Segal-Shale-Weil complex

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$K(H)$ and the Hilbert module

- H be a separable Hilbert space $(,)_H$
- $K(H)$ be the vector space of compact operators on $H =$ completion of finite rank operators in the operator norm
- $K(H) \subseteq$ of bounded operators
- For any $f \in H, a \in K(H)$

$$f \cdot a = a^*(f)$$

defines a right $K(H)$-module

Algebra of compact operators

- $K(H)$ is a C^*-algebra with respect to the adjoint of maps and the norm $|a|_{K(H)} = \sup_{v \in H, |v|=1} |a(v)|_H$, $a \in K(H)$
- $(,) : H \times H \to K(H)$, $(f,g) = f \otimes g^*$, $(f \otimes g^*)(h) = (g,h)_H f$, $f, g, h \in H$
- It maps into rank 1 operator
- **Lemma:** H is a $K(H)$-Hilbert module.
- **Proof.** Check $(f \cdot a, g) = (f,g)a^*$, $(f, g \cdot b) = (f,g)b$ $(f,f) \geq 0$ and $(f,f) = 0$ implies $f = 0$. H is complete wrt. $|f| = \sqrt{|(f,f)_H|_{K(H)}}$, $f \in H$. $\square$

General definitions

- $K(H)$ is a C^*-algebra $=$
- $K(H)$ is associative
- $* : K(H) \to K(H)$ and $*^2 = \mathrm{Id}_{K(H)}$
- $|\,| : K(H) \to [0, \infty)$ is a norm and $|TT^*| = |T|^2$ (C^*-identity)
- addition, multiplication and scalar multiplication are continuous (consequence of triangle $+$ C^*-identity)
- $K(H)$ is complete with respect to $|\,|$ (it is so defined)

Definition of Hilbert and pre-Hilbert A-modules

Definition

Let A be a C^*-algebra and H be a vector space over the complex numbers. We call $(H, (,))$ a pre-Hilbert A-module if

- H is a right A-module, $\cdot : H \times A \to H$
- $(,) : H \times H \to A$ is a $\mathbb{C}$-bilinear mapping
- $(f \cdot T + g, h) = (f, h)T^* + (g, h)$, $f, g, h \in H$, $T \in K(H)$
- $(f, g) = (g, f)^*$
- $(f, f) \geq 0$ and $(f, f) = 0$ implies $f = 0$

We say $T \in A$ is non-negative ($T \geq 0$) if $T = T^*$ and $\mathrm{Spec}(T) \subseteq [0, \infty)$.
$\mathrm{Spec}(T) = \{\lambda \in \mathbb{C}; T - \lambda\overline{1}$ is not invertible in $A^0\}$, where $\overline{1} = (0, 1)$ is the unit in $A^0 = A \oplus \mathbb{C}$ (augmentation)

Definition of Hilbert and pre-Hilbert A-modules

Definition

If $(H, (,))$ is a pre-Hilbert A-module we call it Hilbert A-module if it is complete with respect to the norm $|\,| : H \to [0, \infty)$ defined by $f \ni H \mapsto |f| = \sqrt{|(f, f)|_A}$ where $|\,|_A$ is the norm in A.

Pre-Hilbert A-module is a normed space.
Hilbert A-module is a Banach space.

Examples of C^*-algebras

- X locally compact topological vector space, $A = \mathcal{C}_0(X)$ (continuous complex valued functions vanishing at infinity), $(*f)(x) = \overline{f(x)}$, $x \in X$, $|f| = \sup\{|f(x)|; x \in X\}$
- H Hilbert space, $A = B(H)$ bounded on H, $*T = T^*$, $|T|$ the supremum norm

Examples of Hilbert A-modules

- For A a C^*-algebra, $M = A$, $a \cdot b = ab$ and $(a, b) = a^* b$. Form $(M, (,))$ - it is a Hilbert A-module
- For $A = K(H)$, the C^*-algebra of compact operators on a separable Hilbert space H, $M = H$ is a Hilbert A-module with respect to $(,) : H \times H \to K(H)$ given by $(f, g) = f \otimes g^*$ and the right action given by the evaluation $f \cdot T = T^*(f)$.
- If M is a Hilbert A-module, then $M^n = M \oplus \ldots \oplus M$ is a Hilbert A-module with respect to $(m_1, \ldots, m_n) \cdot a = (m_1 \cdot a, \ldots, m_n \cdot a)$ and the product given by $(m_1, \ldots, m_n) \cdot (m'_1, \ldots, m'_n) = \sum_{i=1}^n (m_i, m'_i)$
- Further generalizes to $\ell^2(M)$ controlled by the convergence in A. Special case $\ell^2(A)$ $(M = A)$

C^*-Hilbert bundles

Definition: Fomenko, Mishchenko [FM]
Attempt: generalize the Atiyah-Singer index theorem

- An A-Hilbert bundle is a Banach bundle the fibers of which are homeomorphic to a fixed Hilbert A-module M and the transition functions are into $\mathrm{Aut}_A(M)$
- If $\mathcal{F} \to M$ is a Hilbert bundle over a compact M, then $\Gamma(\mathcal{E})$ is a pre-Hilbert A-module; canonically $(s \cdot a)(m) = s(m) \cdot a$, $m \in M$; $s \in \Gamma(\mathcal{F})$ and $a \in A$.
- Sobolev type completion of $\Gamma(\mathcal{E})$ exists (over compacts)
- These completions form Hilbert A-modules

- (finite order) differential operators in finite rank vector bundles over a manifold $\longrightarrow$ generalizes
- (finite order) differential operators in A-Hilbert bundles
- symbols of differential operators (as in classical PDE-theory), $\sigma : \triangle \mapsto (\sigma(\triangle) : f \mapsto |x|^2 f)$
  (Differential operators) $D \longrightarrow \sigma(D)$ (Morphisms in the category of A-Hilbert bundles)

Definition

A complex $D = (\Gamma(\mathcal{E}^k), D_k)_k$ of differential operators in A-Hilbert bundles $\mathcal{E}^k$ is called *elliptic* if its symbol sequence is exact in the category of A-Hilbert bundles.

Theorem (Krýsl): Let M be a compact manifold, A a C^*-algebra, $(\mathcal{F}^k)_{k\in\mathbb{N}_0}$ a sequence of finitely generated projective A-Hilbert bundles over M and $D_k : \Gamma(\mathcal{F}^k) \to \Gamma(\mathcal{F}^{k+1})$, $k \in \mathbb{Z}$, a complex D of differential operators. Suppose that the Laplace operators $\triangle_k = D_{k-1}D^*_{k-1} + D^*_k D_k$ of D have closed image in the norm topology of $\Gamma(\mathcal{F}^k)$. If D is elliptic, then the cohomology of D is finitely generated and projective A-module, especially a Banach space and $\Gamma(\mathcal{F}^k) = \mathrm{Ker}\,\triangle_k \oplus \mathrm{Im} D_{k-1} \oplus \mathrm{Im}\, D^*_{k+1}$ and $H^k(M, D) \simeq \mathrm{Ker}\,\triangle_k$. Moreover, the cohomology groups of D are finitely generated and projective Hilbert A-modules.
Theorem (Krýsl, AGAG15): If A is a C^*-subalgebra of the algebra of compact operators $K(H)$, one may drop the closed image assumption on the Laplacians.

Symplectic structures

- (V, ω) a symplectic space of dimension $2n$ (flat phase space of a system with n-degrees of freedom)
- $G = Sp(V, \omega)$ the symplectic group (linear transformation which do not change the form of the Hamilton equations)
- $\pi_1(G) = \mathbb{Z} \implies \exists\ 2:1$ covering $\lambda : \widetilde{G} \to G$
- $\widetilde{G} = Mp(V, \omega)$ the metaplectic group
- non-universal, not compact, non matrix Lie group

Segal-Shale-Weil representation

- L any maximal isotropic (Lagrangian) subspace of (V, ω)
- $J : V \to V$ compatible complex structure, the **bilinear** form $g(v, w) = \omega(Jv, w)$ is positive definite
- $H = L^2(L)$ Lebesgue square integrable functions on L
- $\sigma : \widetilde{G} \to U(L^2(L))$ the Segal-Shale-Weil representation
- oscillator, metaplectic, symplectic spinor
- $\sigma(\widetilde{\omega}) = \mathcal{F} : L^2(L) \to L^2(L)$, $\widetilde{\omega} \in \lambda^{-1}(\omega)$

Properties of SSW

- σ is unitary
- decomposes into an orthogonal sum of two irreducible $\widetilde{G}$-modules
- they are highest weight modules (and especially in the category $\mathcal{O}$)
- multiplicity bounded reps of $\mathfrak{sp}(V, \omega)$

Symplectic parallels of orthogonal spin geometry

- (M, ω) symplectic manifold of dimension $2n$ (phase space of a "curved" system of n freedom degrees)
- $\mathcal{P} = \{e = (e_1, \ldots, e_{2n}) |\, e$ is a symplectic basis of $T^*_x M, x \in M\}$
- $p : \mathcal{P} \to M$ is a principal G-bundle
- (Λ, q), where $q : \widetilde{\mathcal{P}} \to M$ a $\widetilde{G}$-bundle and Λ is a bundle homomorphism, is called metaplectic structure if the diagram commutes

$$\begin{array}{ccccc}
\widetilde{\mathcal{P}} \times \widetilde{G} & \longrightarrow & \widetilde{\mathcal{P}} & & \\
\Big\downarrow {\scriptstyle \Lambda \times \lambda} & & \Big\downarrow {\scriptstyle \Lambda} & \searrow^{q} & \\
 & & & & M \\
 & & & \nearrow_{p} & \\
\mathcal{P} \times G & \longrightarrow & \mathcal{P} & &
\end{array}$$

Higher symplectic spins

$\sigma^k : \widetilde{G} \to \mathrm{Aut}(\bigwedge^k V^* \otimes H),\ k = 0, \ldots, 2n$

$\sigma^k(g)(\alpha \otimes s) = \lambda^{*\wedge k}(g)(\alpha) \otimes \sigma(g)(s),\ g \in \widetilde{G},\ s \in H$

$E^k = \bigwedge^k V^* \otimes H$ "Higher symplectic spinors"

Bundle of **higher symplectic spinors**:
$\mathcal{E}^k = \widetilde{\mathcal{P}} \times_{\sigma^k} (\bigwedge^k V^* \otimes H)$

Higher symplectic spinor fields $\Gamma(\mathcal{E}^k)$

$K(H)$-structure on these fields

$(\alpha \otimes v) \cdot a = \alpha \otimes a^*(v)$

$(\alpha \otimes v, \beta \otimes w) = g(\alpha, \beta) v \otimes w^*,\ \alpha, \beta \in \bigwedge^k V^*,\ a \in K(H),$
$v, w \in H$

Trivialization - Kuiper theorem

$H = L^2(L)$, $\mathcal{E}^0 = \mathcal{H}$,

∇ a flat connection on $\mathcal{E}^0$

Exists because $\mathcal{E}^0$ is trivial: trivialization, horizontal distribution, horizontal directions define the connection

∇ induces $d_k^\nabla : \Gamma(\mathcal{E}^k) \to \Gamma(\mathcal{E}^{k+1})$ by the Leibniz formula

$\nabla_X(s \cdot a) = (\nabla_X s) \cdot a$, $s \in \Gamma(\mathcal{H})$, $k = 0, \ldots, 2n$

$\nabla_X(s, t) = (\nabla_X s, t) + (s, \nabla_X t)$, $s, t \in \Gamma(\mathcal{H})$, $X \in \mathfrak{X}(M)$

It is a hermitian A-connection

The Segal-Shale-Weil complex

Let (M, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a trivial connection on $\mathcal{E}^0$

Definition

The complex $0 \to \Gamma(\mathcal{E}^0) \stackrel{d_0^\nabla}{\to} \Gamma(\mathcal{E}^1) \stackrel{d_1^\nabla}{\to} \dots \stackrel{d_{2n-1}^\nabla}{\to} \Gamma(\mathcal{E}^{2n}) \to 0$ is called the Segal-Shale-Weil complex.

This complex is elliptic, i.e., the symbol sequence is exact (equivalent to symbols of Laplacians are isomorphisms) (out of the zero section of the cotangent bundle)

Laplacians $\triangle_k = d_{k-1}d_{k-1}^* + d_k^* d_k,\ k = 0, \dots, 2n$

Azumaya bundle, matrix densities

$K(H)$ algebra / vector space of compact operators on H

$\rho : \widetilde{G} \to \mathrm{Aut}(K(H))$

$\rho(g)a = \sigma(g)a\sigma(g)^{-1}$, $g \in \widetilde{G}$, $a \in K(H)$

$\mathcal{A} = \widetilde{\mathcal{P}} \times_\rho K(H)$

So called Azumaya bundle, sections form sheaves of Azumaya algebras (Bundle of "matrix densities", "Filtern", "measuring devices")

Kuiper theorem $+$ $K(H) = H\hat{\otimes}H \longrightarrow \mathcal{A}$ is trivial, represented by 0 in $H^3(M, \mathbb{Z})$

Construction in more detail

$\mathcal{E}^{k\prime}$ is $\lambda^{-1}(U(n))$-reduction of $\mathcal{E}^k$
$\cdot : \mathcal{E}^k \otimes \mathcal{A} \to \mathcal{E}^k$

$$[(e, v)] \cdot [(e, a)] = [(e, v \cdot a)]$$

where $e \in \widetilde{\mathcal{P}}$, $v \in E^k$, and $a \in K(H)$
$(,) : \mathcal{E}^{k\prime} \otimes \mathcal{E}^{k\prime} \to \mathcal{A}'$

$$([(e, v)], [(e, w)]) = [(e, (v, w))] \in \mathcal{A}'$$

where $e \in \widetilde{\mathcal{P}}$, and $v, w \in E^k$
$\psi : \mathcal{A} \to M \times K(H)$ trivialization
$\mathfrak{M} : \mathcal{E}^k \times \mathcal{E}^k \to A$
$\mathfrak{M}(s', s'') = \mathrm{pr}_2(\psi((s', s''))),\ s', s'' \in \mathcal{E}^k$
$\mathfrak{A} : \mathcal{E}^k \times A \to \mathcal{E}^k$
$\mathfrak{A}(s, a) = s \cdot \psi^{-1}(p(s), a),\ s \in \mathcal{E}^k,\ a \in A$

Analytic properties of the cohomology of the SSW-complex

Theorem: If M is a compact symplectic, admits a metaplectic structure, and ∇ is a flat connection on $\mathcal{H}$, then the cohomology groups are finitely generated projective Hilbert $K(H)$-modules and the Hodge decomposition holds for it.

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