; x \in X\}$
- H Hilbert space, $A = B(H)$ bounded on H, $*T = T^*$, $|T|$ the supremum norm as above
- For A a C^*-algebra, $M = A$, $a \cdot b = ab$ and $(a, b) = a^* b$. Form $(M, (,))$ - it is a Hilbert A-module
- For $A = K(H)$, the C^*-algebra of compact operators on a separable Hilbert space H, $M = H$ is a Hilbert A-module with respect to $(,) : H \times H \to K(H)$ given by $(f, g) = f^* \otimes g$ and the left action given by the evaluation $T \cdot f = T(f)$.

Examples of Hilbert A-modules

- $A^n = A \oplus \ldots \oplus A$ is a Hilbert A-module with respect to $a \cdot (a_1, \ldots, a_n) = (aa_1, \ldots, aa_n)$ and the product given by $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = \sum_{i=1}^n a_i^* b_i$
- If M is a Hilbert A-module, then $M^n = M \oplus \ldots \oplus M$ is a Hilbert A-module with respect to $a \cdot (m_1, \ldots, m_n) = (a \cdot m_1, \ldots, a \cdot m_n)$ and the product given by $(m_1, \ldots, m_n) \cdot (m_1', \ldots, m_n') = \sum_{i=1}^n (m_i, m_i')$
- Further generalizes to $\ell^2(M)$ controlled by the convergence in A. Special case $\ell^2(A)$ $(M = A)$

The category of pre-Hilbert and Hilbert modules and adjointable maps

Definition

Let PH_A^* be the category of pre-Hilbert A-modules with morphisms continuous adjointable A-module homomorphisms. Let H_A^* be its full subcategory of Hilbert A-modules.

Definition

Morphism $F : U \to U$ in PH_A^* is called *self-adjoint parametrix possessing* if it is self-adjoint and there exist $G : U \to U$ and $P : U \to U$ morphisms in PH_A^* such that $P^* = P$ and

- $FG + P = 1$ (G Green operator)
- $GF + P = 1$
- $FP = 0$

Parametrix complexes

Let $D = (M^i, d_i)_{i \in \mathbb{Z}} \in K(PH^*_A)$ be a co-chain complex in the category PH^*_A.
Laplacians of D**:** $\triangle_i = d_{i-1} d^*_{i-1} + d^*_i d_i$

Definition

We call $D \in K(PH^*_A)$ *self-adjoint parametrix possessing* complex if each Laplacian $\triangle_i : M^i \to M^i$ is self-adjoint parametrix possessing.

Theorem (Krýsl): If $D = (M^i, d_i)_{i \in \mathbb{Z}} \in K(PH^*_A)$ is self-adjoint parametrix possessing complex in PH^*_A, then the following holds

- $M^i = \operatorname{Ker} \triangle_i \oplus \operatorname{Im} d^*_i \oplus \operatorname{Im} d_{i-1}$
- $H^i(D, A) \simeq \operatorname{Ker} \triangle_i$ as pre-Hilbert A-modules
- $\operatorname{Ker} d_i = \operatorname{Ker} \triangle_i \oplus \operatorname{Im} d_{i-1}$
- $\operatorname{Ker} d^*_i = \operatorname{Ker} \triangle_{i+1} \oplus \operatorname{Im} d^*_{i+1}$
- $\operatorname{Im} \triangle_i = \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d^*_i$

C^*-Hilbert bundles

Definition: Fomenko, Mishchenko [FM]
Attempt: generalize the Atiyah-Singer index theorem
Bundles/Fibrations/Bündeln/Fibré (Champs continus, Dixmier)/Stohy of Hilbert A-modules/Snůšky (strange type of scissors, in Czech)

- An A-Hilbert bundle is a Banach bundle the fibers of which are homeomorphic to a fixed Hilbert A-module M and the transition functions are into $\mathrm{Aut}_A(M)$
- If $\mathcal{E} \to M$ is a Hilbert bundle over a compact M, then $\Gamma(\mathcal{E})$ is a pre-Hilbert A-module.
- Sobolev type completion of $\Gamma(\mathcal{E})$ exists (over compacts)
- These completions form Hilbert A-modules

- (finite order) differential operators in finite rank vector bundles over a manifold $\longrightarrow$ generalizes
- (finite order) differential operators in A-Hilbert bundles
- symbols of differential operators (as in classical PDE-theory), $\sigma : \triangle \mapsto (\sigma(\triangle) : f \mapsto |x|^2 f)$
  (Differential operators) $D \longrightarrow \sigma(D)$ (Morphisms in the category of A-Hilbert bundles)

Definition

A complex $D = (\Gamma(\mathcal{E}^k), D_k)_k$ of differential operators in A-Hilbert bundles $\mathcal{E}^k$ is called *elliptic* if its symbol sequence is exact in the category of A-Hilbert bundles.

Theorem (Krýsl, AGAG 2014): Let M be a compact manifold, A a C^*-algebra, $(\mathcal{E}^k)_{k\in\mathbb{N}_0}$ a sequence of finitely generated projective A-Hilbert bundles over M and $D_k : \Gamma(\mathcal{E}^k) \to \Gamma(\mathcal{E}^{k+1})$, $k \in \mathbb{Z}$, a complex D of differential operators. Suppose that the Laplace operators $\triangle_k$ of D have closed image in the norm topology of $\Gamma(\mathcal{E}^k)$. If D is elliptic, then D is a self-adjoint parametrix possessing complex in $K(H_A^*)$.
Moreover, the cohomology groups of D are finitely generated and projective Hilbert A-modules.
Theorem (Krýsl): If A is a C^*-subalgebra of the algebra of compact operators $K(H)$, one may drop the closed image assumption on the Laplacians.

Remarks: This generalizes the classical Hodge theory for finite rank vector bundles, compact manifolds and elliptic complexes to the finitely generated projective bundles over C^*-algebras.
Moreover, one can say that the finiteness and projectiveness of the cohomology is connected to the finiteness and projectiveness of the fibers.
This interpretation was not seen till yet in the Hodge theory for finite rank bundles (bundles with finite dimensional vector spaces as fibers).

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