Symplectic spinors and Hodge theory

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Motivation

- Generalize Hodge theory from finite rank elliptic complexes to infinite rank
- Deformation quantization (non-formal, analytic)
- Connection to parallel transport (Tuckey, Drechsler)
- Analysis for the BRST (Becchi-Rouet-Stora-Tyutin quantum theory with constraints)

Statements of the Hodge theory

- Applies to the following situation
  - $(E^i \to M)_{i\in\mathbb{Z}}$ sequence of **finite rank vector bundles**, equipped by a Hermitian metric, over a **compact** manifold M
  - $D_i : \Gamma(E^i) \to \Gamma(E^{i+1})$ (pseudo-)differential operators forming a complex $D^\bullet = (\Gamma(E^i), D_i)_{i\in\mathbb{Z}}$, $D_{i+1}D_i = 0$
  - Then symbols and principal symbols $\sigma_i(\xi, -) : E^i \to E^{i+1}$ form a complex for any $\xi \in T^*M$
  - If $\sigma_i(\xi, -)$ forms an exact sequence for any $0 \neq \xi \in T^*M =:$ elliptic complexes.
- Then
  - $H^i(E^\bullet)$ is finite dimensional and
  - $H^i(E^\bullet) \simeq \operatorname{Ker} \triangle_i$ where $\Delta_i = D_i^* D_i + D_{i-1}D_{i-1}^*$. The adjoint is with respect to the inner product induced on section spaces $\Gamma(E^i)$ by the metric on E^i.
  - Moreover, $\Gamma(E^i) = \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^* \oplus \operatorname{Ker} \Delta_i$ (Hodge decomposition)

Examples and Non-examples of Hodge theory

- deRham complex over a compact manifold, $\sigma_i(\xi, \alpha) = \xi \wedge \alpha$ forms an exact sequence, $0 \neq \xi \in T^*M$
- Dolbeault complex over a compact complex manifold
- **Not an example**: deRham complex on $\mathbb{R}^n$, $H^0_{dR}(\mathbb{R}^n) = \mathbb{R}$, and $\mathrm{Ker}\,\Delta_0$ is infinite dimensional (e.g., when restricted to polynomials and $n = 2 : 1, x, y, x^2 - y^2, 2xy, \ldots$ (harmonic polynomials); it is of dimension 2 if $n = 1$)
- **Not an example:** $\mathbb{E}$ infinite dimensional Hilbert space, M connected compact manifold $E = M \times \mathbb{E} \to M$, $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ the trivial connection, $\nabla s = 0$; kernel are constant functions with values in $\mathbb{E}$. Kernel is $\{s \in \Gamma(E) | \, \exists e \in \mathbb{E} \, \forall m \in M \, s(m) = (m, e)\} \simeq \mathbb{E}$ - thus infinite dimensional.

Key steps in the proof of classical Hodge theory

- Δ_i is a map on pre-Hilbert space $\Gamma(E^i)$ of smooth sections of E^i
- Constructions of extensions $\widetilde{\Delta_i}$ for Δ_i to the Sobolev spaces completions $W^s(E^i)$ - already Hilbert spaces
- Construction of Green's operators (partial inversions) for $\widetilde{\triangle_i}$
  - orthogonal projections
  - proof that Δ_i are Fredholm
- Regularity $\mathrm{Ker}(\Delta_i) = \mathrm{Ker}(\widetilde{\Delta}_i)$, construction of parametrix on $\Gamma(E^i)$

Tasks for establishing of a 'fiber-wise' infinite Hodge theory

- Choose a suitable class of infinite rank Banach bundles $(E^i \to M)_{i\in\mathbb{Z}}$ over a compact manifold
- Make suitable completions of the pre-Hilbert $\Gamma(E^i) \Longrightarrow$ Sobolev-type or Hardy-type spaces with values in vector spaces (Then the Banach space theory works)
- Make continuous extensions of D_i and Δ_i to these completions (differential operators are of finite integer order)
- Prove **Elliptic implies regular**
- Prove **Elliptic implies: extension is Fredholm in a "suitable sense"**
- We choose Banach bundles with fibers so-called Hilbert A-modules, where A is a C^*-algebra (algebra of measured quantities)

Definition of pre-Hilbert A-modules

Definition

Let A be a C^*-algebra and $\mathbb{E}$ be a vector space over the complex numbers. We call $(\mathbb{E}, (,))$ a pre-Hilbert A-module if

a) $\mathbb{E}$ is a **right complex** A**-module** – operation $\cdot : \mathbb{E} \times A \to \mathbb{E}$

b) $(,) : \mathbb{E} \times \mathbb{E} \to A$ is an A- **and** $\mathbb{C}$**-sesquilinear Hermitian map**

$$(u, v + \lambda w) = (u, v) + \lambda(u, w) \text{ (complex module)}$$
$$(u, v \cdot a) = (u, v)a \text{ - the product is in } A \text{ at R.H.S.}$$
$$(u, v)^* = (v, u)^*$$

c) $(u, u) \geq 0$ and $(u, u) = 0$ implies $u = 0$

We say $a \geq 0$, $a \in A$, if $spec(a) \subseteq [0, \infty)$, where $spec(a) = \{\lambda \in \mathbb{C} | \, a - \lambda 1 \text{ is not invertible in the augmentation } \tilde{A} \text{ of } A\}$

Definition of Hilbert A-modules

Definition

Let $(A, *, ||_A)$ be a C^*-algebra. If $(\mathbb{E}, (,))$ is a pre-Hilbert A-module we call it Hilbert A-module if it is complete with respect to the norm $|| : \mathbb{E} \to [0, \infty)$ defined by $u \ni \mathbb{E} \mapsto |u| = \sqrt{|(u, u)|_A}$ where $||_A$ is the norm on A.

- Closed submodules need not have orthogonal complements: $\{f \in \mathcal{C}([0,1]) | f(0) = 0\} \subseteq \mathcal{C}([0,1])$
- Continuous linear maps need not be adjointable $(=(T^*u, v) = (u, Tv))$
- $\mathrm{Aut}_A(\mathbb{E}) = \{T : \mathbb{E} \to \mathbb{E} | T(u \cdot a) = T(u) \cdot a, T$ is continuous and bijective$\}$

Examples of Hilbert A-modules

- For $A = \mathbb{C}$, Hilbert $\mathbb{C}$-module = Hilbert space
- For A a C^*-algebra, $\mathbb{E} = A$, $a \cdot b = ab$ and $(a, b) = a^* b$.
- $A = \mathbb{E} = C^0([0,1])$, $(f \cdot g)(x) = f(x)g(x)$, $x \in [0,1]$, $(f, g) = fg \in C^0([0,1])$
- $\ell^2(A) = \{(a_i)_{i=1}^{\infty} \subseteq A | \sum_i |a_i|^2 < \infty\}$, $(a_i)_i \cdot b = (a_i \cdot b)_i$, $((a_i)_i, (b_i)_i) = \sum_i a_i^* b_i$.
- Sections of A-Hilbert bundles over compact manifolds form pre-Hilbert modules. (A. S. Mishchenko: Sobolev-type completions of the section spaces)

Bracketing type example or Matrix densities

For $A = K(\mathbb{E})$, the C^*-algebra of compact operators on a separable Hilbert space $\mathbb{E}$ with (Hilbert) product $(,)_{\mathbb{E}}$

The continuous dual $\mathbb{E}^*$ is a right A-module.
$\mathbb{E}^* \times A \to \mathbb{E}^*$, action: $l \cdot a = l \circ a$, $l \in \mathbb{E}^*$, $a \in K(\mathbb{E})$

The A-product $(,) : \mathbb{E}^* \times \mathbb{E}^* \to K(\mathbb{E})$ is defined by
$(k, l)(v) = l(v)k^\#$, $k, l \in \mathbb{E}^*$, $v \in \mathbb{E}$ and
$(k^\#, v)_{\mathbb{E}} = k(v)$; $k^\#$ is the Hilbert dual to k

For $\langle\psi| , \langle\phi|$, the A-product is the projector $|\psi\rangle \langle\phi|$, i.e.,
$(\langle\psi| , \langle\phi|) = |\psi\rangle \langle\phi|$.

Generalization of the Fredholm property

- $f_{u,v}(w) = u \cdot (v, w)$ small operators, called A-rank 1
- $F : \mathbb{E} \to \mathbb{F}$, is called A-finite rank if $F(w) = \sum_i^n \lambda_i f_{u_i,v_i}(v)$ for some $u_i, v_i, w \in \mathbb{E}$, $\lambda_i \in \mathbb{C}$ and $n \in \mathbb{N}$
- A-compact operators := in the closure of A-finite rank, ($\mathbb{E}$ is a Banach space, $B(\mathbb{E})$ is a normed space with operator norm)

Definition (A-Fredholm operator, Atkinson's like definition)

$F : \mathbb{E} \to \mathbb{E}$ is called A-Fredholm := invertible modulo A-compact, i.e., there is an operator G_1 and G_2 such that $FG_1 = 1 + K_1$ and $G_2F = 1 + K_2$ for A-compact K_1, K_2

- Problems with A-Fredholm: The **image of A-Fredholm need not be closed** in contrast to the Banach space case (F $\mathbb{C}$-Fredholm implies $M/\mathrm{Im}\, F < \infty$ implies $\mathrm{Im}\, F$ is closed) $Ff = xf$, $A = \mathbb{E} = \mathcal{C}^0([0,1])$ - counterexample: A-Fredholm but not closed range

Dagger category

Definition

An additive category is called dagger if it is equipped with a contravariant functor $*$ which is the identity on the objects, it is involutive on morphisms, $**F = F$, and it preserves the identity morphisms, i.e., $*\mathrm{Id}_C = \mathrm{Id}_C$ for any object C.

Remark: Additive is weaker than abelian.

Examples:

1) (Separable or non-separable) Hilbert spaces with continuous linear maps and dagger being the adjoint of maps. Especially, finite dimensional inner product spaces.
2) The category PH_A^* or H_A^* of pre-Hilbert A-modules or Hilbert A-modules, respectively, and adjointable maps between the objects.
3) Categorical Quantum mechanics (à la Abramsky, Coecke)

Notation for additive dagger categories

For a morphism F, we denote $*F$ by F^*.

For any additive category $\mathcal{C}$, we denote the category of its complexes by $\mathfrak{K}(\mathcal{C})$ (not the homotopy category of complexes).

If $\mathcal{C}$ is an additive and dagger category and $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}} \in \mathfrak{K}(\mathcal{C})$, we set

$$\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^* : U_i \to U_i,$$

$i \in \mathbb{Z}$, and call it the i-th Laplace operator.

Hodge and parametrix-type complexes

Definition

Let $\mathcal{C}$ be an additive dagger category. We call a complex $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}} \in \mathfrak{K}(\mathcal{C})$ of Hodge-type if for each $i \in \mathbb{Z}$

$$U^i = \operatorname{Ker}\Delta_i \oplus \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*.$$

We call $d^\bullet$ self-adjoint parametrix possessing if for each i, there exist morphisms $G_i : U^i \to U^i$ and $P_i : U^i \to U^i$ such that $\operatorname{Id}_{U^i} = G_i\Delta_i + P_i$, $\operatorname{Id}_{U^i} = \Delta_i G_i + P_i$, $\Delta_i P_i = 0$ and $P_i = P_i^*$.

Remark: In the preceding definition, we suppose that the images of the chain maps, the images of their adjoints, and the kernels of the Laplacians exist as objects in the additive and dagger category $\mathcal{C}$. The sign $\oplus$ denotes the biproduct in $\mathcal{C}$.

Simple properties of self-adjoint parametrix complexes

1) The first two equations from the definition of a self-adjoint parametrix possessing complex are called the parametrix equations.

2) Morphisms P_i from the above definition are idempotent as can be seen by composing the first equation with P_i from the right and using the equation $\Delta_i P_i = 0$. Thus, P_i are projections.

3) Operators G_i are called the Green operators.

4) It is easy to prove that any complex in the category of finite dimensional inner product spaces is self-adjoint parametrix possessing (and of Hodge-type).

Complexes in H_A^* and PH_A^*

To any complex $d^\bullet = (U^i, d_i)_{i \in \mathbb{Z}} \in \mathfrak{K}(PH_A^*)$, the cohomology groups $H^i(d^\bullet) = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$ are assigned. They are A-modules. We consider them to be equipped with the canonical quotient topology.

Theorem

The cohomology groups are pre-Hilbert A-modules with respect to $(,)_{U_i}$ if and only if $\operatorname{Im} d_{i-1}$ has an A-orthogonal complement in $\operatorname{Ker} d_i$ if and only if there is an idempotent p onto $\operatorname{Im} d_{i-1}$ which satisfies $p = p^*$.

Abstract theorems in PH_A^* and H_A^*

Theorem (Krýsl, Annals Glob. Anal. Geom. 2015)

If $d^\bullet = (U_i, d_i)_{i \in \mathbb{Z}} \in \mathfrak{K}(\mathcal{C})$ is a self-adjoint parametrix possessing complex, then $d^\bullet$ is of Hodge type.

In the smaller category H_A^*, the opposite implication holds

Theorem (Krýsl, Journ. Geom. Physics 2016)

If $d^\bullet = (U^i, d_i)_{i \in \mathbb{Z}} \in \mathfrak{K}(H_A^*)$ is a complex of Hodge-type in H_A^* if and only if $d^\bullet$ is self-adjoint parametrix possessing.

Back to motivation

Remark: If the image of d_{i-1} is not closed, the quotient topology on the cohomology group $H^i(d^\bullet)$ is non-Hausdorff.

Non-Hausdorff in TVS implies non-T1, i.e., there are convergent sequences with more than one limit. This makes impossible to compare whether results of a measurement are close to the calculated (by theory estimated) value. Consequently, it is not possible to test the theory by measurements. Thus, the corroboration theory does not apply. Connected: Representing elements of the quotient (cohomology group) by harmonic elements (kernels of the Laplacians)?

One can take a non-canonical topology. Which one? 'What' decides?

Back to motivation

Von Neumann [11] - necessity of topology for state spaces

BRST quantization use (co-)chain complexes (differential denoted Q). State spaces are cohomology groups. They are state spaces or quantum systems $\implies$ they ought to be infinite dimensional linear topological spaces $\implies$ cycle spaces must be infinite dimensional. What if boundary spaces are not closed in the cycles (necessarily infinite dimensional)?

Bundles of Hilbert modules

Banach bundles and champs contunus de C^*-algèbres lead quite naturally to

Definition

Let $\mathbb{E}$ be a Hilbert A-module. $(p : E \to M, \mathcal{E})$ is called an A-Hilbert bundle with typical fiber $\mathbb{E}$ if

- p is a smooth Banach bundle and $\mathcal{E}$ is a smooth bundle atlas of p with typical fiber the Banach space $\mathbb{E}$
- transition maps of the atlas are maps into $\mathrm{Aut}_A(\mathbb{E})$

Example: the Segal-Shale-Weil (metaplectic, Kostant, oscillator, symplectic spinor) bundle $E = \mathcal{P} \times_\sigma \mathbb{E}$ over a symplectic manifold admitting a metaplectic structure with a Kuiper atlas. Explained below.

A theorem of Mishchenko and Fomenko

Theorem (MF, Dokl. Akad. Nauk SSSR, 1979)

Let A be a C^*-algebra, M a compact manifold and $p : E \to M$ a finitely generated projective A-Hilbert bundle. If D is an A-elliptic operator, its **extension** *is an A-Fredholm operator. In particular, $\operatorname{Ker} D$ is a finitely generated projective Hilbert A-module.*

Finitely generated projective := fiber is a finitely generated and a projective A-module. Projective Hilbert A-module := whenever it embeds into $\ell^2(A)$, $\ell^2(A)$ splits into an orthogonal direct sum of the embedded fiber and its OG-complement

Extensions are meant to the Sobolev-type extensions (of Mishchenko, Fomenko) Generalization of the procedures from the classical Hodge theory.

Complexes of differential operators

Theorem (Krýsl, Annals Global Anal. Geom., 2015)

Let M be a compact manifold, A be a C^*-algebra and $D^\bullet = (\Gamma(\mathcal{F}^i), D_i)_{i\in\mathbb{Z}}$ be an elliptic complex on finitely generated projective A-Hilbert bundles over M. Let for each $i \in \mathbb{Z}$, the **image of Δ_i be closed** *in $\Gamma(\mathcal{F}^i)$. Then for any $i \in \mathbb{Z}$*

1) $D^\bullet$ *is of Hodge-type*
2) $H^i(D^\bullet)$ *is a finitely generated projective Hilbert A-module isomorphic to* $\mathrm{Ker}\,\Delta_i$ *as a Hilbert A-module*
3) $\mathrm{Ker}\, D_i = \mathrm{Ker}\, \triangle_i \oplus \mathrm{Im}\, D_{i-1}$
4) $\mathrm{Ker}\, D_i^* = \mathrm{Ker}\, \triangle_{i+1} \oplus \mathrm{Im}\, D_{i+1}^*$
5) $\mathrm{Im}\, \Delta_i = \mathrm{Im}\, D_{i-1} \oplus \mathrm{Im}\, D_i^*$.

Complexes of differential operators

Theorem (Krýsl, J. Geom. and Physics, 2016)

Let M be a compact manifold, K be a C^*-algebra of compact operators and $D^\bullet = (\Gamma(\mathcal{F}^i), D_i)_{i \in \mathbb{Z}}$ be an elliptic complex on finitely generated projective K-Hilbert bundles over M. If $D^\bullet$ is elliptic, then for each $i \in \mathbb{Z}$

1) $D^\bullet$ *is of Hodge-type*
2) *The cohomology group $H^i(D^\bullet)$ is a finitely generated projective Hilbert K-module isomorphic to the Hilbert K-module $\mathrm{Ker}\,\Delta_i$.*
3) $\mathrm{Ker}\, D_i = \mathrm{Ker}\,\Delta_i \oplus \mathrm{Im}\, D_{i-1}$
4) $\mathrm{Ker}\, D_i^* = \mathrm{Ker}\,\Delta_{i+1} \oplus \mathrm{Im}\, D_{i+1}^*$
5) $\mathrm{Im}\,\Delta_i = \mathrm{Im}\, D_{i-1} \oplus \mathrm{Im}\, D_i^*$

Remark: In particular, we see that the cohomology groups share properties of the fibers.

Metaplectic, Symplectic spinor or Segal-Shale-Weil representation

- (V, ω) symplectic vector space of dimension $2n$ (linear/flat phase space)
- $Sp(V, \omega)$ (linear canonical transformations); connected double-cover $G = Mp(V, \omega)$ metaplectic group
- $\lambda : Mp(V, \omega) \to Sp(V, \omega)$ the 2-fold covering map
- $Mp(V, \omega)$ is not a matrix group, no faithful representation on finite dimensional vector space
- L Lagrangian subspace in V, J a compatible positive complex structure on (V, ω), metric $g(-,-) = \omega(-, J-)$
- $\mathbb{E} = L^2(L)$, Lebesgue measure is induced by $g_{|L \times L}$
- $\sigma : G \to U(\mathbb{E})$, unitary operators on $\mathbb{E}$.

Metaplectic, Symplectic spinor or Segal-Shale-Weil

$\widetilde{g} := \lambda^{-1}(g)$ for $g \in Sp(2n, \mathbb{R})$; $\widetilde{g}$ is a two-point set.
Suppose $A, B \in \mathrm{End}\,(L)$, A is invertible and $B^t = B$. The SSW-representation σ of $G = Mp(V, \omega)$ is defined by

$$
\begin{aligned}
(\sigma(h_1)f)(x) &= \pm e^{-\pi\imath g(Bx,x)/2} f(x) \text{ for any } h_1 \in \widetilde{g_1},\ g_1 = \left(\begin{array}{c|c} 1 & 0 \\ \hline B & 1 \end{array}\right) \\
(\sigma(h_2)f)(x) &= \sqrt{\det A^{-1}} f(A^{-1}x) \text{ for any } h_2 \in \widetilde{g_2},\ g_2 = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A^{-1t} \end{array}\right) \\
(\sigma(h_3)f)(x) &= \pm \imath^n e^{\pi\imath n/4} (\mathcal{F}f)(x) \text{ for any } h_3 \in \widetilde{g_3},\ g_3 = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right)
\end{aligned}
$$

Elements in $\mathbb{E} = L^2(L)$ are called symplectic spinors.
The representation is sometimes called Weil [16], oscillator or metaplectic.

Properties of SSW-representation

- Construction: Schrödinger representation of the Heisenberg group $(Sch((q,p,t))f)(x) = e^{2\pi\imath t + 2\pi\imath\omega(p,x) + \pi\imath\omega(q,p)} f(x+q)$ $f \in L^2(\mathbb{R}^n)$, $(q,p,t) \in H(n) = L \oplus L^{\perp\omega} \oplus \mathbb{R}$, $x \in L$.
- Twisting of the representation by action $Sp(V,\omega)$ (on the Heisenberg group), $Sch_g = Sch \circ L_g : H(n) \to U(L^2(L))$, $\{Sch_g | \, g \in Sp(V,\omega)\}$. $L_g(q,p,t) = (g(q,p),t)$, $(q,p,t) \in H(n)$.
- Stone–Neumann theorem on uniqueness of representations of the Heisenberg group with a fixed center action: there are unitary intertwiners $C_g : L^2(L) \to L^2(L)$. They satisfy $C_g \circ C_{g'} = m(g,g')C_{gg'}$, $m : Sp(V,\omega) \times Sp(V,\omega) \to U(1)$. Thus **projective representation** of $Sp(V,\omega)$.
- Cocycle counting of A. Weil: $g \mapsto C_g$ can be made a true representation of the metaplectic group

Metaplectic structures

Metaplectic structures

- (M,ω) symplectic manifold
- $\mathcal{P} = \{f$ is a symplectic basis of $(T^*M, \omega_m)| \, m \in M\}$
- $\mathcal{Q} \to M$ any $Mp(2n, R)$-principal bundle compatible with the projection structures is called metaplectic structure

Basic examples: even dimensional tori, $\mathbb{C}P^{2n+1} \supseteq S^2$, T^*M of an orientable manifold M.

Theorem (Kostant [7])

(M,ω) admits a metaplectic structure iff the first Chern class of (TM, J) is even for a compatible positive almost structure complex J.

Remark: A class $a \in H^2(M,\mathbb{Z})$ is called even := exists $b \in H^2(M,\mathbb{Z})$ such that $a = 2b$.
There is a notion of so-called Mp^c**-structure** that exists on any symplectic manifold (Robinson, Rawnsley [14], Cahen et. al [3])

A priori constructions

- (M,ω) symplectic manifold admitting a metaplectic structure $\mathcal{P} \to M$
- $E = \mathcal{P} \times_\sigma \mathbb{E}$ Segal-Shale-Weil bundle (Kostant)

Definition

Sections $\Gamma(\mathbb{E})$ are called symplectic spinor fields.

- ∇ symplectic connection $(\nabla\omega = 0) \implies Z$ principal connection on $\mathcal{Q} \implies$ lift on $\mathcal{P} \implies \nabla^E$ associated connection on $E \implies$ exterior covariant derivative d_i^∇ on $\Omega^i(E)$ - forms with values in the SSW-bundle
- If ∇^E is flat, then $(d_i^{\nabla^E}, \Omega^i(E))_{i\in\mathbb{Z}}$ forms a complex - SSW-complex

Symplectic Dirac operator of Habermann

- (M, ω) symplectic manifold admitting a metaplectic structure
- ∇ a symplectic connection, D^S symplectic Dirac operator of Habermann
- $D^S : \Gamma(E) \to \Gamma(E)$, $D^S s = \mu \circ \nabla^E s$, $s \in \Gamma(E)$
  $\mu : T^*M \otimes E \to E$
- $\mu(dx^i \otimes s) = \imath x^i s$ and $\mu(dx^{i+n} \otimes s) = \frac{\partial s}{\partial x^i}$, $i = 1, \ldots, n$
  Habermann, Habermann [5]

How to incorporate D^S into the C^*-Hodge theory?
It is Mp-invariant but it is **not** C^*-invariant in general

Modification of SSW-complex

- (M, ω) symplectic manifold admitting a metaplectic structure
- $p : E \to M$ (dual) to the Kostant's symplectic spinor bundle
- E is an infinite rank Hilbert bundle, it is homeomorphic to product $pr_1 : M \times \mathbb{E} \to M$ (Kuiper's theorem)
- $(p : E \to M, \{\psi\})$, $\psi : E \to M \times L^2(L)$ a global map. Suppose it is smooth.
- $\mathbb{E} = L^2(L^2))$ is a finitely generated projective Hilbert A-module (bracket example) for $A = K(L^2(L))$
- ∇^ψ the horizontal connection given by ψ (Kuiper connection)
- $(d_i^{\nabla^\psi} : \Omega^i(E) \to \Omega^{i+1}(E), \Omega^i(E))_{i \in \mathbb{Z}}$,
  $\Omega^i(E) = \Gamma(\bigwedge^i T^*M \otimes E)$

Application of Hodge theory for $\mathfrak{K}(PH^*_{K(\mathbb{E})})$

Historical examples for the Mishchenko, Fomenko theorem use a decomposition of unity on manifold and define operators by a use of it. These ops are dependent on patches. This seems unpleasant from a point of view of Physics.

Theorem (SK 2016, Hodge theory for modified SSW-complex)

If (M,ω) is a compact symplectic manifold admitting a metaplectic struture, ψ is a smooth trivialization and ∇^ψ is the Kuiper connection subordinated to ψ, then the Hodge theory holds for $d^\bullet = (d_i^{\nabla^\psi}, \Omega^i(E))_{i\in\mathbb{Z}}$, cohomology groups $H^i(d^\bullet)$ are Banach spaces and finitely generated projective Hilbert A-modules.

Open questions and outlook

- Nice and "relevant" examples of elliptic complexes with Laplacians whose images are not closed
- G- and C^*-triviality of Hilbert vector bundles
- Use the non-commutativity allowed by the KK-theory?
- How much does the Kuiper connection depend on the map?

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