Elliptic complexes over C*-algebras of compact operators

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Motivation

- Geometry: Hodge theory for de Rham complex □
- Physics: Quantum field theory (deals with 'big' objects and PDE's for them)
- Learn the theory of elliptic operators (on compact manifolds)
- Striking: Some analytical K-theory (K-Fredholm theory of Russian mathematicians A. Mishchenko and A. Fomenko) and symplectic Dirac operators of Katharina Habermann
- Symplectic manifolds as generalization of the phase space $\mathbb{R}^{2n}[q^1, \ldots, q^n, p_1, \ldots, p_n]$ (unconstrained) but also as manifolds with their geometry and topology (homology)

C^*-algebra

Definition (C^*-algebra)

A associative algebra (over $\mathbb{C}$)

$* : A \to A$ is an involution and an anti-automorphism □

$|\,| : A \to \mathbb{R}_{\geq 0}$ norm, such that $|a^*a| = |a|^2$, $a \in A$

A with respect to $|\,|$ is a Banach space

Basic examples:

- $B(H)$ algebra of bounded operators on a separable Hilbert space with $*$ the adjoint, and the operator norm
- $\mathcal{C}_0(X)$ continuous functions on a locally compact space X which vanish in infinity with the supremum norm and $f^*(x) = \overline{f(x)}$, $x \in X$.
- G locally compact unimodular group, then $L^1(G)$ **need not** be a C^*-algebra

'Topological' modules over C^*-algebras

Analysis of classical PDE's: Specific Banach and Hilbert spaces occur, like $\mathbb{C}^\infty(K, \mathbb{R})$ ($K \subseteq \mathbb{R}^n$ compact), $W^{k,p}(\mathbb{R}^n)$, $W^{k,2}(\mathbb{R}^n)$ or $W^{k,p}(\mathbb{R}^n, V)$ for V a vector space

On manifolds: $\mathcal{V} \to M$ vector bundle, M compact, $\Gamma(M, \mathcal{V})$ pre-Hilbert space, $W^{k,p}(M, \mathcal{V})$ Banach, $W^{k,2}(M, \mathcal{V})$ Hilbert (used in the so-called elliptic PDE's)

Aim: Do analysis of PDE's and Quantum Physics when $\mathbb{C}$ or $\mathbb{R}$ is replaced by a C^*-algebra

Objects: not only vector spaces (= modules over $\mathbb{C}$ or $\mathbb{R}$), but modules over C^*-algebras having a convenient topological str.

For elliptic operators: The Hilbert product is appropriately modified

A. T. Fomenko, A. S. Mishchenko use the rule "Change the ground field by the C^*-algebra A", and do it consequently.

Thus $(,) : H \times H \to \mathbb{C} \Longrightarrow (,) : H \times H \to A$

Hilbert C^*-modules

A trivial example ('tautological module'):
Right action: $\cdot : A \times A \to A$, $a \cdot b = ab$
Satisfies $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity)
$a \cdot (b + c) = a \cdot b + a \cdot c$
$(a + b) \cdot c = a \cdot c + b \cdot c$
$a \cdot (kb) = k(a \cdot b)$, $k \in \mathbb{C}$
Thus A is a right module over itself
Product: $(a, b) = a^* b$
Properties of the product:
$(a, b \cdot c) = a^*(b \cdot c) = a^* bc = (a, b)c$
$(a \cdot b, c) = (a \cdot b)^* \cdot c = (ab)^* \cdot c = b^* a^* c = b^*(a, c)$

A further example:

H a separable Hilbert space with scalar product $(,)_H$ (complex conjugate in the first input)

C^*-algebra $A = B(H)$ of bounded operators, A-module $V = H$

1) $v \cdot a = a^*(v)$ - evaluation, $a \in B(H)$ and $v \in H$

2) $(u, v) = u \otimes v^* \in B(H)$, where $(u \otimes v^*)(w) = (v, w)_H u$, $u, v, w \in H$ (Vectors are columns and co-vectors are rows, thus $u \otimes v^*$ is a matrix.)

2.a) $(u, v \cdot a)(w) = (u \otimes (v \cdot a)^*)w = (v \cdot a, w)_H u =$
$(a^*(v), w)_H u = (v, a(w))_H u =$
$(u \otimes v^*)(a(w)) = (u, v)(a(w)) = [(u, v)a](w)$

[[2.b) $(u \cdot a, v)(w) = ((u \cdot a) \otimes v^*)(w) = (v, w)_H (u \cdot a) =$
$(v, w)_H a^*(u) = a^*((v, w)_H u) = [a^*(u \otimes v^*)](w)$]]

Note: $(u, u)(u) = (u \otimes u^*)(u) = (u, u)_H u$, thus
$B(H) \ni (u, u) = c1_{B(H)}$, $c = (u, u)_H \geq 0$

Definition of Hilbert and pre-Hilbert A-modules

Definition (Pre-Hilbert module)

Let A be a C^*-algebra and V be a vector space over the complex numbers. We call $(V, (,))$ a pre-Hilbert A-module if

V is a right A-module - operation $\cdot : V \times A \to V$

$(,) : V \times V \to A$ is a C-sesquilinear map satisfying

$(f, g + h \cdot T) = (f, h)T + (g, h)$

$(f, g) = (g, f)^*$

$(f, f) \geq 0$ and $(f, f) = 0$ implies $f = 0$

We say $T \in A$ is non-negative, $T \geq 0$ if $T = T^*$ and
$\mathrm{Spec}(T) \subseteq [0, \infty)$, where
$\mathrm{Spec}(T) = \{\lambda \in C; T - \lambda 1 \text{ is not invertible in } A\}$.

Definition of Hilbert A-modules

Definition (Hilbert A-module)

If $(V, (,))$ is a pre-Hilbert A-module we call it redHilbert A-module if it is complete with respect to the norm $||: V \to [0, \infty)$ defined by $V \ni f \mapsto |f| = \sqrt{|(f,f)|_A}$, $(f,f) \in A$, where $||_A$ is the norm in A.

Closed submodules need have neither orthogonal nor only a topological complement

Homomorphisms need not be adjointable

$F : V_1 \to V_2$ is called adjointable if there is a map $F' : V_2 \to V_1$ satisfying $(Ff, g) = (f, F'g)$, $f \in V_1$ and $g \in V_2$. If it exists, it is unique: $F' = F^*$. It is called a homomorphism if it is continuous and $F(f \cdot T) = F(f) \cdot T$ for each $f \in V_1$ (equivariant, A-module homomorphism).

Miscellaneous

Important example: H a separable Hilbert space
For $A = K(H)$, the C^*-algebra of bounded operators on H, $V = H$ is a Hilbert A-module with respect to $(,) : H \times H \to K(H)$ given by $(f, g) = f \otimes g^*$, $(f \otimes g^*)(h) = (g, h)_H f$ and the right action given by the evaluation $f \cdot T = T^*(f)$, where $f, g \in V$.
Recall:
An A-module V is projective if when it is embedded in any other module, it splits.
It is called finitely generated if there exist $m_1, \ldots, m_k$ such that for all $m \in V$ there are $a_1, \ldots, a_k$ such that $m = \sum_{i=1}^k m_i \cdot a_i$.

Examples of Hilbert A-modules

1) If V is a Hilbert A-module, then $V^n = V \oplus \ldots \oplus V$ is a Hilbert A-module with respect to $(m_1, \ldots, m_n) \cdot a = (m_1 \cdot a, \ldots, m_n \cdot a)$ and the product given by $((m_1, \ldots, m_n), (m'_1, \ldots, m'_n)) = \sum_{i=1}^n (m_i, m'_i)$
2) Special case: $V = A$ (the tautological module), $V^n = A^n = A \oplus \ldots \oplus A$ is a Hilbert A-module with respect to $(a_1, \ldots, a_n) \cdot a = (a_1 a, \ldots, a_n a)$ and the product given by $((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \sum_{i=1}^n (a_i, b_i) = \sum_{i=1}^n a_i^* b_i$
3) Ex. in item 1 generalizes to $\ell^2(V)$, where V is a Hilbert A-module.
$\ell^2(V) = \{m = (m_1, m_2, \ldots) | \sum_{i=1}^\infty (m_i, m_i) \text{ converges in } A\}$
Product $(m, n) = \sum_{i=1}^\infty (m_i, n_i)$ (finite because of (a kind of) Cauchy-Schwarz inequality)

How to do geometry with these structures?

- $\mathbb{C}$ is replaced by a C^*-algebra, topological vector spaces by Hilbert A-modules
- Notion of manifold - the same, i.e., locally compact Hausdorff topological space locally homeomorphic to $\mathbb{R}^n$
- Vector bundles with fiber a vector space V are replaced by certain bundles with fiber a Hilbert A-module V
- Section spaces and their completions (pre-Hilbert and Hilbert spaces) are then even pre-Hilbert and Hilbert modules
- Differential operators of order r (same definition - formally), elliptic operators ((formally) same definition, i.e., via symbol)

Recall: ☐ de Rham, Laplace

C^*-Hilbert bundle

- An A-Hilbert bundle is a Banach bundle the fibers of which are Hilbert A-modules isomorphic to a fixed Hilbert A-module $(V, (,))$, the transition maps of the bundle chart are into $\mathrm{Aut}_A(V)$.
- If $\mathcal{V} \to M$ is a Hilbert bundle over a compact M, then $\Gamma(M, \mathcal{V})$ is a pre-Hilbert A-module, $(s \cdot a)(m) = s(m) \cdot a$, $s \in \Gamma(M, \mathcal{V}$ and $a \in A$.
- Sobolev type completion of $\Gamma(M, \mathcal{V})$ exists (over compacts) (Fomenko, Mishchenko)
- These completions (denoted by $W^{k,2}(M, \mathcal{V})$) form Hilbert A-modules

Category of pre-Hilbert modules and its complexes

Definition

Let PH_A^* be the category of pre-Hilbert A-modules as objects and adjointable pre-Hilbert A-module homomorphisms as morphisms.

Definition

A complex $D^\bullet = (D_i, E^i)_{i \in \mathbb{N}_0} \in \mathrm{Kom}(PH_A^*)$ ($D_i : E^i \to E^{i+1}$, $D_{i+1}D_i = 0$, $E^i \in \mathrm{Ob}(PH_A^*)$, $D_i \in \mathrm{Mor}_{PH_A^*}(E^i, E^{i+1})$) is called self-adjoint parametrix possessing if the Laplacian operators $\triangle_i = D_{i-1}D_{i-1}^* + D_i^* D_i$ are self-adjoint parametrix possessing, i.e., if there exist maps $G_i, P_i : E^i \to E^i$ such that $1_{E_i} = \triangle_i G_i + P_i = \triangle_i G_i + P_i$, $\triangle_i P_i = 0$ and $P_i = P_i^*$.

Theorem (K): Let M be a compact manifold, A a C^*-algebra, $(\mathcal{V}^k)_{k\in\mathbb{N}_0}$ be a sequence of finitely generated projective A-Hilbert bundles over M and $D_k : \Gamma(M, \mathcal{V}^k) \to \Gamma(M, \mathcal{V}^{k+1})$, $k \in \mathbb{N}_0$, form a complex of differential operators. Suppose that the Laplace operators of $D^\bullet$ have closed image in the norm topology of $\Gamma(M, \mathcal{V}^k)$. If $D^\bullet$ is elliptic, then $D^\bullet$ is a self-adjoint parametrix possessing complex in $\mathrm{Kom}(PH^*_A)$ and

- $\Gamma(M, \mathcal{V}^i) = \mathrm{Ker}\,\triangle_i \oplus \mathrm{Im}\,D_i^* \oplus \mathrm{Im}\,D_{i-1}$
- $H^i(D^\bullet, A) \simeq \mathrm{Ker}\,\triangle_i$ as pre-Hilbert A-modules
- $\mathrm{Ker}\,D_i = \mathrm{Ker}\,\triangle_i \oplus \mathrm{Im}\,D_{i-1}$
- $\mathrm{Ker}\,D_i^* = \mathrm{Ker}\,\triangle_{i+1} \oplus \mathrm{Im}\,D_{i+1}^*$
- $\mathrm{Im}\,\triangle_i = \mathrm{Im}\,D_{i-1} \oplus \mathrm{Im}\,D_i^*$
- Moreover, the cohomology groups of $D^\bullet$ are finitely generated and projective Hilbert A-modules.

Hodge theory for finitely generated projective bundles

Theorem (K): If A is a C^*-subalgebra of the algebra of compact operators $K(H)$, one may drop the closed image assumption on the Laplacian.

Remarks:

1)The thm generalizes the classical Hodge theory for finite rank vector bundles, compact manifolds and elliptic complexes to the finitely generated projective bundles over C^*-algebras.

2) Moreover, one may say that the finiteness and projectiveness of the cohomology is connected to the finiteness and projectiveness of the fibers.

3) This interpretation is usually not mentioned in the Hodge theory for finite rank bundles (bundles with finite dimensional vector spaces as fibers).

Symplectic manifolds, symplectic and metaplectic group

(M^{2n}, ω) symplectic manifold (S^2, even dimensional tori, T^*M, Kähler manifolds, KT-manifold...)
$Sp(2n, \mathbb{R})$ symplectic group (automorphisms of (T_mM, ω_m))
$Mp(2n, \mathbb{R})$ connected two-fold cover of $Sp(2n, \mathbb{R})$,
$\lambda : Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$, the covering
$\pi_1(Sp(2n, \mathbb{R})) = \mathbb{Z}$, the universal covering is '$\mathbb{Z}$-folded'

Symplectic spinor structures

Bundle of symplectic frames:
$\mathcal{P} = \{e = (e_1, \ldots, e_{2n}) | \, e$ is a symplectic basis of $(T_m^* M, \omega_m), \ m \in M\}$. It is a principal $Sp(2n, \mathbb{R})$-bundle.
We call an $Mp(2n, \mathbb{R})$-bundle $\mathcal{Q}$ and a surjective bundle map $\Lambda : \mathcal{Q} \to \mathcal{P}$ a metaplectic structure if the following diagram commutes

$$
\begin{array}{ccccc}
\mathcal{Q} \times Mp(2n, \mathbb{R}) & \longrightarrow & \mathcal{Q} & & \\
\downarrow{\scriptstyle \Lambda \times \lambda} & & \downarrow{\scriptstyle \Lambda} & \overset{q}{\searrow} & \\
 & & & & M \\
 & & & \underset{p}{\nearrow} & \\
\mathcal{P} \times Sp(2n, \mathbb{R}) & \longrightarrow & \mathcal{P} & &
\end{array}
$$

Oscillator representation of Shale and Weil

$\rho : Mp(2n, \mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R}^n))$

- a unitary representation
- a faithful representation of $Mp(2n, \mathbb{R})$
- $H = L^2(\mathbb{R}^n)$ splits into $L^2(\mathbb{R}^n)_+ \oplus L^2(\mathbb{R}^n)_-$, odd and even functions, irreducible summands
- similar to the spinor representation of $Spin(2n, \mathbb{R})$
- a completion of complex valued polynomials $\mathbb{C}[x^1, \ldots, x^n] = \bigoplus_{i=0}^{\infty} S^i(\mathbb{R}^n)$.

Oscillator representation - continuation

but different meaning (in Physics)

constructed through certain intertwiners of the Schrödinger representation of the Heisenberg group

Inventors: David Shale (doctoral student by Irving Segal, KG-fields) and André Weil (number thy), further Berezin at infinitesimal level.

Other names:

Segal-Shale-Weil representation, Shale-Weil, Weil representation, metaplectic representation, symplectic spinor representation (Kostant, Habermann)

Oscillator representation (R. Howe)

Exterior algebra with values in the oscillator rep.

$E = \bigoplus_{k=0}^{2n} E^k = \bigoplus_{k=0}^{2n} (\bigwedge^k (\mathbb{R}^{2n})^* \otimes H)$
$\rho_k : Mp(2n, \mathbb{R}) \to \mathrm{Aut}(E^k)$
$\rho_k(g)(\alpha \otimes f) = \lambda^{*\wedge k}(g)\alpha \otimes \rho(g) f$, $g \in Mp(2n, \mathbb{R})$ and
$\alpha \in \bigwedge^k (\mathbb{R}^{2n})^*$, $f \in H$
$\mathcal{E}^k = \mathcal{Q} \times_{\rho_k} E^k$
$\mathcal{E}^0 = \mathcal{H} = \mathcal{Q} \times_\rho H$ (oscillator bundle)

$\sigma : Mp(2n, \mathbb{R}) \to \mathrm{Aut}(K(H))$
$\sigma(g) T = \rho(g) T \rho(g)^*$
$\mathcal{K} = \mathcal{Q} \times_\sigma K(H)$
Azumaya bundle - bundle of Azumaya algebras (A. Grothendieck)
Bundle of measuring devices ("Filtern"), matrix densities

$K(H)$-structure - a Recall

Hilbert $K(H)$-module structure on the $Mp(2n,\mathbb{R})$-module E
E is a $K(H)$-module with respect to the action
$E \times K(H) \to E$ by $(\alpha \otimes f) \cdot T = \alpha \otimes f \cdot T = \alpha \otimes T^*(f)$
$(,) : E \times E \to K(H)$
$(\alpha \otimes f, \alpha' \otimes f') = g(\alpha, \alpha') f \otimes (f')^* \in K(H)$

Bundle lifts

$\mathcal{E} = \mathcal{Q} \times_\rho E$ is the $Mp(2n, \mathbb{R})$-associated vector bundle
Bundle lifts of the Hilbert $K(H)$-module structures
We need: $\mathcal{E} \times A \to \mathcal{A}, (,) : \mathcal{E} \times \mathcal{E} \to A$
Theorem (N. Kuiper): Any infinite rank Hilbert bundle is trivial (a product) bundle.
Consequence (K): $\mathcal{K} \to M$ is also trivial.
Idea of the proof: $\mathcal{H} \to M$ is trivial: $\exists$ a trivialization $\phi : M \times H \to \mathcal{H}$
It induces $(\exists)$ a trivialization $\psi : M \times A \to \mathcal{K}$
$\psi = \overline{\phi^* \widehat{\otimes} \phi}$ $\square$
$\psi : M \times A \to \mathcal{K}$

Final step of construction of bundle lifts

The A-bundle structure:
$\cdot : \mathcal{E} \times A \to \mathcal{E}$, $[(q, v)] \cdot a = [(q, v \cdot a)]$, where
$q \in \mathcal{Q}$, $v \in E$ and $a \in A$.
The A-product:
1) First reduce the $Mp(2n, \mathbb{R})$-bundle $\mathcal{Q}$ to the structure group
$\widetilde{U(n)}$
(in order the next maps are well defined)
2) $(,) : \mathcal{E} \times \mathcal{E} \to A$ by setting
3)

$$([(q, v)], [(q, w)]) = \mathrm{pr}_2(\psi^{-1}([(q, (v, w))])),$$

where $q \in \mathcal{Q}$ and $v, w \in E$.

Oscillator bundle twisted de Rham complex

Let (M,ω) be a symplectic manifold admitting a metaplectic section and let $\phi : M \to \mathcal{H}$ be a global trivializing section
Choice of the section $\phi \in \Gamma(M,\mathcal{H})$ defines a flat connection

$$\nabla : \mathfrak{X} \times \Gamma(M,\mathcal{H}) \to \Gamma(M,\mathcal{H})$$

on $\mathcal{H}$
Choose a local frame $(e_i)_{i=1}^{2n}$ and the dual co-frame $(\epsilon^i)_{i=1}^{2n}$ $(epsilon^i(e_j) = \delta^i_j)$.

$$d_k^{\nabla}(\alpha \otimes s) = d\alpha \otimes s + (-1)^{\mathrm{deg}(\alpha)} \epsilon^i \wedge \alpha \otimes \nabla_{e_i} s$$

∇ is flat $\Longrightarrow D^\bullet = (d_k^{\nabla}, \Gamma(M,\mathcal{E}^k))_{k\in\mathbb{N}_0}$ is a complex of pseudodifferential operators in finitely generated projective $K(H)$-Hilbert bundles

'Final' theorem

Theorem (K): Let (M, ω) be a compact symplectic manifold admitting a metaplectic structure $\mathcal{Q}$ and ϕ be a trivializing section of the oscillator bundle $\mathcal{H}$, then

$$D^\bullet = (d_k^\nabla, \Gamma(M, \mathcal{E}^k))_{k \in \mathbb{N}_0}$$

is a complex, the cohomology groups of which are **finitely generated projective** Hilbert $K(H)$-modules and the Hodge theory holds for it. In particular,

$$\Gamma(M, \mathcal{E}^k) = \mathrm{Ker} \triangle_k \oplus \mathrm{Im}\, d_k^{\nabla *} \oplus \mathrm{Ker}\ , d_k^\nabla$$
$$H^k(D^\bullet, K(H)) \simeq \mathrm{Ker}\, \triangle_k.$$

Remarks

Remarks:

First specific use of the C^*-bundles in the case not of Dirac (or from Dirac derived) operators

First specific construction of a Dirac-type operator which is not constructable of the previously known Dirac operators in Riemannian (spin) geometry...

Future:

Treat the symplectic Dirac of Habermann and derived operators as symplectic twistor operators

Homological aspects: sheaf homology, Künneth type theorem

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