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ANALYSIS OVER C^*-ALGEBRAS AND THE OSCILLATORY REPRESENTATION

SVATOPLUK KRÝSL

Faculty of Mathematics and Physics, Charles University in Prague
Sokolovská 83, Praha 8 - Karlín, 186 75, Czech Republic

Abstract. Since the last two decades, several differential operators appeared in connection with the so-called oscillatory geometry. These operators act on sections of infinite rank vector bundles. Definitions of the oscillatory representation, metaplectic structure, oscillatory Dirac operator, as well as some fundamental results of the analysis in C^*-Hilbert bundles are recalled in this paper. These results are used for a description of the kernel of a certain second order differential operator arising from oscillatory geometry and of the cohomology groups of the de Rham complex of exterior forms with values in the oscillatory representation.

1. Introduction

In the 60's, when quantizing solutions to the Klein-Gordon equation, Shale found a non-trivial projective unitary representation of the symplectic group $\mathrm{Sp}(V,\omega)$. (See Shale [19].) Short after, Weil in [23] made it a true representation of the metaplectic group, the connected double cover of the appropriate symplectic group. In this paper, this representation is called the *oscillatory representation*. The underlying vector space of this representation is the Hilbert space $S = L^2(L)$ of square Lebesgue integrable functions on a Lagrangian subspace L of the symplectic space (V,ω). For a suitable class of symplectic manifolds (M,ω), Kostant used this representation to derive a quantization procedure for Hamiltonian mechanics by introducing the *metaplectic structure* on M and the *oscillatory bundle* $\mathcal{S} \to M$, often called the symplectic spinor bundle. See Kostant [13]. The fibers of $\mathcal{S}$ are isometrically isomorphic to the carrier $S = L^2(L)$ of the oscillatory representation. Using the oscillatory connection ∇^S on $\mathcal{S}$, which is induced by a symplectic connection ∇ on the underlying manifold, Habermann defined a symplectic analogue of the classical Dirac operator. (See Habermann [8].) This operator, which we denote by $\mathfrak{D}$ and call it the *oscillatory Dirac operator* acts on sections $s \in \Gamma(M, \mathcal{S})$ of the oscillatory bundle $\mathcal{S}$.

One can introduce a further operator by the prescription $\mathfrak{P} = \mathrm{i}(\widetilde{\mathfrak{D}}\mathfrak{D} - \mathfrak{D}\widetilde{\mathfrak{D}})$, where $\widetilde{\mathfrak{D}}$ is an orthogonal version of $\mathfrak{D}$, defined using a compatible almost complex structure on (M, ω). This operator, acting on $\Gamma(M, \mathcal{S})$ as well, is elliptic in the sense that its symbol is an automorphism of $\mathcal{S}$ out of the zero-section of the cotangent bundle. Already on the 2-sphere, the spectrum of $\mathfrak{P}$ turns out to be unbounded from both sides, and the kernel of $\mathfrak{P}$ is infinite dimensional. (See Habermann, Habermann [9].) This is an exact opposite to what holds for elliptic operators in finite rank vector bundles over compact manifolds. Since the time of introducing of $\mathfrak{P}$, further elliptic operators appearing in oscillatory geometry were studied and also similar deflections in their behavior from the behavior of "the classical" elliptic operators were found. By classical operators we mean the ones acting in finite rank vector bundles. The base manifolds are supposed to be compact. See, e.g., Cahen et al. [2] and Korman [11], [12] for a study of Dolbeualt type operators acting in sections of the (infinite rank) oscillatory bundle. They use the trick of Habermann, based on a splitting of $\mathcal{S}$ into certain finite rank unitary bundles, and study spectral properties of these "deflective" operators using the representation theory of compact Lie groups.

We decided to explain the different behavior of these newly appeared operators as a generalization of the behavior of the classical ones. The structures we use in order to do this are C^*-algebras, Hilbert C^*-modules and C^*-Hilbert bundles. Our reference for C^*-algebras is Arveson [1], and for Hilbert C^*-modules the text-book of Lance [17]. For the readers convenience, we recall their definitions in this paper briefly. For the C^*-Hilbert bundles, we refer, e.g., to Solovyov, Troitsky [20]. A Hilbert C^*-bundle is a smooth generalization of the notion of "champs continus de C^*-algèbres" (see Dixmier [3]).

Differential operators acting between sections of C^*-Hilbert bundles and their ellipticity can be defined similarly as in the finite rank case, i.e., by partial derivatives in local coordinates and by symbol maps, respectively. Let us mention, that Fomenko and Mishchenko proved in [6], that the kernels of extensions of elliptic operators to certain **completions** of the space of smooth sections are finitely generated projective Hilbert C^*-modules if the C^*-Hilbert bundles are finitely generated and projective. In Krýsl [15], these results were used in the case of *elliptic complexes* and **smooth sections** to prove a generalization of the Hodge theory of elliptic complexes of operators acting in finite rank vector bundles. In the mentioned article, the author proves that under a certain condition on the so-called associated Laplacians, the cohomology groups of an elliptic complex in finitely generated projective A-Hilbert bundles over a compact manifold are finitely generated A-modules and Banach spaces.

As far as we know, the mentioned differently behaved operators were studied without a use of the *analysis over C^*-algebras*, till yet, and this sort of analysis wasn't used in the case of examples of specific complexes for which classical methods cannot be used. The purpose of this paper is to summarize basic facts on this topic, and use them for the complex of exterior forms tensored by the oscillatory bundle and for the operator $\mathfrak{P}$. We gain information on the cohomology groups of this complex and on the kernel of $\mathfrak{P}$. Under conditions specified in the text, the kernel as well as the cohomology groups appear to be finitely generated as C^*-modules and Banach spaces. Let us notice that we are motivated by the idea of a quantum theory for fields "displaced in the points of the phase space", and by deformation and geometric quantization. See e.g. Kostant [13], Fedosov [4] and Habermann, Habermann [9] for sources of these ideas.

In the second section, we recall notions from symplectic linear algebra, introduce the oscillatory representation and show some of its applications within harmonic analysis (eigenvalues of Fourier transform) and its connections with quantum mechanics (harmonic operator). The eigenvalues of the Fourier transform are computed there using basic properties of the oscillatory representation (Theorem 2). In the third section, we collect information on Fedosov and oscillatory geometry, including a definition of the oscillatory Dirac operator. The fourth section is devoted to a repetition of Hodge theory of elliptic complexes in finite rank bundles. The fifth part of the text starts by a recollection of results of Habermann on the kernel of $\mathfrak{P}$. In the second part of this chapter, we present some basic definitions from the theory of Hilbert C^*-modules and formulate a theorem on elliptic complexes in finitely generated projective C^*-Hilbert bundles (Theorem 8). In the last part of the fifth section, we use this theorem to describe the cohomology groups of de Rham complex with values in the oscillatory representation (Corollary 9) and the kernel of $\mathfrak{P}$ (Corollary 11).

2. Symplectic Linear Algebra and the Oscillatory Representation

Let (V, ω) be a real symplectic vector space of dimension $2n$. In particular, $\omega : V \times V \to \mathbb{R}$ is a non-degenerate anti-symmetric bilinear form. We will need the technical notion of a symplectic basis. This is a basis $(e_i)_{i=1}^{2n}$ of V for which $\omega(e_i, e_j) = 1$ if and only if $i = 1, \ldots, n$ and $j = i + n$; $\omega(e_i, e_j) = -1$ if and only if $i = n+1, \ldots, 2n$ and $j = i - n$; and $\omega(e_i, e_j) = 0$ otherwise. Using a process similar to the Gram-Schmidt orthogonalization (one just uses a symplectic form instead of a scalar product), it is possible to prove the existence of a symplectic basis. We fix such a basis for the rest of this paper and set $\omega_{ij} = \omega(e_i, e_j)$, $i, j = 1, \ldots, 2n$. The numbers ω^{ij} are uniquely defined by the equations

$\sum_{k=1}^{2n} \omega_{ik}\omega^{jk} = \delta_i^j$, $i,j = 1,\ldots,2n$. With respect to a symplectic basis

$$(\omega_{ij})_{i,j=1,\ldots,n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where 0 and 1 denote the $n \times n$ zero and unit matrix, respectively. In symplectic linear algebra, the order of indices i,j in ω^{ij} and ω_{ij} by which we rise or lower indices of tensors matters. We use the following convention. If $S_{ab\ldots c\ldots d}{}^{rs\ldots t\ldots u}$ is a tensor, we denote by $S_{ab\ldots}{}^{i}{}_{\ldots d}{}^{rs\ldots u}$ the tensor $\sum_{c=1}^{2n} \omega^{ic} S_{ab\ldots c\ldots d}{}^{rs\ldots u}$, and by $S_{ab\ldots d}{}^{rs\ldots}{}_{i}{}^{\ldots u}$ the tensor $\sum_{t=1}^{2n} S_{ab\ldots c\ldots d}{}^{rs\ldots t\ldots u}\omega_{ti}$.
It is well known that the symplectic group

$$G = \mathrm{Sp}(V,\omega) = \{A : V \to V\,;\, \omega(Av,Aw) = \omega(v,w) \text{ for each } v,w \in V\}$$

is smoothly retractable onto the unitary group $\mathrm{U}(n)$, the homotopy group of which is $\mathbb{Z}$. Thus, $\mathrm{Sp}(V,\omega)$ has a non-universal connected double covering, the so-called metaplectic group $\widetilde{G} = \mathrm{Mp}(V,\omega)$. Let us denote the appropriate covering homomorphism by $\lambda : \mathrm{Mp}(V,\omega) \longrightarrow \mathrm{Sp}(V,\omega)$. Thanks to this $2:1$ map, the relationship of the group $\mathrm{Mp}(2n,\mathbb{R})$ to the group $\mathrm{Sp}(2n,\mathbb{R})$ is similar to the one of the spin group $Spin(m,\mathbb{R})$ to the special orthogonal group $\mathrm{SO}(m,\mathbb{R})$, $m \in \mathbb{N}$.

2.1. The Segal-Shale-Weil or the Oscillatory Representation

Let us set $L = \mathcal{L}(\{e_i\,;\, i = 1,\ldots,n\})$. In particular, L is a Lagrangian subspace of (V,ω), i.e., a maximal isotropic subspace of (V,ω). There exists a distinguished non-trivial unitary representation ρ of $\mathrm{Mp}(V,\omega)$ which can be realized, for the chosen Lagrangian space L, as a homomorphism

$$\rho : \widetilde{G} \to \mathcal{U}(L^2(L)),$$

where $\mathcal{U}(L^2(L))$ denotes the group of unitary operators on the Hilbert space $S = L^2(L)$. Due to its inventors, this representation is known as the Segal-Shale-Weil representation. Sometimes, it is called metaplectic, symplectic spinor or oscillatory. We use the name *oscillatory representation* in this text. It is known that this representation splits into two irreducible submodules, the spaces of odd and even complex valued square Lebesgue integrable functions on L (modulo the equivalence of being equal almost everywhere). Declaring the chosen symplectic basis $(e_i)_{i=1}^{2n}$ to be orthonormal defines a scalar product g on V. Further, the equation $g(u,v) = \omega(u,Jv)$ determines a linear map $J : V \to V$ due to the non-degeneracy of ω. Its existence can be proved using the basis $(e_i)_{i=1}^{2n}$ by defining the matrix of J with respect to this basis to be equal to the matrix of ω given above and by checking that the defining equation for J and the identity $J^2 = -1_{|V}$ are satisfied. The uniqueness of J (for the chosen g) follows from the non-degeneracy of ω.

Lemma 1. *Let (V, ω) be a symplectic vector space and $J : V \to V$ be an endomorphism of V such that $J^2 = -1_{|V}$, and such that $g(u, v) = \omega(u, Jv)$ $(u, v \in V)$ defines a non-degenerate bilinear form on V. Then*

1) $J = -J^t$
2) $J \in \mathrm{O}(V, g) \cap \mathrm{Sp}(V, \omega)$

Proof: Let us write $g(J^t u, v) = g(u, Jv) = g(Jv, u) = \omega(Jv, Ju) = -\omega(Ju, Jv) = -g(Ju, v)$ from which, due to the non-degeneracy of g, we get $J = -J^t$, i.e., J is anti-symmetric.

Further, $g(Ju, Jv) = g(u, J^t Jv) = -g(u, J^2 v) = g(u, v)$, thus J is orthogonal. Let us compute $\omega(Ju, Jv) = g(Ju, v) = g(u, J^t v) = -g(u, Jv) = -\omega(u, JJv) = \omega(u, v)$ from which $J \in \mathrm{Sp}(V, \omega)$. ■

(Note that the endomorphisms J satisfying the assumption of Lemma 1 are called compatible almost complex structures.) The compatible almost complex structure J described before Lemma 1 by its matrix representation will be fixed for the rest of this section. It is known that there exists an element $\sigma \in \lambda^{-1}(J)$ for which $\sigma^4 = 1 \in \mathrm{Mp}(V, \omega)$ and $\rho(\sigma) = \mathcal{F}^{-1}$, where $\mathcal{F} : L^2(L) \to L^2(L)$ is the Fourier transform on $L^2(L)$. For convenience of the reader, we present a prescription for ρ on other elements of $\widetilde{G}$. For any $g \in \mathrm{Sp}(V, \omega)$, let $\tilde{g} \in \mathrm{Mp}(V, \omega)$ denote an element from the two-point set $\lambda^{-1}(g)$. Further let $A \in \mathrm{End}(L)$ be symmetric $(A^t = A)$ and $B \in \mathrm{GL}(L)$. Then with respect to the symplectic basis $(e_i)_{i=1}^{2n}$, we have

$$g_1 = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}, \quad (\rho(\tilde{g_1})f)(x) = e^{-\mathrm{i}g(Ax,x)/2} f(x)$$

$$g_2 = \begin{pmatrix} B & 0 \\ 0 & (B^t)^{-1} \end{pmatrix}, \quad (\rho(\tilde{g_2})f)(x) = \sqrt{\det B} f(B^t x)$$

$$g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\rho(\tilde{g_3})f)(x) = \pm e^{\mathrm{i}n\pi/4} \mathcal{F}^{-1} f(x),$$

where $f \in S$ and $x \in L$. The elements of type g_1, g_2 and g_3 above generate $\mathrm{Sp}(V, \omega)$. (For it, see, e.g., Folland [5].) Item 2 should be understood in the sense that there exists a branch of the square root, such that the prescription is valid for all $f \in S$ and $x \in L$. The sign in item 3 depends on the choice of $\tilde{g}_3$ in $\lambda^{-1}(g_3)$. The existence of ρ is connected with the Stone-von Neumann theorem on unitary representations of the Heisenberg group deeply. See, e.g., Habermann, Habermann [9] and Kashiwara, Vergne [10] or Weil [23] for more information.

For $i = 1, \ldots, n$, we define the following unbounded operator $e_i. : L^2(L) \rightarrowtail L^2(L)$ acting on a dense subspace of $L^2(L)$. We set

$$(e_i.f)(x) = \mathrm{i}x^i f(x), \quad (e_{i+n}.f)(x) = \frac{\partial f}{\partial x^i}(x),$$

where $f \in L^2(L)$ and $x = \sum_{i=1}^n x^i e_i$. These relations resemble the *canonical quantization prescription*. For a general vector $v \in V$, we set $v.s = \sum_{i=1}^{2n} v^i(e_i.s)$, where $v = \sum_{i=1}^{2n} v^i e_i$, i.e., we extend the canonical quantization prescription linearly. This so-called symplectic Clifford multiplication satisfies for all $v, w \in V$ and $s \in S$ the relation

$$v.w.s - w.v.s = -\mathrm{i}\omega(v,w)s$$

as one can check easily on the basis elements for instance. This relation differs from the one for the orthonormal Clifford multiplication in anti-symmetry of the left-hand side, a property forced by the anti-symmetry of ω. Moreover, this multiplication (of the oscillatory vectors by the phase space vectors) is also equivariant with respect to $\widetilde{G}$, i.e., $\rho(g)(v.s) = (\lambda(g)v).(\rho(g)s)$ holds for each $v \in V$, $s \in S$ and $g \in \widetilde{G}$. (See Lemma 1.4.4. in Habermann, Habermann [9], pp 13, for a proof of this statement.) Note that the symplectic Clifford algebra $sCliff(V,\omega)$, defined as the quotient of the tensor algebra $\mathcal{T}V = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ by the ideal generated by non-homogeneous elements of the form $v \otimes w - w \otimes v + \mathrm{i}\omega(v,w)1$, is infinite dimensional. We have a canonical isomorphism $sCliff(V,\omega) \cong \bigoplus_{i=0}^{\infty} S^i(V)$ as vector spaces, where $S^i(V)$ denotes the i-th symmetric power of V. Thus, $sCliff(V,\omega)$ is isomorphic to the space of polynomials on V. However, at the level of algebras, these structures are different since the polynomials are commutative, whereas $sCliff(V,\omega)$ is not.

2.2. Quantum Harmonic Oscillator

Let us form an unbounded operator $H : S \rightarrowtail S$ by setting

$$Hs = -\frac{1}{2}\sum_{i,j=1}^{2n} \omega^{ij}(Je_i).e_j.s,$$

where $s \in S \cap \mathcal{C}^2(L)$. Operator H is independent on the choice of an orthogonal basis. In coordinates, we have

$$Hs = \frac{1}{2}\sum_{i=1}^{n}\left(-\frac{\partial^2}{\partial(x^i)^2} + (x^i)^2\right)s = \frac{1}{2}(-\triangle + ||x||^2)s, \ s \in S,$$

where $\triangle$ is the Laplace operator on L and $||\ ||$ is the norm on L, both induced by the scalar product g restricted to L. The operator H is the quantum Hamiltonian of the n-dimensional isotropic harmonic oscillator ($\hbar = \omega = m = 1$). It is known to be essentially self-adjoint. (See, e.g., Theorem 8.5 in Teschl [21], pp 179.) For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, the so-called generalized Hermite function $h_\alpha : L \to \mathbb{C}$ is defined by

$$h_\alpha(x) = h_{\alpha_1}(x^1)\ldots h_{\alpha_n}(x^n),$$

where $x = \sum_{i=1}^n x^i e_i$ and $h_k(x) = e^{x^2/2} \frac{\mathrm{d}^k}{\mathrm{d}x^k}(e^{-x^2})$, $x \in \mathbb{R}$, is the k-th Hermite function (in variable x). Notice that usually, one chooses a specific normalization and only the normalized $h_k(x)$ are called Hermite functions. It is known that for each $\alpha \in \mathbb{N}_0^n$, the generalized Hermite function h_α is an eigenfunction of H with eigenvalue $|\alpha| + \frac{n}{2}$, where $|(\alpha_1, \dots, \alpha_n)| = \sum_{i=1}^n \alpha_i$.
The next result goes back to Norbert Wiener at least. Using the properties of ρ, its proof becomes almost trivial. (Of course, the analytic work was done in proving that ρ as given above is a representation.)

Theorem 2. *Each eigenvalue of the Fourier transform belongs to the set $\{\pm 1, \pm \mathrm{i}\}$ and the set of its eigenfunctions coincides with the set of generalized Hermite functions.*

Proof: We have $\mathcal{F}^4 = \rho(\sigma)^4 = \rho(\sigma^4) = \rho(1) = 1_{|S}$. Thus, each eigenvalue of $\mathcal{F}$ is an element of the set $\{1, \mathrm{i}, -1, -\mathrm{i}\}$. Further, for $s \in L^2(L) \cap \mathcal{C}^2(L)$, we have

$$\begin{aligned}
-2\mathcal{F}Hs &= \rho(\sigma) \sum_{i,j=1}^{2n} \omega^{ij} (Je_i).e_j.s = \sum_{i,j=1}^{2n} \omega^{ij} (\lambda(\sigma) Je_i).(\rho(\sigma)(e_j.s)) \\
&= \sum_{i,j=1}^{2n} \omega^{ij} (JJe_i).(\lambda(\sigma)e_j).(\rho(\sigma)s) = -\sum_{i,j=1}^{2n} \omega^{ij} e_i.(Je_j).(\mathcal{F}s) \\
&= -\sum_{i,j=1}^{2n} \omega^{ij} (-\mathrm{i}\omega(e_i, Je_j) + (Je_j).e_i.)(\mathcal{F}s) \\
&= -\sum_{i,j=1}^{2n} \omega^{ij} (-\mathrm{i}g(e_i, e_j) + (Je_j).e_i.)(\mathcal{F}s) = -\sum_{i,j=1}^{2n} \omega^{ij} (Je_j).e_i.(\mathcal{F}s) \\
&= \sum_{i,j=1}^{2n} \omega^{ij} (Je_i).e_j.(\mathcal{F}s) = -2H\mathcal{F}s,
\end{aligned}$$

where we used the fact that $J^2 = -1_{|V}$. Thus, we see that the Fourier transform and the Hamiltonian H commute. Now, let us consider $n = 1$. Denoting the eigenvalue $(k + \frac{1}{2})$ of H corresponding to h_k by μ_k, we have $\mathcal{F}Hh_k = \mu_k \mathcal{F}h_k = H(\mathcal{F}h_k)$. It is known, that any eigenvector of H is a complex multiple of some h_k with eigenvalue μ_k. According to the above computation, the vector $\mathcal{F}h_k$ is an eigenfunction of H with eigenvalue μ_k. Hence, we get $\mathcal{F}h_k = c_k h_k$ for a complex number c_k. For a general $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, we have

$$\begin{aligned}
\mathcal{F}h_\alpha &= \mathcal{F}(h_{\alpha_1} h_{\alpha_2} \dots h_{\alpha_n}) = (\mathcal{F}_{x^1} h_{\alpha_1}) \dots (\mathcal{F}_{x^n} h_{\alpha_n}) \\
&= c_{\alpha_1} \dots c_{\alpha_n} h_{\alpha_1} \dots h_{\alpha_n} = c h_\alpha,
\end{aligned}$$

where $c = c_{\alpha_1} \dots c_{\alpha_n}$ and $\mathcal{F}_{x^i}$ denotes the Fourier transform in the variable x^i, $i = 1, \dots, n$. The above factorization of the multi-dimensional Fourier transform is possible due to the shape of the generalized Hermite functions and the Fubini theorem. Thus, h_α are eigenfunctions of $\mathcal{F}$ for each $\alpha \in \mathbb{N}_0^n$. Since S equals the completed (Hilbert) sum $\widehat{\bigoplus}_{k=0}^{\infty}(\bigoplus_{\alpha \in \mathbb{N}_0^n, |\alpha|=k} \mathbb{C}h_\alpha)$, the theorem follows. ■

3. Oscillatory Geometry

Let (M, ω) be a symplectic manifold, i.e., for each $m \in M$, the pair $(T_m^* M, \omega_m)$ is a symplectic vector space and $d\omega = 0$. Typical examples of symplectic manifolds are Kähler manifolds or cotangent bundles. There exist compact symplectic manifolds which are not Kähler. Recall, e.g., the Kodaira-Thurston manifold which is historically the first known example of a compact non-Kähler symplectic manifold.

Due to a theorem of Darboux, for any point $m \in M$, there exists a neighborhood $U \ni m$ and coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ on U such that $\omega_{|U} = \sum_{i=1}^n dp_i \wedge dq^i$. Notice, that in the case of a Riemannian manifold (N, g), a similar local normalization cannot be done in general. (Of course, due to the quadratic forms inertia theorem, one can do an appropriate normalization point-wise.)

Definition 1. *An affine connection ∇ on a symplectic manifold (M, ω) is called symplectic if $\nabla\omega = 0$, and it is called a Fedosov connection if in addition, ∇ is torsion-free, i.e., $T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$ for all vector fields $X, Y \in \mathfrak{X}(M)$.*

As of any covariant derivative, the curvature tensor R^∇ of a symplectic or Fedosov connection ∇ is also defined by the formula $R^\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, where $X, Y, Z \in \mathfrak{X}(M)$. The curvature tensor R^∇ of a Fedosov connection possesses symmetries similar however not identical to the ones of the curvature tensor of a Riemannian connection. Let us define the tensor coordinates R_{ijkl} by setting $R(e_i, e_j)e_k = R_{ijk}{}^l e_l$, where $(e_i)_{i=1}^{2n}$ is a local symplectic frame. We have (see, e.g., Habermann, Habermann [9])

$$R_{ijkl} = -R_{jikl}$$
$$R_{ijkl} = R_{ijlk}$$
$$R_{ijkl} + R_{lijk} + R_{klij} + R_{jkli} = 0.$$

Note that in Vaisman [22], where the symmetry relations for the curvature of a Fedosov connection are investigated using the representation theory of symplectic groups, a different convention for the indices ordering in the completely covariant form of the tensor R^∇ is used.

In the case of a Riemannian manifold (N, g) and its Riemann connection ∇^g, the equation $R^{\nabla^g} = 0$ holds if and only if N is locally isometric to a Euclidean space.

Let us consider the 2-sphere with a fixed metric. Note that its Riemannian connection is also a Fedosov connection for the volume form of the metric as the symplectic form. (See also Example 4 below.) We know that the curvature of this connection is non-zero. However, due to the Darboux theorem, the sphere is locally isomorphic (symplectomorphic) to the standard symplectic plane. Thus, we cannot have the same interpretation of the Fedosov connection as of the Riemannian one. Moreover, it is known that the space of Fedosov connections is isomorphic to the infinite dimensional affine space modeled on the vector space of symmetric tensor fields of type $(3, 0)$ on M. (See Gelfand, Retakh, Shubin [7].)
Now, let us concentrate to the oscillatory structures. Let (M^{2n}, ω) be a symplectic manifold. For any point $m \in M$, we define

$$P_m = \{b = (e_1, \dots, e_{2n})\,;\, b \text{ is a symplectic basis of } (T^*_m M, \omega_m)\}$$

and set $P = \bigcup_{m \in M} P_m$ for the space of symplectic repères. Let $p : P \to M$ denote the foot-point projection. The topology on P is, by definition, the coarsest one for which p is continuous. Obviously, P is equipped with an appropriate action of $\mathrm{Sp}(V, \omega)$ from the right.

Definition 2. *Let $q : Q \to M$ be a principal $\mathrm{Mp}(V, \omega)$-bundle over M and $\Lambda : Q \to P$ be a surjective bundle homomorphism. A pair (Q, Λ) is called a metaplectic structure if the following diagram commutes*

$$\begin{tikzcd}
Q \times \mathrm{Mp}(V,\omega) \arrow[r] \arrow[dd, "\Lambda\times\lambda"] & Q \arrow[dd, "\Lambda"] \arrow[dr, "q"] & \\
& & M \\
P \times \mathrm{Sp}(V,\omega) \arrow[r] & P \arrow[ur, "p"] &
\end{tikzcd}$$

The horizontal arrows in the diagram represent the actions of the appropriate groups on the corresponding principal bundles.

It is known that for each symplectic manifold (M, ω), there exists a *compatible almost complex structure* $J : TM \to TM$, i.e., a map that satisfies $J^2 = -1_{|TM}$ and such that $g(X, Y) = \omega(X, JY)$, $X, Y \in \mathfrak{X}(M)$, is a Riemannian metric on M. Further, it is known that (M, ω) admits a metaplectic structure if and only if the first Chern class $c_1(TM)$ of the hermitian bundle (TM, J) is an even element in the $\mathbb{Z}$-module $H^2(M, \mathbb{Z})$, i.e., there exists an element $a \in H^2(M, \mathbb{Z})$ such that $c_1(TM) = 2a$. Moreover, the Chern class does not depend on the choice of the compatible almost complex structure J. If a metaplectic structure exists, the elements of the set of their equivalence classes are parametrized by the cohomology group $H^1(M, \mathbb{Z}_2)$. Two metaplectic structures (Q, Λ) and $(\widetilde{Q}, \widetilde{\Lambda})$ are equivalent if there exists a principal bundle isomorphism $\phi : Q \to \widetilde{Q}$ of the principal $\mathrm{Mp}(V,\omega)$-bundles $Q \to M$ and $\widetilde{Q} \to M$ such that $\tilde{\Lambda} \circ \phi = \Lambda$. See Kostant [13] for results mentioned in this paragraph.

Remark 3. *One can define the so-called complex metaplectic (or* Mp^c*) structure which in known to exist on any symplectic manifold. See Robinson, Rawnsley [18].*

Now, want to proceed from the principal bundles to vector bundles. At any point $m \in M$, we replace $Q_m = q^{-1}(\{m\})$ by $S = L^2(L)$, and do it equivariantly with respect to the representation ρ. Formally, one sets

$$\mathcal{S} = Q \times_\rho S = (Q \times L^2(L))/\sim,$$

where $\sim$ is an equivalence relation on $Q \times L^2(L)$ defined by $(r, f) \sim (t, h)$ if and only if $r = tg$ and $h = \rho(g)f$ for an element $g \in \mathrm{Mp}(V,\omega)$, $(r, h), (t, f) \in Q \times S$. We call $\mathcal{S} \to M$ the *oscillatory bundle*. The topology, we take on $\mathcal{S}$ is the quotient one.

Because the symplectic Clifford multiplication is equivariant (see section 2.1, it lifts to the oscillatory bundle. Thus we get a map $TM \times \mathcal{S} \to \mathcal{S}$. Let ∇ be a symplectic connection on (M,ω). This connection induces a principal connection on the principal $\mathrm{Sp}(V,\omega)$-bundle $P \to M$. If (M,ω) possesses a metaplectic structure, we can lift this connection to a principal connection on Q. Any connection on a principal bundle induces a connection on its associated bundles. In the case of the bundle $Q \to M$ and the associated bundle $\mathcal{S} \to M$, we denote the resulting connection by $\nabla^S : \Gamma(M,\mathcal{S}) \to \Gamma(M, T^*M \otimes \mathcal{S})$ and call it the *oscillatory derivative*. Its curvature is given by the formula

$$R^S(X,Y)s = \nabla^S_X \nabla^S_Y s - \nabla^S_Y \nabla^S_X s - \nabla^S_{[X,Y]} s,\ X, Y \in \mathfrak{X}(M), s \in \Gamma(M,\mathcal{S}).$$

See Habermann, Habermann [9] for more information on the facts in this paragraph.

The curvature can be computed using the following

Theorem 4. *If* R^∇ *is the curvature tensor of a Fedosov connection* ∇ *of a symplectic manifold* (M,ω) *admitting a metaplectic structure, then the curvature* R^S *of the oscillatory derivative* ∇^S *fulfills locally, on a neighborhood* $U \subseteq M$*,*

$$R^S s = \frac{\mathrm{i}}{2} \sum_{i,j,k,l=1}^{2n} R_{ij}{}^{kl} \epsilon^i \wedge \epsilon^j \otimes e_k.e_l.s,$$

where $(\epsilon_i)_{i=1}^{2n}$ *is the co-frame dual to a symplectic frame* $(e_i)_{i=1}^{2n}$ *on* U*.*

Proof: See Habermann, Habermann [9]. ■

3.1. Oscillatory Dirac Operator

For symplectic geometry, we would like to define a differential operator playing a similar role as the Dirac operator in Riemannian geometry. Unfortunately, we cannot expect this operator to have a similar simple interpretation as the Riemannian Dirac operator which can be thought as a square root of the Laplacian, at least on a plane. The scalar Laplacian in symplectic geometry would be of the local form $\sum_{i,j=1}^{2n} \omega^{ij}\partial_i\partial_j$ (with respect to some local Darboux coordinates), which is the zero map.

A symplectic analogue of the Riemannian Dirac operator was introduced by Habermann in [8] using the oscillatory derivative ∇^S. Let us sketch this construction briefly. Let (M,ω) be a symplectic manifold, ∇ be a symplectic connection and (Q,Λ), if it exists, be a metaplectic structure on (M,ω). Let $\mathcal{S} \to M$ denote the oscillatory bundle and ∇^S the oscillatory derivative. Then for $s \in \Gamma(M,\mathcal{S})$, we define the symplectic spinor or *oscillatory Dirac operator* $\mathfrak{D} : \Gamma(M,\mathcal{S}) \to \Gamma(M,\mathcal{S})$ by the formula $\mathfrak{D} = Y \circ \nabla^S$, where $Y : \Gamma(M, T^*M \otimes \mathcal{S}) \to \Gamma(M,\mathcal{S})$ is given by $Y(\alpha \otimes s) = \sum_{i,j=1}^{2n} \omega^{ij}(\iota_{e_i}\alpha)e_j.s$, where $\alpha \otimes s \in \Gamma(M, T^*M \otimes \mathcal{S})$ and ι denotes the insertion of a vector field into a differential form. Thus, locally, the oscillatory Dirac operator is given by the formula

$$\mathfrak{D}s = \sum_{i,j=1}^{2n} \omega^{ij} e_i.\nabla^S_{e_j} s,$$

where $U \subseteq M$ is a neighborhood in M, $s \in \Gamma(U,\mathcal{S})$ and $(e_i)_{i=1}^{2n}$ is a local symplectic frame on $(U, \omega_{|U})$.

Example 1. For the canonical symplectic vector space $(\mathbb{R}^{2n},\omega)$, we have $H^2(\mathbb{R}^{2n},\mathbb{Z}) = 0$. Thus necessarily, $c_1(T\mathbb{R}^{2n}) = 0$ which is an even element. Due to the universal coefficient theorem, we have the exact sequence

$$0 \to \mathrm{Ext}(H_0(\mathbb{R}^{2n},\mathbb{Z}),\mathbb{Z}_2) \to H^1(\mathbb{R}^{2n},\mathbb{Z}_2) \to \mathrm{Hom}(H_1(\mathbb{R}^{2n},\mathbb{Z}),\mathbb{Z}_2) \to 0.$$

Evaluating the homology and cohomology groups, we get $0 \to \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_2) \to H^1(\mathbb{R}^{2n},\mathbb{Z}_2) \to \mathrm{Hom}(0,\mathbb{Z}_2) \to 0$, i.e., $0 \to 0 \to H^1(\mathbb{R}^{2n},\mathbb{Z}_2) \to 0 \to 0$, which implies $H^1(\mathbb{R}^{2n},\mathbb{Z}_2) = \{0\}$. Thus up to an isomorphism, there is only one metaplectic structure, and consequently, it is the product one, i.e., $Q = \mathrm{Mp}(2n,\mathbb{R}) \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. It follows that $\mathcal{S} \cong S \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Thus, any element $s \in \Gamma(\mathbb{R}^{2n},\mathcal{S})$ can be represented by a function $\tilde{s} : \mathbb{R}^{2n} \times \mathbb{R}^n \to \mathbb{C}$ by setting $\tilde{s}(v,x) = (s(v))(x)$ for $v \in \mathbb{R}^{2n}$ and $x \in \mathbb{R}^n$. For $n = 1$, the oscillatory Dirac operator $\mathfrak{D}s = e_1.\nabla_{e_2}s - e_2.\nabla_{e_1}s$ thus gains the form

$$\mathfrak{D}\tilde{s}(p,q,x) = \mathrm{i}x\frac{\partial \tilde{s}}{\partial q}(p,q,x) - \frac{\partial^2 \tilde{s}}{\partial x \partial p}(p,q,x)$$

See Habermann, Habermann [9], pp 51, for solutions of $\mathfrak{D}s = 0$ in this case.

4. Elliptic Operators in Finite Rank Bundles

In this section, we recall basic results of Hodge theory for elliptic complexes in finite rank vector bundles.

Let $p : \mathcal{E} \to M$ and $\mathcal{F} \to M$ be two vector bundles over a manifold M. For any point $m \in M$, we denote the fiber $p^{-1}(\{m\})$ of $\mathcal{E}$ at m by $\mathcal{E}_m$. For each differential operator $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F})$ of order $k \in \mathbb{N}_0$, $m \in M$ and $\xi \in T_m^* M$, one defines a symbol $\sigma(D, \xi)(m) : \mathcal{E}_m \to \mathcal{F}_m$ of D in the following way. Let U be an open neighborhood of m in M, $v \in \mathcal{E}_m$, $e \in \Gamma(U, \mathcal{E})$ such that $e(m) = v$ and $g : U \to \mathbb{C}$ be a function defined on U such that $(\mathrm{d}g)_m = \xi$. The symbol is defined by $[\sigma(D, \xi)(m)]v = [D(\frac{\mathrm{i}^k}{k!}(g - g(m))^k e)](m) \in \mathcal{F}_m$. In particular, $\sigma(D, \xi)(m) : \mathcal{E}_m \to \mathcal{F}_m$. One can easily show that the symbol $\sigma(D, \xi) : \mathcal{E} \to \mathcal{F}$ is a vector bundle homomorphism. If D is a first order differential operator, its symbol fulfills $[\sigma(D, \xi)(m)]e(m) = \mathrm{i}([D, g]e)(m) = \mathrm{i}[D(ge) - gDe](m)$, $e \in \Gamma(M, \mathcal{E})$, $\xi_m = (\mathrm{d}g)_m$.

Example 2. 1) *Exterior differentiation.* Let $\mathrm{d} : \Omega^i(M) \to \Omega^{i+1}(M)$, $i = 0, \ldots, \dim M$, be the de Rham differential, $\alpha \in \Omega^i(M)$ and $g \in \mathcal{C}^\infty(M)$. Then $\mathrm{d}(g\alpha) - g\mathrm{d}(\alpha) = \mathrm{d}g \wedge \alpha + g\mathrm{d}\alpha - g\mathrm{d}\alpha = \mathrm{d}g \wedge \alpha$. Therefore $\sigma(\mathrm{d}, \xi)\alpha = \mathrm{i}\xi \wedge \alpha$, i.e., the symbol of the de Rham derivative is basically the exterior multiplication in direction ξ.

2) *Laplace-Beltrami operator.* Let (M, g) be a Riemannian manifold, and $\triangle_g : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ be the Laplace-Beltrami operator associated to it. Here, the bundle is the trivial line bundle $M \times \mathbb{R} \to M$. It is known that $\sigma(\triangle_g, \xi)f = -g(\xi^\flat, \xi^\flat)f$, $f \in \mathcal{C}^\infty(M)$, where $\xi^\flat \in TM$ is defined by $g(\xi^\flat, v) = \xi(v)$, $v \in TM$. This can be computed, e.g., using the formula $\triangle_g = \mathrm{d}^*\mathrm{d}$, where d^* is the adjoint of d with respect to the scalar products $(f, g) = \int_M fg\mathrm{vol}_M$ and $(\alpha, \beta) = \int_M g(\alpha^\flat, \beta^\flat)\mathrm{vol}_M$, where $f, g \in \mathcal{C}^\infty(M)$, $\alpha, \beta \in \Omega^1(M)$ and vol_M is a volume element of (M, g).

3) *Dolbeault operator.* Let (M, J) be a complex manifold and $\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ be the Dolbeault operator, $p, q \in \mathbb{N}_0$. Then the symbol $\sigma(\overline{\partial}, \xi)$ of $\overline{\partial}$ is given by $\sigma(\overline{\partial}, \xi)\alpha = \mathrm{i}\xi^{(0,1)} \wedge \alpha$, $\alpha \in \Omega^{p,q}(M)$. See Wells [24], pp 117, for details and notation.

We want to study a more general situation, namely that of *complexes* of differential operators. We consider the following data

1) a compact manifold M,
2) a sequence $(p_i : \mathcal{E}^i \to M)_{i \in \mathbb{N}_0}$ of finite rank vector bundles over M and
3) a co-chain complex $D^\bullet = (\Gamma(M, \mathcal{E}^i), D_i)_{i \in \mathbb{N}_0}$ of differential operators $D_i : \Gamma(M, \mathcal{E}^i) \to \Gamma(M, \mathcal{E}^{i+1})$, $i \in \mathbb{N}_0$.

Recall that a sequence

$$0 \to \Gamma(M,\mathcal{E}^0) \xrightarrow{D_0} \Gamma(M,\mathcal{E}^1) \xrightarrow{D_1} \dots \xrightarrow{D_{n-1}} \Gamma(M,\mathcal{E}^n) \xrightarrow{D_n} \dots$$

is called a co-chain complex if $D_{i+1}D_i = 0$ for all $i \in \mathbb{N}_0$. Next we use the word complex only.

Definition 3. *A complex*

$$0 \to \Gamma(M,\mathcal{E}^0) \xrightarrow{D_0} \Gamma(M,\mathcal{E}^1) \xrightarrow{D_1} \dots \xrightarrow{D_{n-1}} \Gamma(M,\mathcal{E}^n) \xrightarrow{D_n} \dots$$

of differential operators is called elliptic if for any $m \in M$ and any non-zero co-vector $\xi \in T_m^*M \setminus \{0\}$, the symbol sequence

$$0 \to (\mathcal{E}^0)_m \xrightarrow{\sigma(D_0,\xi)(m)} (\mathcal{E}^1)_m \xrightarrow{\sigma(D_1,\xi)(m)} \dots \xrightarrow{\sigma(D_{n-1},\xi)(m)} (\mathcal{E}^n)_m \xrightarrow{\sigma(D_n,\xi)(m)} \dots$$

is exact.

Recall that a complex is called *exact* if the kernel of each map in the complex equals the image of the preceding map. Maps from 0 as well as maps into 0 are zero homomorphisms.

Remark 5. *A single differential operator $D : \Gamma(M,\mathcal{E}) \to \Gamma(M,\mathcal{F})$ is considered as the complex $0 \to \Gamma(M,\mathcal{E}) \xrightarrow{D} \Gamma(M,\mathcal{F}) \to 0$. Consequently, a differential operator is elliptic iff its symbol is an isomorphism, which coincides with the classical notion of an elliptic operator. Thus the definition of an elliptic complex extends the classical one.*

Example 3. 1) Using the result of item 2 in Example 2, one finds that the Laplace-Beltrami operator $\triangle_g$ is elliptic since g is positive definite and since multiplying by a non-zero function is a vector bundle isomorphism.

2) The de Rham complex is elliptic because of the following reasons. Since $\sigma(\mathrm{d},\xi)\alpha = \mathrm{i}\xi \wedge \alpha$, we have $\sigma(\mathrm{d}_{|\Omega^{i+1}(M)},\xi) \circ \sigma(\mathrm{d}_{|\Omega^i(M)},\xi) = 0$ proving $\mathrm{im}\,\sigma(\mathrm{d}_{|\Omega^i(M)},\xi) \subseteq \ker\sigma(\mathrm{d}_{|\Omega^{i+1}(M)},\xi)$. For the opposite inclusion, suppose $\xi \wedge \beta = 0$, $\xi \in T_m^*M$ and $\beta \in \bigwedge^i T_m^*M$, $m \in M$. Applying the insertion operator $\iota_{\xi^\flat}$ on this equation, we get $g(\xi^\flat,\xi^\flat)\beta - \xi \wedge \iota_{\xi^\flat}\beta = 0$, i.e., $\beta = (g(\xi^\flat,\xi^\flat))^{-1}\xi \wedge \iota_{\xi^\flat}\beta$ since $g(\xi^\flat,\xi^\flat) \neq 0$ for $\xi \neq 0$. Thus, $\beta \in \mathrm{im}\,\sigma(\mathrm{d},\xi)$, proving the ellipticity of the de Rham complex.

2) It can be proved that the Dolbeault complex is elliptic as well. See Example 2.6 in Wells [24], pp 117, for instance.

The cohomology group of a complex $D^\bullet = (\Gamma(M,\mathcal{E}^i), D_i)_{i\in\mathbb{N}_0}$ is the vector space

$$H^i(D^\bullet,\mathbb{C}) = \frac{\ker(D_i : \Gamma(M,\mathcal{E}^i) \to \Gamma(M,\mathcal{E}^{i+1}))}{\mathrm{im}(D_{i-1} : \Gamma(M,\mathcal{E}^{i-1}) \to \Gamma(M,\mathcal{E}^i))}.$$

Notice, that we do not know whether this vector space is a topological vector space because the denominator is not closed *a priori*. In particular, $H^i(D^\bullet, \mathbb{C})$ need not be a Hausdorff space.

Suppose now, that a compact manifold M equipped with a Riemannian metric g, and a sequence $(p_i : \mathcal{E}^i \to M)_{i\in\mathbb{N}_0}$ are given such that for any $i \in \mathbb{N}_0$, each fiber of $\mathcal{E}^i$ is equipped with a scalar product, which varies smoothly when going through the individual fibers. These metric structures enable us to make adjoints of differential operators acting in sections of $\mathcal{E}^i$. Therefore to any complex $D^\bullet = (\Gamma(M, \mathcal{E}^i), D_i)_{i\in\mathbb{N}_0}$ of differential operators in vector bundles over M, we may associate a sequence $\triangle_i = D_{i-1}D_{i-1}^* + D_i^* D_i$ of the so-called *associated Laplacians*.

Now, we recall the following result, the core of the Hodge theory, on elliptic complexes of operators acting in sections of finite rank vector bundles over compact manifolds. For a proof, see Theorem 4.12 in Wells [24], pp 141.

Theorem 6. *Let M be a compact manifold, $(p_i : \mathcal{E}^i \to M)_{i\in\mathbb{N}_0}$ be a sequence of finite rank vector bundles over M and $D^\bullet = (\Gamma(M, \mathcal{E}^i), D_i)_{i\in\mathbb{N}_0}$ be an elliptic complex of differential operators. Then for each $i \in \mathbb{N}_0$,*

1) $dim\,(Ker\,\triangle_i) < +\infty$ *and*
2) $H^i(D^\bullet, \mathbb{C}) \simeq Ker\,\triangle_i$ *as vector spaces.*

Remark 7. *Notice that especially, $H^i(D^\bullet, \mathbb{C})$ is a Banach space. The property of being finite dimensional and complete can be seen as inherited from the fibers, which possess both of these properties.*

5. Analysis over C^*-Algebras

We start by giving a definition of a certain second order elliptic operator acting in the oscillatory bundle and present some quantitative information on its kernel computed by Habermann. Despite its ellipticity, its kernel is infinite dimensional. We follow the presentation in Habermann, Habermann [9].

Let J be a compatible almost complex structure on a symplectic manifold (M, ω) and let g denotes the corresponding Riemannian metric on M, i.e., $g(X, Y) = \omega(X, JY)$, $X, Y \in \mathfrak{X}(M)$. Suppose that ∇ is a symplectic connection and that (M, ω) admit a metaplectic structure. Then one can define an operator $\widetilde{\mathfrak{D}}$ by the following local formula

$$\widetilde{\mathfrak{D}}s = g^{ij} e_i.\nabla^S_{e_j} s,$$

where $(e_i)_{i=1}^{2n}$, is a symplectic frame on $(U, \omega_{|U})$, (g^{ij}) is the inverse of $(g_{ij}) = g(e_i, e_j)$, $i, j = 1, \dots, 2n$, and $s \in \Gamma(M, \mathcal{S})$. We set $\mathfrak{P} = \mathrm{i}(\widetilde{\mathfrak{D}}\mathfrak{D} - \mathfrak{D}\widetilde{\mathfrak{D}})$. This operator turns out to be elliptic. Namely, its symbol $\sigma(\mathfrak{P}, \xi)(m)$ is the multiplication of elements in $\mathcal{S}_m$ by $-g(\xi^\flat, \xi^\flat)$ which is an isomorphism of $\mathcal{S}_m$ for any

$0 \neq \xi \in T^*_m M$. For a computation of the symbol, see Corollary 5.1.4. in Habermann, Habermann [9], pp 68.

Example 4. Let $M = \mathbb{S}^2$ be the two dimensional sphere considered as the complex projective space $\mathbb{CP}^1$ equipped with the chordal metric $h(z) = (1 + |z|^2)^{-2}\mathrm{d}z^2$. The volume form ω (of the chordal metric) is a symplectic form since it is non-degenerate and a two-form. Thus, $(\mathbb{S}^2, \omega)$ is a symplectic manifold. Computing the curvature of the Riemannian connection of the chordal metric and using the Weil definition of Chern classes, we get

$$c_1(TM) = \left[\frac{\mathrm{i}}{\pi(1 + |z|^2)^2} dz \wedge \overline{dz}\right] \in H^2(M, \mathbb{R}).$$

(See Wells [24], pp 95.) Using polar coordinates, one easily computes that $\int_{\mathbb{S}^2} c_1(TM) = 2$ due to which $c_1(TM)$ is an even class. For a calculation of the isomorphism classes of metaplectic structures on $(\mathbb{S}^2, \omega)$, we use the universal coefficient theorem as we did in the case of $(\mathbb{R}^{2n}, \omega)$, getting $0 \to H^1(\mathbb{S}^2, \mathbb{Z}_2) \to 0$ since $H_0(\mathbb{S}^2, \mathbb{Z}) = \mathbb{Z}$ and $H_1(\mathbb{S}^2, \mathbb{Z}) = 0$. Thus there exists only one metaplectic structure Q on the sphere. For its (homogeneous) realization see Habermann, Habermann [9]. Associating the oscillatory representation to the principal $Mp(2, \mathbb{R})$-bundle of the metaplectic structure, we get $\mathcal{S} \cong S \times \mathbb{S}^2 \to \mathbb{S}^2$.

The Riemannian connection ∇ of the chordal metric is torsion-free and preserves the symplectic form ω, since ω is a volume form for h. Therefore $(\mathbb{S}^2, \omega, \nabla)$ is a Fedosov manifold. The construction of Habermann applies and we have the oscillatory Dirac operator on $\mathfrak{D} : \Gamma(\mathbb{S}^2, \mathcal{S}) \to \Gamma(\mathbb{S}^2, \mathcal{S})$ and also the operator $\mathfrak{P} : \Gamma(\mathbb{S}^2, \mathcal{S}) \to \Gamma(\mathbb{S}^2, \mathcal{S})$ at our disposal.

The decomposition $L^2(\mathbb{R}) = S = \widehat{\bigoplus}_{k=0}^{\infty} \mathbb{C}h_k$ (mentioned already in Section 2.2) translates to the bundle level as $\mathcal{S} = \widehat{\bigoplus}_{k=0}^{\infty} \mathcal{S}_k$, where $\mathcal{S}_k$ denotes the line bundle corresponding to the vector space $\mathbb{C}h_k$. When one restricts the oscillatory representation $\rho : \mathrm{Mp}(2, \mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R}))$ to the λ-preimage of $\mathrm{U}(1) \subseteq \mathrm{Sp}(2, \mathbb{R})$, S decomposes exactly into the spaces $\mathbb{C}h_k$, which are irreducible with respect to the group $\lambda^{-1}(\mathrm{U}(1))$. Using harmonic analysis for compact groups, Habermann was able to compute (see Habermann, Habermann [9]) eigenvalues of $\mathfrak{P}$. In particular, she obtained a monotone sequence $(l_i)_{i=0}^{\infty}$ such that $\ker \mathfrak{P} \cap \Gamma(\mathbb{S}^2, \mathcal{S}_{l_i}) \neq 0$ and $\dim(\ker \mathfrak{P} \cap \Gamma(\mathbb{S}^2, \mathcal{S}_{l_i})) = 2(i + l_i + 2)$. Consequently, the kernel of $\mathfrak{P}$ is infinite dimensional.

We might say that the infinite dimensionality of the kernel contradicts Theorem 6 if the fibers of $\mathcal{S}$ were finite dimensional. Since only the finite rank condition was not satisfied in the studied example, it is natural to ask how one can modify this assumption to still obtain an information on the cohomology groups. This will be done in the next section.

5.1. C^*-Algebras and Hilbert C^*-Modules

We generalize Theorem 6 to the case of finitely generated projective A-Hilbert bundles, where A is a unital C^*-algebra. We keep the compactness assumption on the underlying manifold. Let us start by a definition of a C^*-algebra.

Definition 4. *An associative algebra A over $\mathbb{C}$ with a norm $|\,| : A \to \mathbb{R}_0^+$ and a vector space antihomomorphism $^* : A \to A$ is called a C^*-algebra if*

1) $|ab| \leq |a||b|$ *for all* $a, b \in A$,
2) $^* : A \to A$ *is an antiinvolution,*
3) $|a|^2 = |aa^*|$ *for all* $a \in A$, *and*
4) $(A, |\,|)$ *is a Banach space.*

Example 5. 1) Let $A = \mathcal{C}_c^0(X)$ be the algebra of complex valued functions vanishing at infinity, on a locally compact Hausdorff space X with the point-wise multiplication. The involution $^* : A \to A$ is defined by $f^*(x) = \overline{f(x)}$, $x \in X$, and the norm $|f| = \sup\{|f(x)|;\, x \in X\}$ is the supremum norm, $f \in A$. Then A is a (commutative) C^*-algebra.

2) Let $\mathcal{H}$ be a Hilbert space and $A = \mathrm{End}(\mathcal{H})$ be the algebra of continuous endomorphisms of $\mathcal{H}$ with the product being the composition of maps. The involution is $A^* = A^\dagger$ (the adjoint of A). In order * is everywhere defined, we suppose that $\mathcal{H}$ is separable. Finally, the norm is given by

$$|A| = \sup\{\frac{|Af|_{\mathcal{H}}}{|f|_{\mathcal{H}}};\, f \in \mathcal{H}, f \neq 0\},$$

where $|\,|_{\mathcal{H}}$ is the norm induced by the scalar product on $\mathcal{H}$. The norm $|\,|$ is well defined since any continuous operator in a Hilbert space is bounded.

3) $A = \mathrm{Mat}(\mathbb{C}^n)$, $A^* = A^\dagger$, $|A| = \max\{|\lambda|;\, \lambda \text{ is an eigenvalue of } A\}$ is a special case of the preceding example as one learns in courses of functional analysis.

>From now on, we suppose that A contains a unit, $1a = a1 = a$. Then for each element $a \in A$, the spectrum $\mathrm{spec}(a)$ of a is defined by

$$\mathrm{spec}(a) = \{\lambda \in \mathbb{C};\, a - \lambda 1 \text{ does not possess an inverse}\} \subseteq \mathbb{C}.$$

We set $A_0^+ = \{a \in A;\, a = a^* \text{ and } \mathrm{spec}(a) \subseteq \mathbb{R}_0^+\}$ (for the set of non-negative elements in A). Now, let us define a generalization of Hilbert spaces, the Hilbert C^*-modules.

Definition 5. *Let U be a vector space with a left action of a C^*-algebra A. Suppose that there exists a map $(,)_U : U \times U \to A$ such that for each $u, v, w \in U$, $a \in A$ and $r \in \mathbb{C}$,*

1) $(u + rv, w)_U = (u, w)_U + r(v, w)_U$,
2) $(a.u, v)_U = a(u, v)_U$,
3) $(u, v)_U = (v, u)^*_U$,
4) $(u, u)_U \in A_0^+$, *and*
5) *if* $(u, u)_U = 0$, *then* $u = 0$.

Then the pair $(U, (,)_U)$ *equipped with the topology induced by the norm* $||_U : u \in U \mapsto |(u, u)_U|_A^{1/2} \in \mathbb{R}_0^+$ *is called a pre-Hilbert A-module. If this topology is complete,* $(U, (,)_U)$ *is called a Hilbert A-module.*

Let us notice, that $||_A$ denotes the norm on the C^*-algebra A. The map $(,)_U : U \times U \to A$ satisfying the conditions above is called an A-*product*. If $(U, (,)_U)$ is a Hilbert A-module, we call $(,)_U$ a *Hilbert* A-*product*.

Homomorphisms $L : U \to V$ between pre-Hilbert A-modules U, V are supposed to be A-linear, i.e., for each $a \in A$ and $u \in U$, $L(a.u) = a.L(u)$, and continuous with respect to the topologies induced by $||_U$ and $||_V$. An adjoint of a pre-Hilbert A-module homomorphism $L : U \to V$ is a map $L^* : V \to U$ satisfying $(Lu, v)_V = (u, L^*v)_U$ for each $u \in U, v \in V$. The adjoint of a pre-Hilbert A-module homomorphism need not exist. If it exists, it is unique and moreover, it is a pre-Hilbert module homomorphism. For it, see, e.g., Lance [17], pp 8.

When we consider a pre-Hilbert A-submodule V of a pre-Hilbert A-module U, we suppose that in particular, it is closed in U and the A-product in V is the restriction of the A-product in U. For any pre-Hilbert A-submodule $V \subseteq U$, we set $V^\perp = \{u \in U | (u, v)_U = 0 \text{ for all } v \in V\}$. Unfortunately, it is not in general true that $V \oplus V^\perp = U$. A Hilbert A-module U is called *finitely generated projective*, if $U \oplus U^\perp \cong A^n$, where A^n is the direct sum of n copies of A. In more detail, $A^n = \underbrace{A \oplus \ldots \oplus A}_{n}$ as a vector space, the action is given by $a.(a_1, \ldots, a_n) = (aa_1, \ldots, aa_n)$ and the Hilbert A-product $(,)_{A^n}$ is defined by the formula

$$((a'_1, \ldots, a'_n), (a_1, \ldots, a_n))_{A^n} = \sum_{i=1}^{n} a'_i a_i^*,$$

where $a, a_i, a'_i \in A$, $i = 1, \ldots, n$.

5.2. Complexes of Differential Operators in C^*-Hilbert Bundles

In the preceding subsection, C^*-algebras and finitely generated projective C^*-modules were introduced. In this chapter, these two types of objects shall play a similar role as the field of scalars and finite dimensional vector spaces (over this field) play in the theory of differential operators acting in finite rank vector bundles.

Let $\mathcal{E}$ and $\mathcal{F}$ be Banach manifolds modeled on Banach spaces X and Y, respectively. We call a continuous map $A : \mathcal{E} \to \mathcal{F}$ *smooth* if for each manifold charts $\phi : U \to X$ of $\mathcal{E}$, $U \subseteq \mathcal{E}$, and $\phi' : V \to Y$ of $\mathcal{F}$, $V \subseteq \mathcal{F}$, the composed mapping $\phi \circ A \circ \phi'^{-1} : \phi'(A^{-1}(V) \cap U) \to X$ is smooth, i.e., possess infinitely many Fréchet derivatives in each point.

Let Z be a Banach space and M be a manifold. We say that $p : \mathcal{E} \to M$ is a Banach bundle with typical fiber Z, if

1) $\mathcal{E}$ is Banach manifold and p is a smooth submersion of $\mathcal{E}$ onto M,
2) there exists an open covering $(U_\alpha)_{\alpha \in I}$ of M and for each $\alpha \in I$, we have a diffeomorphism $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times Z$ (called a *bundle chart*) such that $p_1 \circ \phi_\alpha = p$, where $p_1 : M \times Z \to M$ denotes the projection onto the first component, and
3) for each $\alpha, \beta \in I$, the map $\psi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ (called a *transition map*) defined by $\psi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}{}_{|\phi_\alpha(U_\alpha \cap U_\beta)}$ is a smooth homeomorphism.

The transition maps induce mappings $\overline{\psi_{\alpha\beta}} : Z \to Z$ defined by $(m, \overline{\psi_{\alpha,\beta}}(v)) = (\phi_\beta \circ \phi_\alpha^{-1})(m, v)$, $m \in \phi_\alpha(U_\alpha \cap U_\beta), v \in Z$.

Definition 6. *Let $(U, (,)_U)$ be a Hilbert A-module. A Banach bundle $p : \mathcal{E} \to M$ with a bundle atlas $\mathcal{A}$ is called an A-Hilbert bundle with typical fiber $(U, (,)_U)$ if*

1) *for each $m \in M$, the fiber $\mathcal{E}_m = p^{-1}(\{m\})$ is equipped with a Hilbert A-module structure and as such, it is isomorphic to $(U, (,)_U)$*
2) *for each $m \in M$, the subset topology on $\mathcal{E}_m \subseteq \mathcal{E}$ is equivalent to the norm topology on $(U, | \, |_U)$ and*
3) *the transition maps between the bundle charts of $\mathcal{A}$ are maps into the group $Aut_A(U)$ of Hilbert A-module automorphisms of U.*

Let $p : \mathcal{E} \to M$ be an A-Hilbert bundle. In the same way as for a smooth Banach bundle, one defines the space of smooth sections $\Gamma(M, \mathcal{E})$ for an A-Hilbert bundle. The space of sections admits a left action A defined by $(a.s)(m) = a.(s(m))$, where $a \in A$, $s \in \Gamma(M, \mathcal{E})$ and $m \in M$. Suppose that M is compact. We choose a Riemannian metric g on M and a volume element vol_M for this metric. An A-product on $\Gamma(M, \mathcal{E})$ is defined by

$$(s', s)_0 = \int_{m \in M} (s'(m), s(m))_m (\mathrm{vol}_M)_m,$$

where $s', s \in \Gamma(M, \mathcal{E})$ and $(,)_m$ denotes the Hilbert A-product in fiber $p^{-1}(\{m\})$. This makes $\Gamma(M, \mathcal{E})$ a pre-Hilbert A-module. We denote the completion of the normed space $(\Gamma(M, \mathcal{E}), | \, |_0)$ by $W^0(\mathcal{E})$ and call it the *zeroth Sobolev type completion*. Let us denote the Laplace-Beltrami operator for g by $\triangle_g$. For each $r \in \mathbb{N}_0$, we define an A-product $(,)_r$ on $\Gamma(M,\mathcal{E})$ by

$$(s',s)_r = \int_{m\in M} (s'(m),(1+\triangle_g)^r s(m))_m (\mathrm{vol}_M)_m,$$

where $s', s \in \Gamma(M,\mathcal{E})$. We denote the completion of $\Gamma(M,\mathcal{E})$ with respect to the norm $|\,|_r$ induced by $(,)_r$ by $W^r(\mathcal{E})$ and call it the *Sobolev type completion* (of order r). Differential operators in A-Hilbert bundles are defined by local coordinates and partial derivatives with respect to these coordinates in the same way as in the finite rank bundles. They possess continuous extensions to the Sobolev type completions, and they as well as their extensions to the Sobolev type completions are adjointable. Ellipticity is defined as in the finite rank case and is called the A-ellipticity in this case. See Solovyov, Troitsky [20] for more details on the facts in this paragraph.

The announced generalization of the Hodge theory is presented in the next theorem. The order of the Laplacian $\triangle_k = D_{k-1}D_{k-1}^* + D_k^* D_k$ is denoted by r_k. The adjoints are taken with respect to $(\Gamma(M,\mathcal{E}^i),(,)_0)$.

Theorem 8. *Let $(p_i : \mathcal{E}^i \to M)_{i\in\mathbb{N}_0}$ be a sequence of finitely generated projective A-Hilbert bundles over a compact manifold M. If $D^\bullet = (\Gamma(M,\mathcal{E}^i),D_i)_{i\in\mathbb{N}_0}$ is an A-elliptic complex of differential operators and for each $k\in\mathbb{N}_0$, the image of the r_k-th extension of the associated Laplacian $\triangle_k$ to the Sobolev type completion $W^{r_k}(\mathcal{E}^k)$ is closed, then for any $i\in\mathbb{N}_0$,*

1) $H^i(D^\bullet, A) \cong \ker \triangle_i$ *as A-modules,*
2) $H^i(D^\bullet, A)$ *is a Banach space with respect to $|\,|_0$, and*
3) $H^i(D^\bullet, A)$ *is a finitely generated A-module.*

Proof: See Krýsl [15]. ■

5.3. De Rham Complex with Values in the Oscillatory Module

As in Section 2, we suppose that (V,ω) is a $2n$ dimensional symplectic vector space, $J\in \mathrm{Sp}(V,\omega)$ is a compatible almost complex structure on V, $(e_i)_{i=1}^{2n}$ is an orthonormal basis for the scalar product induced by ω and J, and $L = \mathcal{L}(\{e_i\,;\, i = 1,\ldots,n\})$ is a Lagrangian space. (See Section 2 for details if necessary.) The ordering of the basis $(e_i)_{i=1}^{2n}$ induces volume forms vol_V and vol_L on V and L, respectively.

For $i\in\mathbb{N}_0$, we set $E^i = \bigwedge^i V^* \otimes S$ and consider E^i with the canonical Hilbert space topology (the space $\bigwedge^i V^*$ is finite dimensional). Further, we consider the tensor product representation of $\widetilde{G} = \mathrm{Mp}(V,\omega)$ on E^i, i.e.,

$$\widetilde{\rho}(g)(\alpha\otimes s) = \lambda^{*\wedge i}(g)\alpha\otimes\rho(g)s,$$

where $\alpha \in \bigwedge^i V^*$, $g \in \mathrm{Mp}(V,\omega)$ and $s \in S = L^2(L)$, and extend it to non-homogeneous elements by linearity. (The decomposition of $\widetilde{\rho}$ into irreducible $\mathrm{Mp}(V,\omega)$-modules was computed in Krýsl [14].)

Let A be the algebra of bounded operators on the Hilbert space $S = L^2(L)$. As described in Example 5 item 2 above, A is a C^*-algebra. On spaces E^i, we define a left A-module structure by setting

$$a.(\alpha \otimes s) = \alpha \otimes a(s)$$

for $a \in A$, $s \in S$, $\alpha \in \bigwedge^i V^*$ and extend it linearly to non-homogeneous elements. The volume form vol_V induces a scalar product on $\bigwedge^\bullet V^* = \bigoplus_{i=0}^{2n} \bigwedge^i V^*$ which we denote by $\tilde{g}$. (See, e.g., Krýsl [16] for more details.) Now, we can define an A-product $(,)_{E^i} : E^i \times E^i \to A$ by setting

$$(\alpha \otimes s, \beta \otimes t)_{E^i} = \widetilde{g}(\alpha,\beta) t \otimes s^*,$$

where $t \otimes s^* \in A$ is given by $(t \otimes s^*)(h) = (s,h)t$, $s,t,h \in S$. In the last formula, $(s,h) = \int_L \overline{s} h \mathrm{vol}_L$. It is easy to show that the resulting structure $(E^i, (,)_{E^i})$ is a pre-Hilbert A-module for each $i \in \mathbb{N}_0$. For a proof that $(E^i, (,)_{E^i})$ is a finitely generated projective Hilbert A-module, see Krýsl [16].

Now, let us proceed to the appropriate geometric version. Let (M^{2n},ω) be a symplectic manifold admitting a metaplectic structure (Q,Λ) and let ∇ be a flat Fedosov connection on (M,ω). We set $\mathcal{E}^i = Q \times_{\widetilde{\rho}} E^i$ (the bundle of *exterior forms with values in the oscillatory representation*). Note that $\mathcal{E}^0 = \mathcal{S} = Q \times_\rho S$ is the oscillatory bundle. The connection ∇ induces the oscillatory derivative ∇^S. Extending ∇^S according to the Leibniz rule, we get exterior covariant derivative $\mathrm{d}_i^S : \Gamma(M,\mathcal{E}^i) \to \Gamma(M,\mathcal{E}^{i+1})$. (See Example 6 below.) Gluing these exterior covariant derivatives together, we obtain

$$0 \to \Gamma(M,\mathcal{E}^0) \overset{\mathrm{d}_0^S}{\to} \Gamma(M,\mathcal{E}^1) \overset{\mathrm{d}_1^S}{\to} \dots \overset{\mathrm{d}_{n-1}^S}{\to} \Gamma(M,\mathcal{E}^n) \to 0.$$

Since any Hilbert bundle (though not any A-Hilbert bundle) over a manifold is trivial due to the Kuiper theorem, one can choose a bundle atlas for each $\mathcal{E}^i \to M$ such that its transition maps equal the identity on S. Since $1_{|S} \in \mathrm{Aut}_A(S)$, the oscillatory bundle is an A-Hilbert bundle.

Example 6. Let $(U, x = (x^1,\dots,x^{2n})) \subseteq M$ be a Darboux coordinate chart, $e_i = \frac{\partial}{\partial x^i}$ and $\epsilon_i = \mathrm{d}x^i$, $i = 1,\dots,2n$. Then, in particular, $(e_i)_{i=1}^{2n}$ is a symplectic frame and $(\epsilon^i)_{i=1}^{2n}$ is its dual co-frame. Because $(e_i)_{i=1}^{2n}$ is a coordinate frame, $[e_i,e_j] = 0$ for each $i,j = 1,\dots,2n$. Consequently, $R^\nabla(e_i,e_j) = \nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}$. Since $\epsilon^i = \mathrm{d}x^i$, we have $d\epsilon^i = 0$. Now, let $\alpha \in \Omega^r(U)$ and $s \in \Gamma(U,\mathcal{S})$. Using the

Einstein summation convention, we get

$$\begin{aligned}
\mathrm{d}^S_{r+1}\mathrm{d}^S_r(\alpha\otimes s) &= \mathrm{d}^S_{r+1}(\mathrm{d}\alpha\otimes s+\epsilon^i\wedge\alpha\otimes\nabla^S_{e_i}s)\\
&= \mathrm{d}\mathrm{d}\alpha\otimes s+\epsilon^i\wedge\mathrm{d}\alpha\otimes\nabla^S_{e_i}s\\
&\quad +(-1)\epsilon^i\wedge\mathrm{d}\alpha\otimes\nabla^S_{e_i}s\\
&\quad +\epsilon^j\wedge\epsilon^i\wedge\alpha\otimes\nabla^S_{e_j}\nabla^S_{e_i}s\\
&= \frac{1}{2}\epsilon^j\wedge\epsilon^i\wedge\alpha\otimes(\nabla^S_{e_j}\nabla^S_{e_i}-\nabla^S_{e_i}\nabla^S_{e_j})s\\
&= \frac{\mathrm{i}}{4}R_{ij}{}^{kl}\epsilon^i\wedge\epsilon^j\wedge\alpha\otimes e_k.e_l.s,
\end{aligned}$$

where we used Theorem 4 in the last step. In particular, $\mathrm{d}^{S^\bullet}=(\mathcal{E}^i,\mathrm{d}^S_i)_{i=0}^{2n}$ is a complex if the curvature R^∇ of ∇ vanishes, i.e., if (M,ω,∇) is a flat Fedosov manifold.

As a consequence of Theorem 8, we have the following

Corollary 9. *Let (M^{2n},ω) be a symplectic manifold and ∇ be flat symplectic connection. Suppose that M is compact, admits a metaplectic structure and the extensions of the associated Laplacians $\triangle_k$ to the Sobolev completions $W^2(\mathcal{E}^k)$ have closed images for all $k=0,\dots,2n$. Then for each $i=0,\dots,2n$, the cohomology group $H^i(\mathrm{d}^{S^\bullet},A)$ is a finitely generated A-module and a Banach space.*

Proof: The order of d^S_k is one and the order of $\triangle_k=(\mathrm{d}^S_k)^*\mathrm{d}^S_k+\mathrm{d}^S_{k-1}(\mathrm{d}^S_{k-1})^*$ is two. Because each E^i is a finitely generated projective Hilbert A-module, $\mathrm{d}^{S^\bullet}=(\Gamma(M,\mathcal{E}^i),\mathrm{d}^S_i)_{i=0}^{2n}$ is a sequence of finitely generated projective A-Hilbert bundles. Since the Fedosov connection ∇ is supposed to be flat, this sequence is a complex according to Example 6. Now, using Theorem 8 the result follows. ■

Remark 10. *The condition on the images of the Laplacians seems to be technically difficult to verify and we would like to focus our attention to this phenomenon in the future.*

Corollary 11. *Let (M^{2n},ω) be a compact symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection. Suppose that the extension of $\mathfrak{P}^2$ to $W^2(\mathcal{S})$ is closed. Then the kernel of $\mathfrak{P}$ is a finitely generated A-module and a Banach space.*

Proof: Since $\mathfrak{P}$ is self-adjoint and elliptic the corollary follows from Theorem 8. ■

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