Induced C^*-complexes in metaplectic geometry

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Abstract

For a symplectic manifold admitting a metaplectic structure and for a Kuiper map, we construct a complex of differential operators acting on exterior differential forms with values in the dual of Kostant's symplectic spinor bundle. Defining a Hilbert C^*-structure on this bundle for a suitable C^*-algebra, we obtain an elliptic C^*-complex in the sense of Mishchenko–Fomenko. Its cohomology groups appear to be finitely generated projective Hilbert C^*-modules. The paper can serve as a guide for handling of differential complexes and PDEs on Hilbert bundles.

1 Introduction

An important assumption is made in classical works on the Hodge theory of elliptic complexes beside the compactness of the base manifold. Namely, the bundles in the complexes are supposed to be of finite rank. (See Hodge [24] and Palais [52].) Under these two assumptions, elements of the complexes can be decomposed into closed, co-closed and harmonic parts uniquely. It follows that cohomology groups of the complexes are bijectively representable by harmonic elements. Moreover, they are finite dimensional vector spaces. The study of elliptic complexes has a long history which goes back to W. Hodge and G. de Rham. See Maurin [46] and also Klein [29] for references and for the (pre-)history of this subject. Elliptic complexes are studied also in areas 'extraterritorial' to differential geometry and global analysis or in cases violating the compactness. See Tsai at al. [67], Schmid, Vilonen [62], Li [41], Nekovář, Scholl [50], Hain [21] and Albin et al. [1] for recent contributions.

However, elliptic complexes can be studied in the case of infinite rank bundles as well. Troitsky in [66] and Schick, subsequently, in [61] elaborate a theory for elliptic complexes on finitely generated projective Hilbert C^*-bundles, which is based on the index theory of Mishchenko and Fomenko described in [48] and

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[14]. In [66] and [61], C^*-compact perturbations of the complexes' differentials are allowed. For the infinite rank bundles, one cannot expect that, in general, the space of harmonic elements represents the cohomology groups homeomorphically (quotient topology) and linearly (quotient projections). Therefore, it makes sense to find conditions when this happens. This was investigated by the author in the past years (see [35], [36] or an overview in [37]).

In the connection to sheaf cohomology, Banach and Fréchet bundles are studied in the papers of Illusie [25] and Röhrl [57]. To present a sample of the broad context (foremost connected to C^*-algebras, K-theory and homological algebra) in which such bundles are considered, let us mention the papers of Maeda, Rosenberg [43], Freed, Lott [16], Larraín-Hubach [39], the author [36] and Fathizadeh, Gabriel [12]. There are also works in which holomorphic Banach bundles and their sheaf cohomology groups are treated. See Lempert [40] and Kim [27]. A further reason for an investigation of these complexes might originate in BRST-theories ([22]), which we do not touch here explicitly, although we give a modest interpretation of the introduced structures in the realm of quantum theory.

The aim of our paper is to study elliptic complexes for the case of infinite rank Hilbert C^*-bundles which are induced by Lie group representations, and to show as well how the theory of connections and appropriately generalized elliptic complexes apply in this situation. In order to make the considered situation not too general, we choose a specific Lie group representation (the Segal–Shale–Weil representation of the metaplectic group). We hope that the reader may follow the text easier. Our further aim is to find examples for the theory of Mishchenko and Fomenko ([14]) that would not be trivial or covered by the theory developed around the classical Atiyah–Singer index theorem, and to show how the Hodge theory for certain C^*-bundles (derived in [35] and [36]) make us able to describe the cohomology of elliptic complexes on Hilbert bundles easily.

Let (V, ω_0) be a symplectic vector space and L be a Lagrangian subspace of it. The Segal–Shale–Weil representation is a non-trivial representation of the connected double cover $\widetilde{G}$ of the symplectic group on a complex Hilbert space by unitary operators. The Hilbert space is formed by the L^2-functions defined on L. For references to this representation see, e.g., Weil [70], Borel, Wallach [5], Wallach [68] and Folland [13]. Let us denote the dual of the Segal–Shale–Weil representation by σ and the continuous dual of the Hilbert space by H. We call the resulting representation the *oscillator representation*. The double cover, the metaplectic group, acts by $\sigma_k : \widetilde{G} \to \mathrm{Aut}(H^k)$ on the tensor products $H^k = \bigwedge^k V^* \otimes H$ in a natural way. Let CH denote the algebra of compact operators on the vector space carrier of the Segal–Shale–Weil representation, i.e., on $L^2(L)$. On the vector space space CH, we consider the representation, denoted by ρ, that is given by the conjugation by the Segal–Shale–Weil representation.

We introduce a right CH-module structure $H \times CH \to H$ on H and a CH-product $H \times H \to CH$ obtaining a Hilbert CH-module – a mixture of a module and of a normed vector space as described by Paschke [53] and Rieffel [55]. The spaces H^k, $k \geq 1$, are made Hilbert CH-modules as well. Inducing (σ_k, H^k)

to a principal $\widetilde{G}$-bundle over a symplectic manifold M and in particular, to the so-called metaplectic bundle of a metaplectic structure (Kostant [32]), one gets Banach bundles $\mathcal{H}^k$ (Sect. 2). Further, we associate (ρ, CH) to the metaplectic bundle, getting a continuous Banach bundle. We denote it by $\mathcal{CH}$ and call it the bundle of measurements. Then we define certain intermediate maps $\mathcal{H} \times_M \mathcal{CH} \to \mathcal{H}$ and $\mathcal{H} \times_M \mathcal{H} \to \mathcal{CH}$, called pointwise structures, which are families of C^*-algebra actions and of C^*-algebra valued products. We are not aware that any substantial modification was done regarding the assumptions in the theory of Mishchenko–Fomenko from [14]. Thus, we adapt the pointwise structures to get the honest required ones. Recall that we have to define maps $\mathcal{H} \times CH \to \mathcal{H}$ and $\mathcal{H} \times_M \mathcal{H} \to CH$ instead of $\mathcal{H} \times_M \mathcal{CH} \to \mathcal{H}$ and $\mathcal{H} \times_M \mathcal{H} \to \mathcal{CH}$, respectively. This can be assured by choosing a trivialization J of $\mathcal{H}$, called a Kuiper map, which is possible in the case of a unitary representation. Nevertheless, let us note that we can choose the maps $\mathcal{H} \times CH \to \mathcal{H}$ and $\mathcal{H} \times_M \mathcal{H} \to CH$ by using a Kuiper map without defining the pointwise structures. We do not do so and work with the pointwise maps as far as possible instead. Both the CH-action and the CH-valued fiber product on $\mathcal{H}^\bullet$ are defined in Sect. 4.2.

We focus our attention on the continuity and differentiability of bundles induced by unitary or isometric representations of Lie groups on infinite dimensional Hilbert or Banach spaces, respectively. We show that one may speak about C^∞-differentiable structures when a particular smooth principal bundle atlas is chosen, but only on C^0-differentiable structures for general representations. We treat connections, vertical lifts, connectors and induced covariant derivatives in the context of Banach bundles, showing that the general approach to connections as given, e.g., in Kolář, Michor, Slovák [31] is possible to apply in the case of infinite bundles as well. Since the oscillator representation is known to be not differentiable, we do not get a C^∞-structure that is induced canonically. As already mentioned, it is possible to choose a Kuiper map. The Kuiper structure on $\mathcal{H}$ is defined, in a non-canonical way, as the maximal smooth atlas containing the Kuiper map. Some facts from Lie groups' representations based on Bruhat [6] ($l = \infty$) and Goodman [17] ($l \in \mathbb{N}_0$) provide us with further C^l-structures as well. We define them in Sect. 3.2. However, only the Kuiper structure seems to have some purpose in the theory of C^*-indices and consequently, for C^*-elliptic complexes.

Let us mention that we don't use other algebras than the C^*-algebras for several reasons. The first one is that we would like to stay close to the approach of quantum physics, for which C^*-algebras are fundamental objects until now. The second one is more pragmatic. As far as we know, the Hodge theory has not yet been developed for bundles over other algebras in a satisfactory form. Let us mention that a different approach might be represented possibly by considering bundles over Fréchet $*$-algebras (see Keyl et al. [26] for these algebras and their connection to quantum physics). This approach eliminates problems with the smoothness of the representations.

Further, we give a straightforward proof of the triviality of the bundle of measurements (Sect. 3.3), which is not easy to find in the literature, but which is colloquial if one is acquainted with certain constructions in non-abelian sheaf cohomology adapted for Banach bundles.

In the last section of the paper, we use the Kuiper map of the oscillator bundle $\mathcal{H}$ to define a product type connection Φ which is flat by construction and induces a cochain complex $d^{\Phi}_{\bullet} = (\Gamma(\mathcal{H}^k), d^{\Phi}_k)_{k\in\mathbb{Z}}$ of exterior covariant derivatives acting between sections of $\mathcal{H}^k$. When the symplectic manifold is compact, we prove the central result of the paper (Theorem 18) that the cohomology groups of $d^{\Phi}_{\bullet}$ are isomorphic to the tensor product of the de Rham groups and H. We show also how to use the Hodge theory for projective and finitely generated Hilbert CH-bundles in the case of $d^{\Phi}_{\bullet}$, that is elliptic, for to produce a substantially shorter proof of Thm. 18.

In order to specify our terminology, we start with a preamble to which the reader may come back during the reading.

Preamble

- a1) Let X be a topological space. The symbol $\mathrm{Aut}(X)$ denotes the set of all homeomorphisms of X.
- a2) All vector spaces are considered to be Fréchet. Hilbert spaces are considered with the norm topologies induced by scalar products and the scalar products are anti-linear in the left variable (physicists' convention).
- a3) If X is a vector space, maps in $\mathrm{Aut}(X)$ are additionally supposed to be linear. By $\mathrm{End}(X)$, we understand continuous linear maps $X \to X$. In particular, $\mathrm{End}(X) = B(X)$ where $B(X)$ are bounded operators on X. By a functional on a Banach space, we mean a continuous linear map of this space into the field of complex numbers
- b) By a C^l-differentiable structure on a manifold, we understand a maximal C^l-atlas with respect to the inclusion. A C^∞-differentiable structure is called a smooth structure.
- c) All manifolds are C^0-differentiable (topological) Banach manifolds and all base spaces of fiber bundles are finite dimensional smooth manifolds. All bundles are locally trivial.
- d) Representations of Lie groups are considered continuous in the classical representation theory sense. For a homomorphism $\sigma : G \to \mathrm{Aut}(X)$, of a Lie group G into the linear homeomorphisms of a normed vector space X to be a representation means that $\widetilde{\sigma} : G \times X \to X$ given by $\widetilde{\sigma} : G \times H \ni (g, f) \mapsto \sigma(g)f \in X$ is continuous.[1]
- e) When an index of a labeled object or a labeled morphism exceeds its allowed range, we consider it as zero.

[1] If more details needed, see, e.g., Knapp [30].

2 $\widetilde{G}$-module and CH-module structures

In this section, we introduce the higher oscillator representation of the metaplectic group and the Hilbert CH-module structure on the underlying space of this representation, and we prove some properties of these structures.

Let (V, ω_0) be a real symplectic vector space of dimension $2n \geq 1$ and G denote the symplectic group $Sp(V, \omega_0)$. Let us choose a compatible positive complex structure J_0 on V. By a compatible positive complex structure, we mean an $\mathbb{R}$-linear map $J_0 : V \to V$ such that $J_0^2 = -\mathrm{Id}_V$ and such that g_0 defined by $g_0(u, v) = \omega_0(J_0 u, v)$, $u, v \in V$, is a positive definite bilinear form. Denoting the orthogonal group of (V, g_0) by $O(V, g_0)$, we set $K = O(V, g_0) \cap Sp(V, \omega_0)$ for the unitary group of the hermitian vector space (V, J_0, g_0).

The symplectic group is known to retract smoothly onto K (see, e.g., [13]) the first fundamental group of which is isomorphic to $\mathbb{Z}$. By the theory of covering spaces, there exists a unique smooth connected 2-fold covering $\lambda : \tilde{G} \to G$ of G up to a smooth covering isomorphism. Choosing an element in the λ-preimage of the neutral element of G and declaring it as the neutral element of $\widetilde{G}$, define a unique Lie group structure on $\widetilde{G}$. This is the so called *metaplectic group*, denoted by $Mp(V, \omega_0)$. Further, we set $\widetilde{K} = \lambda^{-1}(K)$ for the corresponding double cover of the unitary group.

Basics on oscillator representation: After Shale [64] and Weil [70], it is known that there exists a specific non-trivial unitary representation of $\widetilde{G}$, called metaplectic, Shale–Weil, Segal–Shale–Weil, oscillator or symplectic spinor representation. It is called a spinor representation probably because it does not descend to a true representation of the symplectic group. Since $\widetilde{G}$ is non-compact, the space carrying this representation is an infinite dimensional Hilbert space. The representation intertwines so-called twisted Schrödinger representations and can be made a representation (of the metaplectic group) on the Hilbert space $L^2(L)$ of complex valued square Lebesgue integrable functions defined on a Lagrangian subspace $L \subseteq (V, \omega_0)$. The Lebesgue measure is determined by the norm on L induced by the inner product g_0 restricted to $L \times L$. For the uniqueness of the representation, we refer to Wallach [68], p. 224. We denote the Hilbert space $L^2(L)$ by $\check{H}$ and fix a Segal–Shale–Weil representation, denoting it by $\check{\sigma} : \widetilde{G} \to U(L^2(L))$ where $U(X)$ denotes the space of unitary operators on a Hilbert space X.

Notation for dualities & musical isomorphisms: Let $(\check{H}, (,)_{\check{H}})$ be a complex Hilbert space, and let us denote its continuous dual by H. For each $f \in H$, we denote its dual by $f^\sharp$, which is the unique element in $\check{H}$ such that $f(v) = (f^\sharp, v)_{\check{H}}$ for any $v \in \check{H}$. By the Riesz's representation theorem for Hilbert spaces, the element $f^\sharp$ exists also for non-separable Hilbert spaces. For any $v \in \check{H}$, we denote by $v^\flat$ the element of H defined by $v^\flat(w) = (v, w)_{\check{H}}$, $v, w \in \check{H}$. We set $(f, g)_H = (g^\sharp, f^\sharp)_{\check{H}}$ in order the product be anti-linear in the left variable. With this notation, $(H, (,)_H)$ is a complex Hilbert space and

$\sharp : \check{H} \to H$ and $\flat : H \to \check{H}$ are anti-unitary maps, i.e., they are complex anti-linear and intertwine the corresponding scalar products, i.e., $(v^\flat, w^\flat)_H = (w,v)_{\check{H}}$, $(f^\sharp, g^\sharp)_{\check{H}} = (g,f)_H$. For any pair $u \in \check{H}$ and $v \in H$, we denote by $u \otimes v$ the endomorphism $u \otimes v : w \ni \check{H} \mapsto v(w)u \in \check{H}$. For any linear continuous map $A : X \to Y$ between Hilbert spaces X and Y, $A^* : Y \to X$ denotes its adjoint.

Because we would like to consider right C^*-modules, it is more convenient to use the dual of the Segal–Shale–Weil representation. We denote it by σ and call it the *oscillator representation.* Using the notation defined, we have a homomorphism

$$\sigma : \widetilde{G} \to U(H)$$

The homomorphism is a representation, i.e., it is continuous in the sense described in Preamble, Par. d.

Let us notice that the oscillator representation decomposes into two irreducible subrepresentations, the space of functionals on even and that of odd functions in $\check{H}$. See Wallach [68] and Folland [13] for more information on $Mp(V, \omega_0)$ and the Segal–Shale–Weil representation.

Since V is equipped with g_0 and ω_0 (the latter gives an orientation to V which is induced by $\omega_0^{\wedge n}$), we get a canonical scalar product (Hodge type product) on $\bigwedge^k V^*$, $k = 0, \ldots, 2n$. We denote it by g_0 as well. For $k = 0, \ldots, 2n$, the tensor product $H^k = \bigwedge^k V^* \otimes H$ is equipped with the norm topology induced by the canonical Hilbert space inner product on the tensor product of Hilbert spaces. Let $\lambda^{*\wedge k}$ denote the k-th wedge product of the dual of the representation (and covering) $\lambda : \widetilde{G} \to Sp(V, \omega_0) \subseteq \mathrm{Aut}(V)$. The tensor product representations $\sigma^k : \widetilde{G} \to \mathrm{Aut}(H^k)$ are given by

$$\sigma^k(g)(\alpha \otimes f) = \lambda^{*\wedge k}(g)\alpha \otimes \sigma(g) f$$

where $g \in \widetilde{G}$, $\alpha \in \bigwedge^k V^*$ and $f \in H$. The above prescription for σ^k is extended linearly to non-homogeneous elements. Let us remark that $H^0 = H$ is the representation space of the oscillator representation $\sigma = \sigma^0$. We set $H^\bullet = \bigoplus_{k=0}^{2n} H^k = \bigoplus_{k=0}^{2n} \bigwedge^k V^* \otimes H$. In a parallel to ordinary spinors, we call $H^\bullet = \bigoplus_{k=0}^{2n} H^k$ the *higher oscillator module* and the representation $\sigma^\bullet = \bigoplus_{k=0}^{2n} \sigma^k$ the *higher oscillator representation.*

Let us denote the space of compact operators on $\check{H}$ equipped with the operator norm by CH, and consider the representation

$$\rho : \widetilde{G} \to \mathrm{Aut}(CH)$$

of the metaplectic group $\widetilde{G}$ on CH given by $\rho(g)a = \check{\sigma}(g) \circ a \circ \check{\sigma}(g)^{-1}$, where $g \in \widetilde{G}$ and $a \in CH$. The representation is well established since CH is a closed two-sided ideal in the space $\mathrm{End}(\check{H})$ of bounded operators on $\check{H}$.

With the adjoint of a Hilbert space map as the $*$-operation, $(CH, || \, ||_{CH}, *)$ becomes a C^*-algebra, where $|| \, ||_{CH}$ is the restriction of the operator norm on $B(\check{H})$ to CH.

Definition: The *right action* $H^\bullet \times CH \to H^\bullet$ of CH on $H^\bullet$ is defined on homogeneous elements by

$$(\alpha \otimes u) \cdot a = \alpha \otimes (u \circ a)$$

where $\alpha \otimes u \in \bigwedge^\bullet V^* \otimes H$ and $a \in CH$. It is extended linearly to all elements of $H^\bullet$. The *CH-product* $(,) : H^\bullet \times H^\bullet \to CH$ is given by

$$(\alpha \otimes u, \beta \otimes v) = g_0(\alpha, \beta) u^\sharp \otimes v$$

where $\alpha, \beta \in \bigwedge^\bullet V^*$ and $u, v \in H$. The product is extended by linearity. It maps $H^\bullet \times H^\bullet$ into finite rank operators, especially, into CH. We denote the induced (C^*-module) norm on $H^\bullet$ by $||\,||$. It is given by $||f|| = \sqrt{||(f,f)||_{CH}}$, $f \in H^\bullet$.

Lemma 1: For $k = 0, \ldots, 2n$ and $\alpha \otimes f \in H^k$

$$||\alpha \otimes f|| = \sqrt{g_0(\alpha, \alpha)}||f||_H$$

In particular, the norm $||\,||$ coincides on homogeneous elements with the norm induced by the inner product on H^k.

Proof. The appearance of the first factor, $\sqrt{g_0(\alpha,\alpha)}$, is easy to show. For the second one, we compute the operator norm $||\,||_{CH}$ of the map $f^\sharp \otimes f : \check{H} \ni k \mapsto f(k) f^\sharp$, where $f \in H$. On one hand, we have $||(f,f)(k)||^2_{\check{H}} = ||f^\sharp f(k)||^2_{\check{H}} = |f(k)|^2 (f^\sharp, f^\sharp)_{\check{H}} = |f(k)|^2 (f,f)_H \leq ||f||^2_H \; ||k||^2_{\check{H}} \; ||f||^2_H$. Consequently, $||(f,f)||_{CH} \leq ||f||^2_H$. On the other hand, for $k = f^\sharp \neq 0$, we get $||f^\sharp f(f^\sharp)||^2_{\check{H}} / \; / ||f^\sharp||^2_{\check{H}} = ||f||^2_H (f,f)_H / ||f||^2_H = ||f||^2_H$. Thus, $||(f,f)||_{CH} = ||f||^2_H = (f,f)_H$. For the norm induced by the CH-product on H, we obtain $||f|| = \sqrt{||(f,f)||_{CH}} = ||f||_H$, and consequently, $||f|| = \sqrt{(f,f)_H}$. ☐

Lemma 2: The pair $(H^\bullet, (,))$ is a finitely generated projective Hilbert CH-module.

Proof. It is obvious that $H^\bullet$ is a right CH-module. Proving that $(,)$ is hermitian symmetric and CH-sesquilinear, is a routine check. Indeed, for $a \in CH$, $\alpha \otimes f, \beta \otimes h \in H^k$, and $u \in \check{H}$

$$\begin{aligned}
((\alpha \otimes f) \cdot a, \beta \otimes h)(u) &= g_0(\alpha,\beta) \left((f \cdot a)^\sharp \otimes h\right)(u) = g_0(\alpha,\beta) \left((f \circ a)^\sharp \otimes h\right)(u) \\
&= g_0(\alpha,\beta) \left(a^*(f^\sharp) \otimes h\right)(u) = g_0(\alpha,\beta) h(u) a^*(f^\sharp) \\
&= a^* \left(g_0(\alpha,\beta) h(u) f^\sharp\right) = a^* \left(g_0(\alpha,\beta) \left(f^\sharp \otimes h\right)(u)\right) \\
&= \left(a^* \circ \left(g_0(\alpha,\beta) \left(f^\sharp \otimes h\right)\right)\right)(u) \\
&= \left(a^* \circ (\alpha \otimes f, \beta \otimes h)\right)(u)
\end{aligned}$$

Thus, $(,)$ is anti-linear over CH in the left variable. The hermitian symmetry is proved similarly. The right CH-linearity follows from the hermitian symmetry and left CH-linearity.

The projectivity follows from the Magajna theorem (Theorem 1, Magajna [44]).

For the existence of a finite set of generators, let us choose a basis $\mathfrak{C}$ of $\bigwedge^\bullet V^*$ and take a non-zero vector $w \in H$. For any $v \in H^\bullet$, there exist scalars $\lambda_\mathfrak{a} \in \mathbb{C}$ and vectors $v_\mathfrak{a} \in H$, $\mathfrak{a} \in \mathfrak{C}$, such that $v = \sum_{\mathfrak{a}\in\mathfrak{C}} \lambda_\mathfrak{a} \mathfrak{a} \otimes v_\mathfrak{a}$. For each $v' \in H$, let us set $p_{v',w}(k) = \frac{v'(k)}{||w||_H^2} w^\sharp$, $k \in \check{H}$. Then $v = \sum_{\mathfrak{a}\in\mathfrak{C}} (\mathfrak{a} \otimes w) \cdot (\lambda_\mathfrak{a} p_{v_\mathfrak{a},w})$. Thus, $\{\mathfrak{a} \otimes w | \, \mathfrak{a} \in \mathfrak{C}\}$ represents a finite set of generators of $H^\bullet$.

Since the C^*-module norm $|| \, ||$ restricted to H^0 is the dual of the Hilbert norm on $\check{H}$ (Lemma 1), we see that $H^\bullet$ is complete. □

Remark: A similar proof can be found in [34] where, though, a determination of the inducing scalar product for the C^*-norm is omitted. The above construction of a Hilbert CH-module structure can be done for an arbitrary Hilbert space.

Note that in the bra–ket convention, $p_{v',w} = \frac{|w\rangle\langle v'|}{\langle w|w\rangle} = \frac{|w\rangle\langle v'|}{||w||^2}$.

Definition: For $a, b \in CH$, $u \in \check{H}$ and $h \in H$, let us set

$\circ : CH \times CH \to CH$	$\circ(a,b) = a \circ b$	(composition)
$\otimes : \check{H} \times H \to CH$	$\otimes(u,h) = u \otimes h$	(tensor product)
$ev : H \times \check{H} \to \mathbb{C}$	$ev(h,u) = h(u)$	(evaluation)

Lemma 3: The CH-action, the composition, the tensor product and the evaluation are $\widetilde{G}$-equivariant. The CH-product $(,) : H \times H \to CH$ and the induced norm $|| \, || : H^\bullet \to \mathbb{R}$ are $\widetilde{K}$-equivariant. For any $g \in \widetilde{G}$, $v \in \check{H}$ and $h \in H$

$$\rho(g)(v \otimes h) = \check{\sigma}(g)v \otimes \sigma(g)h \tag{1}$$

Proof. For $g \in \widetilde{G}$, $v, w \in \check{H}$ and $h \in H$

$$\begin{aligned} \rho(g)(v \otimes h)(w) &= \left(\check{\sigma}(g) \circ (v \otimes h) \circ \check{\sigma}(g)^{-1}\right)(w) \\ &= \check{\sigma}(g)(v) h(\check{\sigma}(g)^{-1} w) \\ &= (\check{\sigma}(g)v \otimes \sigma(g)h)(w) \end{aligned}$$

obtaining formula (1).

(1) is used to prove the $\widetilde{G}$-equivariance of the tensor product readily. We check the $\widetilde{G}$-equivariance of the action $H^\bullet \times CH \to H^\bullet$ and the $\widetilde{K}$-equivariance of the CH-product. For $g \in \widetilde{G}$, $f = \alpha \otimes u \in H^k$, $k = 0, \ldots, 2n$, and $a \in CH$

$$\begin{aligned} (\sigma^k(g)f) \cdot (\rho(g)a) &= \left(\lambda^{*\wedge k}(g)\alpha \otimes \sigma(g)u\right) \cdot (\check{\sigma}(g) \circ a \circ \check{\sigma}^{-1}(g)) \\ &= \lambda^{*\wedge k}(g)\alpha \otimes \left(\sigma(g)u \circ (\check{\sigma}(g) \circ a \circ \check{\sigma}^{-1}(g))\right) \\ &= \lambda^{*\wedge k}(g)\alpha \otimes \left((u \circ \check{\sigma}^{-1}(g)) \circ (\check{\sigma}(g) \circ a \circ \check{\sigma}^{-1}(g))\right) \\ &= \lambda^{*\wedge k}(g)\alpha \otimes (u \circ a \circ \check{\sigma}^{-1}(g)) \\ &= \lambda^{*\wedge k}(g)\alpha \otimes \sigma(g)(u \circ a) = \sigma^k(g)(\alpha \otimes u \cdot a) = \sigma^k(g)(f \cdot a) \end{aligned}$$

The equivariance on non-homogeneous elements follows from linearity.

Let $g \in \widetilde{K}$ and $h = \beta \otimes v \in H^k$. Since $g \in \widetilde{K}$, we have $g_0(\lambda^{*\wedge k}(g)\alpha, \lambda^{*\wedge k}(g)\beta) = g_0(\alpha, \beta)$. Further

$$\begin{aligned}
(\sigma^k(g)f, \sigma^k(g)h) &= g_0(\lambda^{*\wedge k}(g)\alpha, \lambda^{*\wedge k}(g)\beta)\left((\sigma(g)u)^\sharp \otimes (\sigma(g)v)\right) \\
&= g_0(\alpha, \beta)\check{\sigma}(g)u^\sharp \otimes \sigma(g)v \\
&= g_0(\alpha, \beta)\rho(g)(u^\sharp \otimes v) = \rho(g)g_0(\alpha, \beta)(u^\sharp \otimes v) \\
&= \rho(g)(\alpha \otimes u, \beta \otimes v) = \rho(g)(f, h)
\end{aligned}$$

where formula (1) is used in the second row. For the norm, consider $||\sigma^k(g)f||^2 = ||(\sigma^k(g)f, \sigma^k(g)f)||_{CH} = ||\rho(g)(f, f)||_{CH} = ||(f, f)||_{CH} = ||f||^2$, $g \in \widetilde{K}$, since $\rho(g)$ is isometric.

$\widetilde{G}$-equivariances of $\circ$ and ev can be proved in a similar way. $\square$

3 Geometric lifts of $\widetilde{G}$- and CH-module structures

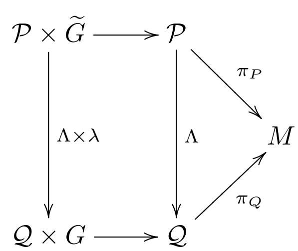
In this chapter, we introduce the higher oscillator bundle and the bundle of measurements. We investigate continuity and smoothness of canonical atlases of these bundles. For this, we examine the Segal–Shale–Weil representation from the analytic point of view.

For a symplectic manifold (M^{2n}, ω), we consider

$$\mathcal{Q} = \{A : V \to T_m^*M \,|\, \omega_m(Au, Av) = \omega_0(u, v) \text{ for any } u, v \in V, m \in M\}$$

together with the bundle projection $\pi_Q(A) = m$ if and only if $A : V \to T_m^*M$. For the topology on $\mathcal{Q}$, we take the final one with respect to inverses of all bundle maps for the projection π_Q. The right action of $G = Sp(V, \omega_0)$ on $\mathcal{Q}$ is given by the composition from the right, i.e., $(A, g) \mapsto A \circ g$ for $A \in \mathcal{Q}$ and $g \in G$. This make $\pi_Q : \mathcal{Q} \to M$ a principal G-bundle.

Let us consider pairs (π_P, Λ), where $\pi_P : \mathcal{P} \to M$ is a principal $\widetilde{G}$-bundle over M and $\Lambda : \mathcal{P} \to \mathcal{Q}$ is a smooth map. Such a pair is called a *metaplectic structure* if Λ intertwines the actions of G and $\widetilde{G}$ on $\mathcal{Q}$ and $\mathcal{P}$, respectively, with respect to the homomorphism $\lambda : \widetilde{G} \to G$. In other words, the following diagram has to commute.



(The horizontal arrows represent the right actions of the groups $\widetilde{G}$ and G on the total spaces $\mathcal{P}$ and $\mathcal{Q}$, respectively.) The bundle $\mathcal{P}$ is called a *metaplectic bundle*.

The bundle $\mathcal{Q}$ is isomorphic to the quotient of $\mathcal{P}$ by the action of the 2-element group $\mathrm{Ker}\,\lambda$. Further, we may consider a version of a metaplectic structure over the complex numbers $\mathbb{C}$ as well, getting a so-called Mp^c-structure. Notice that a metaplectic structure is formally similar to that of a spin structure.

Remark: Let us mention that a metaplectic structure exists on a symplectic manifold (M, ω) if and only if the second Stiefel–Whitney class of TM vanishes. Their set modulo an appropriate equivalence relation is isomorphic to the singular cohomology group $H^1(M, \mathbb{Z})$ as a set. See Kostant [32]. Note that the Mp^c-structures are known to exist without any restriction on the topology of the underlying symplectic manifold (Forger, Hess [15] or Robinson, Rawnsley [56]). Compare approaches of Habermann, Habermann in [20] and Cahen, Gutt, La Fuente-Gravy, Rawnsley in [7].

Similarly as in the orthogonal spinor case, we obtain by induction from representations $\sigma^k : \widetilde{G} \to \mathrm{Aut}(H^k)$ and the principal bundle $\mathcal{P} \to M$ bundles

$$p_k : \mathcal{H}^k = \mathcal{P} \times_{\sigma^k} H^k \to M$$

where $k = 0, \ldots, 2n$. We call the bundle $\mathcal{H}^\bullet = \bigoplus_{k=0}^{2n} \mathcal{H}^k$ the *higher oscillator bundle*. The basic oscillator bundle $\mathcal{H} = \mathcal{H}^0 = \mathcal{P} \times_\sigma H$ coincides with the dual of the Kostant's or symplectic spinor bundle as it is called by Habermann, Habermann [20]. Their continuity and differentiability are discussed in the next sections.

Definition: The Banach bundle $\mathcal{CH}$ associated to $\mathcal{P}$ via $\rho : \widetilde{G} \to \mathrm{Aut}(CH)$, i.e.,

$$\mathcal{CH} = \mathcal{P} \times_\rho CH$$

is called the *bundle of measurements*.

The bundle $\mathcal{CH}$ exists regardless of the existence of the metaplectic structure because $\mathrm{Ker}\,\lambda = \mathrm{Ker}\,\rho$.

Remark (Terminology related to measurements in Quantum theory): The names 'bundle of measuring devices' and 'bundle of filters' for the bundle of measurements $\mathcal{CH}$ seem appropriate as well. The word *filter* is used in quantum theory for a theoretical measuring device. See Ludwig [42] for the role of filters, and still topical texts of Birkhoff, von Neumann [3] and von Neumann [51] for a mathematical framework for quantum (and also classical) physical theories.

In each point m of a symplectic manifold (M, ω) (a 'curved' phase space), we imagine a measuring device (filter) constituting of filtering channels, which are represented by elements of $\mathcal{CH}_m$, the fiber in m. Assuming that for any quantum measurement's result, there is a corresponding measuring device, $\mathcal{CH}_m$ presents a stack for all possible results of quantum measurements that can be obtained on a system found classically in the point m. An ideal filter in point m is a measuring device corresponding to a rank 1 operator in $\mathcal{CH}_m$. Let $k : \mathcal{CH} \to M$ be the bundle projection of the bundle of measurements onto M. Suppose that we have measuring devices filling-up a submanifold $\iota : M' \hookrightarrow M$ (possibly only a discrete set of points). Then we have the pull-back bundle $k_{M'} : \iota^*(\mathcal{CH}) \to M'$. In real measurements, each measuring device corresponds to a finite linear combination of rank 1 operators in the fibers $(\iota^*(\mathcal{CH}))_m, m \in M'$.

3.1 Continuity of G-structures

We make a comment on a canonical continuity of atlases of induced Banach bundles. Let us recall that for a principal Q-bundle $\pi : \mathcal{R} \to M$ and a representation $\kappa : Q \to \mathrm{Aut}(X)$ of a Lie group Q on a topological vector space X, the induced (associated) bundle is the quotient $\mathcal{F} = (\mathcal{R} \times X)/\simeq$ together with the appropriate bundle projection onto M. The equivalence relation $\simeq$ is defined for any $r, s \in \mathcal{R}$ and $v, w \in X$ by $(r, v) \simeq (s, w)$ if and only if there exists an element $g \in Q$ such that $s = rg^{-1}$ and $w = \kappa(g)v$. The topology induced on the quotient is the final topology, i.e., the finest one for which the canonical projection $q : \mathcal{R} \times X \to (\mathcal{R} \times X)/\simeq$ is continuous. The bundle projection $p : \mathcal{F} \to M$ is given by $p([(r, v)]) = \pi(r)$, where $[(r, v)] \in \mathcal{F}$. For $U \subseteq M$, let $\mathcal{F}_U$ denote the set $p^{-1}(U)$. Especially, $\mathcal{F}_{\{m\}}$, denoted by $\mathcal{F}_m$, is the fiber of $\mathcal{F}$ in $m \in M$.

Let us equip the higher oscillator bundle and the bundle of measurements with the final topologies and prove that they form Hilbert and Banach bundles, respectively. Actually, it is convenient to prove a more general theorem.

Theorem 4: Let $\kappa : Q \to \mathrm{Aut}(X)$ be an isometric or a unitary representation of a Lie group Q on Banach or Hilbert space X, respectively. Let $p : \mathcal{F} \to M$ be the associated bundle to a principal Q-bundle $\pi : \mathcal{R} \to M$. Then p is canonically a continuous Banach or a Hilbert bundle, respectively.

Proof. Let us take a principal bundle atlas $(U_\alpha, \phi_\alpha)_{\alpha \in I}$ in the differentiable structure of the principal bundle π. In particular, $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times Q$. For to get a bundle atlas on $p : \mathcal{F} \to M$, we define maps $\psi'_\alpha : U_\alpha \times X \to p^{-1}(U_\alpha)$ by $\psi'_\alpha(m, v) = q(\phi_\alpha^{-1}(m, e), v)$, where e is the unit element in Q, $m \in U_\alpha$ and $v \in X$. The maps ψ'_α are continuous because ϕ_α^{-1} and q are continuous. They are fiber preserving in the sense that $p \circ \psi'_\alpha = \mathrm{pr}_1$, where $\mathrm{pr}_1 : U_\alpha \times X \to U_\alpha$ is the projection onto the first factor in the product. Their inverse maps, $\psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times X$ are given by $\psi_\alpha([(r, v)]) = (\pi(r), \kappa(g^{-1})v)$ where g is the unique element of Q such that $rg = \phi_\alpha^{-1}(m, e)$ and $m = \pi(r)$. The independence on the chosen representative of an equivalence class follows from the definition of the associated bundle. Let $U \subseteq U_\alpha$ and $Y \subseteq X$ be open sets. The ψ_α-preimage of $U \times Y$ is $S = \{[(\phi_\alpha^{-1}(m, e), w)] |\, m \in U, w \in Y\}$. Consequently, the q-preimage $q^{-1}(S) = \{(\phi_\alpha^{-1}(m, e)g^{-1}, \kappa(g)w) |\, m \in U, g \in Q, w \in Y\}$. The preimage can be written as $\bigcup_{g\in Q}(\phi_\alpha^{-1}(U,e)g^{-1},\kappa(g)Y)$.[2] For each $g \in Q$, both of the sets $\{\phi_\alpha^{-1}(m,e)g^{-1} | \, m \in U\}$ and $\{\kappa(g)y | \, y \in Y\}$ are open since ϕ_α^{-1} is a homeomorphism and the map $\kappa(g)$ is a homeomorphism as well. Due to the definition of the topology on Cartesian products, $q^{-1}(S)$ is open and thus ψ_α is continuous. Summing-up, ψ_α is a homeomorphism and $\mathfrak{A} = (U_\alpha, \psi_\alpha)_{\alpha\in I}$ is a bundle atlas.

Choosing a different atlas in the differentiable structure of $\mathcal{R}$, leads to an induced atlas $\widetilde{\mathfrak{A}} = (\widetilde{U}_\alpha, \widetilde{\psi}_\alpha)_{\alpha\in\widetilde{I}}$ that is different from $\mathfrak{A}$ in general. However, the transition functions $\psi_{\alpha\beta}$ for $\mathfrak{A}\cup\widetilde{\mathfrak{A}}$ are given by composition of the representation with the transition functions $(\phi_{\alpha\beta})_{\alpha,\beta}$ of the principal bundle, i.e., $\psi_{\alpha\beta} = \kappa\circ\phi_{\alpha\beta}$. Since κ is continuous (as a map $Q \times X \to X$) and unitary, it is continuous as a map $Q \to U(X)$ where $U(X)$ is considered with the strong topology (see Schottenloher [63] or Knapp [30], pp. 10 and 11). Thus we have defined an independent continuous bundle structure.

Suppose that κ is a unitary representation on a Hilbert space X. The fiber-wise scalar product is defined by $([(r,v)],[(r,w)])_S = (v,w)_X$ where at the right-hand side, the scalar product on X is meant. The independence of the scalar product on the chosen representatives follows from the fact that the representation is unitary. Let $|| \, ||_S : \mathcal{F} \to \mathbb{R}$, given by $||a||_S = \sqrt{(a,a)_S}$, $a \in \mathcal{F}$, be the induced norm, and let $(a_n)_n$ converge to an element a in the quotient topology of $\mathcal{F}$. By definition of the quotient topology, there exists a selector $(b_n)_n$ on $\{q^{-1}(\{a_n\}) | \, n \in \mathbb{N}\}$ which converges in $\mathcal{R} \times X$. Denoting its limit by b, we have that 1) $c_n = \mathrm{pr}_2(b_n)$ converges to $c = \mathrm{pr}_2(b)$ in X, and 2) $a = [b]$, where pr_2 denotes the projection $\mathcal{R} \times X \to X$ onto the second factor. Summing-up, $||a_n||_S = \sqrt{(a_n,a_n)_S} = \sqrt{(c_n,c_n)_X} \to ||c||_X = ||[b]||_S = ||a||_S$. Thus, $|| \, ||_S$ is continuous. The continuity of $(,)_S$ in each of the arguments follows from the polarization identity.

Rewriting the definition of $(,)_S$, we have $(q(r,v),q(r,w))_S = (v,w)_X$. Further $(\psi'_\alpha(m,v),\psi'_\alpha(m,w))_S = (q(\phi_\alpha^{-1}(m,e),v),q(\phi_\alpha^{-1}(m,e),w))_S = (v,w)_X$, proving the unitarity of ψ'_α. Consequently, $\mathcal{F}$ is a Hilbert bundle.

Similarly one proceeds in the case of the isometric Banach representation.

□

Let us call any atlas $(U_\alpha, \psi_\alpha)_{\alpha\in I}$ on an induced bundle $\mathcal{F}$ constructed as in the preceding proof the *canonically induced atlas* or, in more detail, the atlas canonically induced by the principal bundle atlas $(U_\alpha, \phi_\alpha)_{\alpha\in I}$. As follows from the above proof, canonical atlases induced by different but compatible atlases are C^0-compatible. In particular, one can speak about the induced C^0-differentiable structure.

Corollary 5: For a symplectic manifold (M,ω) and a metaplectic structure (π_P, Λ) on it, the higher oscillator bundle and the bundle of measurements, both equipped with the canonically induced atlases by a principal bundle atlas, are

[2] More properly, the image equals to $\bigcup_{g\in Q}\{(\phi_\alpha^{-1}(m,e)g^{-1},\kappa(g)y) | \, m \in U, y \in Y\}$.

continuous Banach bundles. The basic oscillator bundle is a continuous Hilbert bundle.

3.2 Differentiable structures on oscillator bundle

A substantial obstacle for analysis on induced Banach or Hilbert bundles of infinite rank is that they need not be smooth or not even l-times differentiable for some $l \in \mathbb{N}$. When speaking about C^l-sections, one has to take care to which atlas the degree of differentiability refers.

Let us recall that for a representation $\kappa : Q \to \mathrm{Aut}(X)$ of a Lie group Q on a Banach space X, a vector $v \in X$ is called smooth if $Q \ni g \mapsto \kappa(g)v \in X$ is smooth, i.e., it has each Fréchet differential in each point g.

Let $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to Q$ be a cocycle of transition functions of a principal Q-bundle $\mathcal{R}$ corresponding to a smooth atlas $(U_\alpha, \phi_\alpha)_{\alpha \in I}$ of $\mathcal{R}$. Using the definition of the canonically induced atlas, we obtain that its transition maps $\psi_{\alpha\beta} : (U_\alpha \cap U_\beta) \times X \to (U_\alpha \cap U_\beta) \times X$ are given by $\psi_{\alpha\beta}(m, v) = (m, \kappa(\phi_{\alpha\beta}(m))v)$. We agree that a Banach bundle atlas is called smooth if the transition functions $\psi_{\alpha\beta}$ considered as maps of $U_\alpha \cap U_\beta \to \mathrm{Aut}(X) \subseteq \mathrm{End}(X)$ are smooth. On $\mathrm{Aut}(X)$, we consider the topology induced from the strong operator topology on $\mathrm{End}(X)$. This happens if and only if $m \mapsto \kappa(\phi_{\alpha\beta}(m))$ is smooth. For that it is sufficient the representation to be smooth as a map into $\mathrm{End}\,(X)$ with the strong topology.

Thus we get

Lemma 6: The bundle atlas canonically induced by a principal bundle atlas is smooth if the inducing representation is smooth.

The opposite implication does not hold. For it, take an arbitrary manifold M, and the metaplectic group $Mp(V, \omega)$ for the Lie group Q. Let us consider the product bundle $M \times Q \to M$ as the principal Q-bundle and the oscillator representation σ as the representation κ. For the smooth atlas on the product bundle take the one containing the single chart $U = M$ and the only bundle map $\phi(m, g) = (m, g)$, where $m \in M$ and $g \in Q$. The set of transition functions of this atlas consists just of the constant map $m \mapsto e$, where e is the unit element in the metaplectic group. The set of transition functions of the canonically induced atlas on $\mathcal{F} = (M \times Q) \times_\sigma \check{H}$ contains only $\psi(m, v) = (m, \kappa(e)v) = (m, v)$, which is clearly smooth, although the representation is known not to be. (See Borel, Wallach [5], p. 153, or below.)

Smooth Hilbert bundle à la Kuiper

Because the unitary group of H is contractible in the strong operator topology (Dixmier, Douday [9])[3], there exists an isomorphism $J : \mathcal{H} \to M \times H$ of

[3] It is contractible also in the operator norm topology (Kuiper [38]).

topological vector bundles. We declare the smooth structure of $\mathcal{H}$ as the maximal set of smoothly compatible atlases containing the chart (M, J) and call it the *Kuiper structure* induced by J.

Smooth Fréchet bundle à la Bruhat

Let $S(L)$ denote the Schwartz space of rapidly decreasing smooth functions on L. Vectors in $\check{H} \setminus S(L)$ are exactly the ones which are not smooth for $\check{\sigma}$ (Borel, Wallach [5]). Consequently, smooth vectors of H^k acted on by σ^k form spaces $S^k = \bigwedge^k V^* \otimes S$, where $S \subseteq H$ denotes the $\flat$-image of $S(L) \subseteq L^2(L)$ in $L^2(L)'$. Note that S is not complete in the norm topology inherited from H. To get a complete space, we retopologize it as it is usual in representation theory (see below).

We recall and combine some known facts from Lie groups' representations (based on the Bruhat's dissertation thesis [6]) and some known facts specific for the Segal–Shale–Weil representation.

1) Representations σ^k preserve the space of smooth vectors S^k.

2) Let us denote the restrictions σ^k to S^k by σ_0^k and consider S^k to be equipped with the topology inherited by the injection $S^k \to C^\infty(\widetilde{G}, H^k)$, $S^k \ni v \mapsto (\pi^v : \widetilde{G} \ni g \mapsto \sigma^k(g)(v) \in H^k)$. Here, $C^\infty(\widetilde{G}, H^k)$ is considered with the compact C^∞-topology. (See, e.g., Warner [69], p. 484 for a definition of compact C^∞-topology if necessary.) The restricted representation σ_0^k is continuous in the new topology ([69], Sect. 4.4.).

3) The compact C^∞-topology on S coincides with the classical semi-norm Fréchet topology on this space (see Borel, Wallach [5]).

4) Since (a) σ_0^k is a representation on S^k with the compact C^∞-topology, which is Fréchet, and since (b) the space $(E_\infty)_\infty$ of smooth vectors of the space E_∞ of smooth vectors of a continuous representation on a Fréchet space E[4] is topologically isomorphic to E_∞, we conclude that $\sigma_0^k : \widetilde{G} \to \mathrm{Aut}\,(S^k)$ are smooth maps when $\mathrm{Aut}(S^k) \subseteq \mathrm{End}(S^k)$ is considered with the strong topology. Technically speaking, this is because the assumption of Thm. 4.4.1.7 (see Warner [69]) is satisfied, and thus Remark (3) at pages 258 and 259 of [69] applies.

Taking representations σ_0^k instead of σ^k, we get Fréchet bundles $\mathcal{S}^k = \mathcal{P} \times_{\sigma_0^k} S^k$. By what was mentioned in item 4 above and by Lemma 6, each of the bundles $\mathcal{S}^k$ are smooth with respect to the atlas induced canonically by any smooth principal bundle atlas (on $\mathcal{P}$). We call the bundles $\mathcal{S}^k \to M$ with the smooth atlases *Bruhat structures.* The atlases are constructed as in the proof of Theorem 4.

C^0-Hilbert bundle à la Goodman

[4]Both E_∞ and $(E_\infty)_\infty$ are considered with the compact C^∞-topology.

We can consider spaces W^l of C^l-vectors in H as well, $l \geq 1$, obtaining representations $\sigma^l : \widetilde{G} \to \mathrm{Aut}(W^l)$. See Goodman [17] or Neeb [49]. Consider the tensor product representations $\sigma^{k,l} : \widetilde{G} \to \mathrm{Aut}(\bigwedge^k V^* \otimes W^l)$.

From the known form of the differential of the Segal–Shale–Weil representation (see Habermann [19], or Kirillov [28], p. 184 for the case $n = 1$) and from the definition of the so-called Hermite–Sobolev spaces (as defined, e.g., in Bongioanni, Torrea [4]), spaces W^l are isomorphic to the Hermite–Sobolev spaces denoted by W_2^{2l} or sometimes $W^{2l,2}$ in [4]. Moreover, the compact C^l-topology on W^l, as described in Goodman [17], coincides with the norm topology, generated by the oscillator ladder operators, considered in Bongioanni, Torrea [4]. (Notice that it is easy to see that spaces W^l, with this topology, are Hilbert spaces as well as that $\sigma^{0,l}$ is not C^1.)

The associated bundles $\mathcal{W}^{k,l} = \mathcal{P} \times_{\sigma^{k,l}} (\bigwedge^k V^* \otimes W^l)$ when equipped with the canonically induced atlases are Hilbert bundles with in general, only continuous structures. We call these bundles $\mathcal{W}^{k,l} \to M$ with the continuous atlases the *Goodman structures.*

Remark: In the case of the Bruhat or the Goodman structures, we either loose the completeness with respect to a norm or the smoothness of the representation, respectively.

3.3 Differentiable structure on bundle of measurements

Operations $\sharp : H \to \check{H}$ and $\flat : \check{H} \to H$ lift to bundles $\mathcal{H} \to M$ and $\check{\mathcal{H}} \to M$, respectively. We denote them by the same symbols, i.e., $\sharp : \mathcal{H} \to \check{\mathcal{H}}$ and $\flat : \check{\mathcal{H}} \to \mathcal{H}$. We shall denote the operator $\mathrm{Id}_M \times \sharp : M \times H \to M \times \check{H}$ by $\sharp_M$ and the operator $\mathrm{Id}_M \times \flat : M \times \check{H} \to M \times H$ by $\flat_M$. Let us consider the map

$$\otimes_{\mathcal{CH}} : \check{\mathcal{H}} \times_M \mathcal{H} \to \mathcal{CH} \text{ defined by } [(r,v)] \otimes_{\mathcal{CH}} [(r,f)] = [(r, v \otimes f)]$$

where $v \in \check{H}, f \in H$ and $r \in \mathcal{P}$. Since the operator norm topology on CH is coarser than the Hilbert tensor product topology on $\check{H} \otimes H$, described equivalently by the Hilbert–Schmidt norm, $\otimes_{\mathcal{CH}}$ is continuous. For a moment, let us denote by $\underline{\mathbb{C}}$ the trivial complex line bundle $M \times \mathbb{C}$ over M. This bundle is isomorphic to the bundle associated to $\mathcal{P}$ by the trivial representation of $\widetilde{G}$ on $\mathbb{C}$. Further, we set

$$ev_{\mathcal{H}} : \mathcal{H} \times_M \check{\mathcal{H}} \to \underline{\mathbb{C}} \qquad ev_{\mathcal{H}}([(r,f)],[(r,v)]) = (\pi_P(r), f(v))$$

Instead of $\mathrm{pr}_2\left(ev_{\mathcal{H}}([(r,f)],[(r,v)])\right)$, we write $[(r,f)]([(r,v)])$, where $\mathrm{pr}_2 : M \times \mathbb{C} \to \mathbb{C}$ denotes the projection onto the second factor. The operations $\sharp : \mathcal{H} \to \check{\mathcal{H}}, \flat : \mathcal{H} \to \check{\mathcal{H}}, \otimes_{\mathcal{CH}} : \check{\mathcal{H}} \times_M \mathcal{H} \to \underline{\mathcal{CH}}$ and $ev_{\mathcal{H}} : \mathcal{H} \times_M \check{\mathcal{H}} \to \underline{\mathbb{C}}$ are well defined. The proof of this is based on the $\widetilde{G}$-equivariance (Lemma 3) and goes in the same lines as the one of the correctness of definitions of the $\mathcal{CH}$-product and the $\mathcal{CH}$-action, explained below (Sect. 4. 1).

Notation for bundle $\underline{H}$: For the trivial bundle $\underline{H} = M \times H \to M$, we set $(m, f') +_M (m, f'') = (m, f' + f'')$, $t \odot_M (m, f) = (m, tf)$, and $((m, f'), (m, f'')) = (f', f'')_H \in \mathbb{C}$ where $t \in \mathbb{R}$, $f, f', f'' \in H$, and $m \in M$. This notation may look like superfluous. However, we use it later (sect. 4.3) when it appears to be helpful.

Normalized Kuiper maps: Since σ is a unitary representation of $\widetilde{G}$ on H, the associated bundle $\mathcal{H}$ is a vector bundle with fiber H and structure group $U(H)$ with the strong operator topology (Thm. 4). The set of isomorphism classes of vector bundles with fiber H and structure group $U(H)$ is in a set bijection with the first Čech cohomology group $H^1(M, U(H))$ (see, e.g., Raeburn, Williams [54], pp. 96–97). Because the unitary group of an infinite dimensional Hilbert space is contractible, the Čech cohomology group consists only of one element. Consequently, each bundle must be isomorphic to the trivial one. This means that by definition (p. 93, [54]), there is a fiber preserving homeomorphism J_N of $\mathcal{H}$ and $\underline{H}$ which is a map into $U(H)$ when considered composed with bundle maps. In particular, this implies that J_N is a unitary map in each fiber. We call such a map a normalized Kuiper map. **By a Kuiper map, we shall only mean a normalized Kuiper map**, i.e., a unitary isomorphism.

Since the bundle of measurements $\mathcal{CH}$ is a specific associated bundle to the principal bundle $\mathcal{P}$, the transition functions of $\mathcal{CH}$ are compositions of the transition functions of $\check{\mathcal{H}}$, the representation σ, and the Ad-representation. Consequently, $\mathcal{CH}$ is a trivial bundle by the Dixmier–Douady theory. (See Raeburn, Williams [54], Thm. 4.85 (a), pp. 109–110.) Because the cited result is often scattered through texts, we give an elementary proof based only on the triviality of $\mathcal{H}$ and $\check{\mathcal{H}}$ and functional analysis' methods.

Theorem 7: Let (M, ω) be a symplectic manifold admitting a metaplectic structure (π_P, Λ). The bundle of measurements $\mathcal{CH} \to M$ and the product bundle $M \times CH \to M$ are isomorphic as Banach bundles. Consequently, the bundle of measurements is trivial.

Proof. Let us consider the Banach bundle $\check{\mathcal{H}} = \mathcal{P} \times_{\check{\sigma}} \check{H}$ and let J be a Kuiper map of the bundle $\mathcal{H} \to M$, i.e., $J : \mathcal{H} \to M \times H$ is an isomorphism of continuous Hilbert bundles which is isometric on each fiber. It induces a map $\check{J} : \check{\mathcal{H}} \to M \times \check{H}$ (trivializing $\check{\mathcal{H}}$) given by the formula $\check{J}(\tilde{v}) = (J(\tilde{v}^\flat))^{\sharp_M}$, where $\tilde{v} \in \check{\mathcal{H}}$. Because J, $\sharp_M$ and $\flat$ are continuous, so is $\check{J}$. The inverse to $\check{J}$ is given by $\check{J}^{-1}(v) = (J^{-1}(v^{\flat_M}))^\sharp$, $v \in M \times \check{H}$. Since $\flat_M$, J and $\sharp$ are continuous, we get that $\check{J}$ is a homeomorphism and thus $(M, \check{J})$ is a bundle chart for $\check{\mathcal{H}}$.

For bundle rank 1 operators, we set

$$J^0_{CH}([(r, v \otimes f)]) = \big(\pi_P(r), \check{\mathrm{pr}}_2(\check{J}([(r, v)])) \otimes \mathrm{pr}_2(J([(r, f)]))\big)$$

where $f \in H$, $v \in \check{H}$, $r \in \mathcal{P}$, $\mathrm{pr}_2 : M \times H \to H$ and $\check{\mathrm{pr}}_2 : M \times \check{H} \to \check{H}$ denote the projections onto the second factors of the respective products. To prove that J^0_{CH} is well defined, we use formula (1) from Lemma 3. Indeed,

$$\begin{aligned} J^0_{CH}([(rg^{-1}, \rho(g)(v \otimes f))]) &= J_{CH}\left([(rg^{-1}, \check{\sigma}(g)v \otimes \sigma(g)f)]\right) \\ &= \left(\pi_P(rg^{-1}), \check{\mathrm{pr}}_2 \check{J}([(rg^{-1}, \check{\sigma}(g)v)]) \otimes \mathrm{pr}_2 J([(rg^{-1}, \sigma(g)f)])\right) \\ &= \left(\pi_P(r), \check{\mathrm{pr}}_2 \check{J}([(r, v)]) \otimes \mathrm{pr}_2 J([(r, f)])\right) = J^0_{CH}([(r, v \otimes f)]) \end{aligned}$$

Since defined by composition of continuous maps, J^0_{CH} is continuous.

The inverse of J^0_{CH} on rank 1 operators is given by $(J^0_{CH})^{-1}(m, \tilde{v} \otimes \tilde{f}) = \check{J}^{-1}(m, \tilde{v}) \otimes_{\mathcal{CH}} J^{-1}(m, \tilde{f})$, $m \in M$, $\tilde{v} \in \check{H}$ and $\tilde{f} \in H$, which is a composition of continuous maps. On finite rank operators, J^0_{CH} is extended linearly. Obviously, $(J^0_{CH})^{-1}$ on finite rank operators is continuous as well.

Finally for $r \in \mathcal{P}$ and $a \in CH$, we set

$$J_{CH}([(r, a)]) = \left(\pi_P(r), \lim_i \mathrm{pr}_2^{CH} J^0_{CH}([(r, a_i)])\right)$$

where $(a_i)_{i \in \mathbb{N}}$ is any sequence of finite rank operators converging to a in the operator norm topology, and pr_2^{CH} denotes the projection of $M \times CH$ onto the second factor. The map J_{CH} is independent of the chosen sequence. Indeed, let $(a_i)_{i \in \mathbb{N}}$ and $(b_i)_{i \in \mathbb{N}}$ be sequences of finite rank operators which converge to a in the operator norm. Their difference $c_i = a_i - b_i$ converges to the zero operator. Moreover, each c_i is a finite rank operator. Therefore, there are $\lambda_i^{jk} \in \mathbb{C}$ such that $c_i = \sum_{j,k=1}^{n_i} \lambda_i^{jk} h_j \otimes \eta_k$, where $(h_j)_j$ and $(\eta_k)_k$ are complete orthonormal systems of $\check{H}$ and H, respectively. We have $||c_i||^2_{CH} = \sum_{j,k} |\lambda_i^{jk}|^2$. Since $c_i \to 0$, we get $\lambda_i^{jk} \to 0$ for all j, k. This fact and the linearity of J and $\check{J}$ make us able to conclude that $J^0_{CH}([(r, c_i)]) = \lim_i \sum_{j,k} \lambda_i^{jk} \left(\check{\mathrm{pr}}_2 \check{J}([(r, h_j)]) \otimes \mathrm{pr}_2 J([(r, \eta_k)])\right) = 0$. This proves that the definition of J_{CH} is correct.

Map J_{CH} is continuous since it is defined as the continuous extension in the second argument and since π is continuous. The inverse of J_{CH} is given by $J^{-1}_{CH}(m, a) = \lim_i (J_0^{CH})^{-1}(m, a_i)$, where $(a_i)_{i \in \mathbb{N}}$ is a sequences of finite rank operators converging to a in the operator norm. Summing-up, $J_{CH} : \mathcal{CH} \to M \times CH$ is a homeomorphism of $\mathcal{CH}$ and $M \times CH$. Since J_{CH} is fiber preserving, J_{CH} is an isomorphism of continuous bundles. Since the Kuiper maps are taken to be normalized, J_{CH} is a fiber-wise isometry and thus, it is an isomorphism of Banach bundles and consequently, $\mathcal{CH}$ is trivial as a Banach bundle. □

Remark: It is known that the set of equivalence classes of bundles of Banach spaces, with fibers the algebra of compact operators CH on a Hilbert space H and whose structure group is the projective unitary group of H, is isomorphic to the third singular cohomology group $H^3(M, \mathbb{Z})$ of M. The equivalence is the isomorphism of continuous fiber bundles. See Dixmier, Douady [9] or Dixmier [10]. We refer also to Mathai, Melrose and Singer [45] and Schottenloher [63] for a context and for a nice topological study, respectively. (Moreover, the set of such Banach fiber bundles forms a group under the spatial tensor product (the

Brauer group). This group is isomorphic to the additive structure on $H^3(M,\mathbb{Z})$. See Rosenberg [59].)

4 Hilbert C^*-bundle and Kuiper complex

In this section, we first introduce pointwise structures which we use for defining a Hilbert C^*-structure on the higher oscillator bundle. Then we use a product type connection to establish the elliptic complex, mentioned in the Introduction.

4.1 Pointwise analytical structures

Let us define a map $\cdot_{\mathcal{H}} : \mathcal{H}^\bullet \times_M \mathcal{CH} \to \mathcal{H}^\bullet$ by

$$[(q,f)] \cdot_{\mathcal{H}} [(q,a)] = [(q, f \cdot a)]$$

where $q \in \mathcal{P}$, $f \in H^\bullet$, and $a \in CH$. We call it the *$\mathcal{CH}$-action*. The independence on the representative of an equivalence class is proved below.

For to define a pointwise $\mathcal{CH}$-valued product on $\mathcal{H}^\bullet$, we reduce the structure group of $\mathcal{P}$ from $\widetilde{G}$ to $\widetilde{K}$ which is possible since this group is a deformation retract of the structure group $Mp(V,\omega_0)$ by the lifting property for coverings and the known fact that $Sp(V,\omega_0)$ deformation retracts onto K. The reduction is non-unique. Therefore, we choose a compatible positive almost complex structure on (M,ω) and consider the Λ-preimage of the bundle of unitary frames. We denote the resulting principal $\widetilde{K}$-bundle by $\mathcal{P}_R$. The corresponding associated bundles are denoted by $\mathcal{H}^\bullet_R$ and $\mathcal{CH}_R$. We call the former one the reduced higher oscillator bundle. The total spaces of these vector bundles coincide, as topological spaces, with $\mathcal{H}^\bullet$ and $\mathcal{CH}$, respectively. Using this reduction, we define the pointwise *$\mathcal{CH}$-product* $(,)_{\mathcal{CH}} : \mathcal{H}^\bullet_R \times_M \mathcal{H}^\bullet_R \to \mathcal{CH}_R$ by the formula

$$([(q,v)],[(q,w)])_{\mathcal{CH}} = [(q,(v,w))] \in \mathcal{CH}_R$$

where $q \in \mathcal{P}_R$, and $v, w \in H^\bullet$. Note that whereas at the right-hand side, $(v,w) \in CH$, at the left-hand side, a map $(,)_{\mathcal{CH}} : \mathcal{H}^\bullet_R \times_M \mathcal{H}^\bullet_R \to \mathcal{CH}_R$ is prescribed as we prove yet.

Lemma 8: For a symplectic manifold (M,ω) admitting a metaplectic structure (π_P,Λ), the maps

$$\begin{aligned} \cdot_{\mathcal{H}} &: \mathcal{H}^\bullet \times_M \mathcal{CH} \to \mathcal{H}^\bullet \\ (,)_{\mathcal{CH}} &: \mathcal{H}^\bullet_R \times_M \mathcal{H}^\bullet_R \to \mathcal{CH}_R \end{aligned}$$

are correctly defined.

Proof. We have to check that these maps do not depend on the choice of representatives.

1) Let us verify that the definition of the action $\cdot_{\mathcal{H}}$ is correct. For $q \in \mathcal{P}$, $g \in \widetilde{G}$, $k = 0, \ldots, 2n$, $a \in CH$, and $v = \alpha \otimes h \in H^\bullet$

$$\begin{aligned}[(q,v)] \cdot_{\mathcal{H}} [(qg, \rho(g)^{-1}a)] &= [\big(qg, \sigma^k(g^{-1})v\big)] \cdot_{\mathcal{H}} [\big(qg, (\rho(g)^{-1}a)\big)] \\ &= [\big(qg, \big(\sigma^k(g)^{-1}(v)\big) \cdot \big(\rho(g)^{-1}a\big)\big)] \\ &= [\big(qg, \sigma^k(g)^{-1}(v \cdot a)\big)] \\ &= [(q,v)] \cdot_{\mathcal{H}} [(q,a)]\end{aligned}$$

where the equivariance of the CH-action (Lemma 3) is used in the second row. In a similar way, one proves that the definition of $\cdot_{\mathcal{H}}$ does not depend on the choice of a representative of the element acted upon.

2) We prove the correctness of the definition of $(,)_{\mathcal{CH}}$. For a frame $q \in \mathcal{P}_R$, $\alpha, \beta \in \bigwedge^k V^*$, $v, w \in H$ and $g \in \widetilde{K}$

$$\begin{aligned}\big([(qg, \sigma^k(g^{-1})v)], [(q,w)]\big)_{\mathcal{CH}} &= \big([\big(qg, \sigma^k(g^{-1})v\big)], [\big(qg, \sigma^k(g^{-1})w\big)]\big)_{\mathcal{CH}} \\ &= \big[\big(qg, \big(\sigma^k(g^{-1})v, \sigma^k(g^{-1})w\big)\big)\big] \\ &= \big[\big(qg, \rho(g)^{-1}(v,w)\big)\big] = [(q,(v,w))]\end{aligned}$$

In the second row above, the $\widetilde{K}$-equivariance of the CH-product from Lemma 3 is used. The independence on representatives of the second argument of $(,)_{\mathcal{CH}}$ follows from the hermitian symmetry of $(,) : H^\bullet \times H^\bullet \to CH$.

$\square$

Remark: Consider the composition

$$h \in \mathcal{H} \xmapsto{(,)_{\mathcal{CH}}} (h,h)_{\mathcal{CH}} \in \mathcal{CH} \xmapsto{||\,||_{\mathcal{CH}}} ||(h,h)_{\mathcal{CH}}||_{\mathcal{CH}} \in \mathbb{R}_{\geq 0}$$

where $||[(q,a)]||_{\mathcal{CH}} = ||a||_{CH}$ for $q \in \mathcal{P}_R$ and $a \in CH$. This map is well defined since ρ is an isometry. Let us recall that there is also a fiber-wise scalar product $(,)_S$ on $\mathcal{H} \to M$ as constructed in the proof of Thm. 4 that makes $\mathcal{H}$ a Hilbert bundle. One can prove easily that $||h||_S^2 = ||(h,h)_{\mathcal{CH}}||_{\mathcal{CH}}$ where $||\,||_S$ is the fiber-wise norm induced by $(,)_S$.

4.2 Hilbert C^*-bundles

According to Fomenko, Mishchenko [14], we shall define a smooth action of CH on the total space $\mathcal{H}_R^\bullet$, which restricts to a Hilbert CH-module action on each fiber; and a smooth CH-valued map $(,) : \mathcal{H}_R^\bullet \times_M \mathcal{H}_R^\bullet \to CH$, which restricts to a Hilbert CH-product on each fiber. Further, on these bundles, Hilbert C^*-atlases have to be fixed such that their transition functions are maps into $D^\bullet = \mathrm{Aut}_{CH}(H^\bullet)$, i.e., the set of all homeomorphisms commuting with the action of CH on $H^\bullet$.

We define a Hilbert C^*-bundle atlas on $\mathcal{H}_R^0$ by requiring that it is the maximal smooth atlas satisfying the following conditions

1) it contains (M, J) and

2) its transition functions are maps into D^0.

Note that $D^0 = \mathrm{Aut}_{CH}(H^0) \simeq \mathbb{C}^\times$, which is quite restrictive but it gives a clear picture of the situation. Namely, the atlas equals

$$\mathcal{A} = \{(U, \mu J_{|(p_0)^{-1}(U)}) |\, \mu \in \mathbb{C}^\times,\, U \text{ open in } M\}$$

This atlas is a subset of the Kuiper structure on $\mathcal{H}^0$ induced by J. On $\mathcal{H}^k$ we consider the tensor product atlases $\mathcal{A}_k$, which are the tensor products of the C^∞-structure on $\bigwedge^k T^*M$ and of $\mathcal{A}$. We call them the *Hilbert C^*-bundle structure induced by J.*

Definition: For a symplectic manifold (M, ω) possessing a metaplectic structure (π_P, Λ), a compatible positive almost complex structure, a Kuiper map $J : \mathcal{H} \to M \times H$, $m \in M$, $f, h \in (\mathcal{H}_R^\bullet)_m$, and $a \in CH$, we set

$$\begin{aligned} h \cdot a &= h \cdot_{\mathcal{H}} J_{CH}^{-1}(m, a) \quad \text{ and} \\ (f, h) &= \mathrm{pr}_2^{CH} \left(J_{CH}(f, h)_{\mathcal{CH}}\right) \end{aligned}$$

where $\mathrm{pr}_2^{CH} : M \times CH \to CH$ denotes the projection onto the second factor.

Remark: The symbols $\cdot$ and $(,)$ should not be confused with the ones referring to the action $H^\bullet \times CH \to H^\bullet$ and the CH-product $H^\bullet \times H^\bullet \to CH$.

Let us introduce the following operations. For $r \in \mathcal{P}$, $a, b \in CH$, $v \in \check{H}$ and $f \in H$, we set

$$\begin{aligned} \circ_{\mathcal{CH}} : \mathcal{CH} \times_M \mathcal{CH} \to \mathcal{CH} &\quad [(r, a)] \circ_{\mathcal{CH}} [(r, b)] = [(r, ab)] \\ \circ_{CH} : (M \times CH) \times_M (M \times CH) \to M \times CH &\quad (m, a) \circ_{CH} (m, b) = (m, ab) \\ \otimes_{CH} : (M \times \check{H}) \times_M (M \times H) \to M \times CH &\quad (m, v) \otimes_{CH} (m, f) = (m, v \otimes f) \end{aligned}$$

These maps are well defined by Lemma 3.

Lemma 9: Let $a, b \in CH$, $f \in H$ and $m \in M$. Then

$$\begin{aligned} &J_{CH}^{-1}(m, a \circ b) = J_{CH}^{-1}(m, a) \circ_{\mathcal{CH}} J_{CH}^{-1}(m, b) \text{ and} \\ &J^{-1}(m, f) \cdot_{\mathcal{H}} J_{CH}^{-1}(m, a) = J^{-1}(m, f \circ a) \end{aligned}$$

Proof. For any $q \in \mathcal{P}$, $v_a, v_b \in \check{H}$, and $f_a, f_b \in H$, $([(q, v_a)] \otimes_{\mathcal{CH}} [(q, f_a)]) \circ_{\mathcal{CH}} ([(q, v_b)] \otimes_{\mathcal{CH}} [(q, f_b)]) = [(q, (v_a \otimes f_a) \circ (v_b \otimes f_b))] = [(q, f_a(v_b) v_a \otimes f_b)]$. Thus for $v, w \in \check{\mathcal{H}}_m$ and $f, h \in \mathcal{H}_m$

$$(v \otimes_{\mathcal{CH}} f) \circ_{\mathcal{CH}} (w \otimes_{\mathcal{CH}} h) = f(w)(v \otimes_{\mathcal{CH}} h) \tag{2}$$

For $a = v_a \otimes f_a \in CH$ and $b = v_b \otimes f_b \in CH$

$$J^{-1}_{CH}(m, a \circ b) = J^{-1}_{CH}(m, (v_a \otimes f_a) \circ (v_b \otimes f_b)) = f_a(v_b) J^{-1}_{CH}(m, v_a \otimes f_b)$$

Since J is a fiber-wise isometry

$$\begin{aligned} f_a(v_b) J^{-1}_{CH}(m, v_a \otimes f_b) &= f_a(v_b) \check{J}^{-1}(m, v_a) \otimes_{\mathcal{CH}} J^{-1}(m, f_b) \\ &= (J^{-1}(m, f_a), J^{-1}((m, v_b)^{\flat_M}))_S \check{J}^{-1}(m, v_a) \otimes_{\mathcal{CH}} J^{-1}(m, f_b) \\ &= (J^{-1}(m, f_a)[\check{J}^{-1}(m, v_b)]) \check{J}^{-1}(m, v_a) \otimes_{\mathcal{CH}} J^{-1}(m, f_b) \end{aligned}$$

Using (2), we have $\left(J^{-1}(m, f_a)[\check{J}^{-1}(m, v_b)]\right) \check{J}^{-1}(m, v_a) \otimes_{\mathcal{CH}} J(m, f_b)$

$$\begin{aligned} &= (\check{J}^{-1}(m, v_a) \otimes_{\mathcal{CH}} J^{-1}(m, f_a)) \circ_{\mathcal{CH}} (\check{J}^{-1}(m, v_b) \otimes_{\mathcal{CH}} J^{-1}(m, f_b)) \\ &= J^{-1}_{CH}(m, v_a \otimes f_a) \circ_{\mathcal{CH}} J^{-1}_{CH}(m, v_b \otimes f_b) = J^{-1}_{CH}(m, a) \circ_{\mathcal{CH}} J^{-1}_{CH}(m, b) \end{aligned}$$

Summing-up, $J^{-1}_{CH}(m, a \circ b) = J^{-1}_{CH}(m, a) \circ_{\mathcal{CH}} J^{-1}_{CH}(m, b)$ for rank 1 operators a and b. For a linear combination of finite rank maps, the result follows by linearity. For a product of compact operators, one proceeds by taking limits which is justified by noting that J_{CH} is a homeomorphism (Sect. 3.3, proof of Thm. 7).

The second equation is proved in a similar way.

□

Theorem 10: Let (M, ω) be a symplectic manifold admitting a metaplectic structure (π_P, Λ) and J be a Kuiper map. Then the reduced higher oscillator bundle $\mathcal{H}^{\bullet}_R \to M$ with the Hilbert C^*-bundle structure induced by a Kuiper map J is a smooth Hilbert CH-bundle for any compatible positive almost complex structure.

Proof. First, we have to verify that the map $\cdot : \mathcal{H} \times CH \to \mathcal{H}$ is an action. Recall that according to the definitions of the $\mathcal{CH}$-action and the composition $\circ_{\mathcal{CH}}$, we have $h \cdot_{\mathcal{H}} (a \circ_{\mathcal{CH}} b) = (h \cdot_{\mathcal{H}} a) \cdot_{\mathcal{H}} b$, whenever these operations make sense. For $m \in M$, $h \in (\mathcal{H}^{\bullet}_R)_m$ and $a, b \in CH$, we get according to Lemma 9

$$\begin{aligned} h \cdot (ab) = h \cdot_{\mathcal{H}} J^{-1}_{CH}(m, ab) &= h \cdot_{\mathcal{H}} \left(J^{-1}_{CH}(m, a) \circ_{\mathcal{CH}} J^{-1}_{CH}(m, b)\right) \\ &= \left(h \cdot_{\mathcal{H}} J^{-1}_{CH}(m, a)\right) \cdot_{\mathcal{H}} J^{-1}_{CH}(m, b) = (h \cdot a) \cdot b \end{aligned}$$

Second, using Lemma 3, it is easy to verify that

$$(f, h \cdot_{\mathcal{H}} b)_{\mathcal{CH}} = (f, h)_{\mathcal{CH}} \circ_{\mathcal{CH}} b$$

for suitable $h, f \in \mathcal{H}^{\bullet}_R$ and $b \in \mathcal{CH}_R$. For such elements and $a \in CH$ we get, using Lemma 9

$$
\begin{aligned}
(f, h \cdot a) &= \mathrm{pr}_2^{CH}\left(J_{CH}\left(f, h \cdot_{\mathcal{H}} J_{CH}^{-1}(m,a)\right)\right)\\
&= \mathrm{pr}_2^{CH}\left(J_{CH}\left((f,h)_{\mathcal{CH}} \circ_{\mathcal{CH}} J_{CH}^{-1}(m,a)\right)\right)\\
&= \mathrm{pr}_2^{CH}\left(J_{CH}\left[J_{CH}^{-1}\left(J_{CH}(f,h)_{\mathcal{CH}}\right) \circ_{\mathcal{CH}} J_{CH}^{-1}(m,a)\right]\right)\\
&= \mathrm{pr}_2^{CH}\left(J_{CH}\left[J_{CH}^{-1}\left(m, \mathrm{pr}_2^{CH}\left(J_{CH}\left(f,h\right)_{\mathcal{CH}}\right) \circ a\right)\right]\right)\\
&= \mathrm{pr}_2^{CH}\left(m, \mathrm{pr}_2^{CH}(J_{CH}(f,h)_{\mathcal{CH}}) \circ a\right)\\
&= \mathrm{pr}_2^{CH}\left(J_{CH}\left(f,h\right)_{\mathcal{CH}}\right) \circ a = (f,h) \circ a
\end{aligned}
$$

The Hilbert C^*-bundle structure induced by J on $\mathcal{H}^k$, $k = 0, \ldots, 2n$, is a maximal set of smooth Hilbert C^*-bundle atlases smoothly compatible with (M, J). Let us verify that the CH-action is smooth. The CH-action $\mathcal{H} \times CH \to \mathcal{H}$ in the map (M, J) is given by $(m, f) \mapsto (m, f \circ a)$, where $m \in M$, $f \in (H_R^{\bullet})_m$ and $a \in CH$. This map is infinitely many times Fréchet differentiable. Similarly, the CH-product is given by $((m, f), (m, h)) \mapsto (f, h) \in CH$ in (M, J), $m \in M$ and $f, h \in (H_R^{\bullet})_m$, which is thus Fréchet smooth. $\square$

Hilbert C^*-bundle structure on $\underline{H}$: Let g be the (Hodge type) Riemannian metric on the fibers of $\bigwedge^k T^*M$ induced by ω and a chosen compatible positive almost complex structure. The tensor product bundle $\bigwedge^{\bullet} T^*M \otimes \underline{H} \to M$ of $\bigwedge^{\bullet} T^*M \to M$ and of the trivial bundle $\underline{H} = M \times H \to M$ is a Hilbert C^*-bundle by setting $(\alpha \otimes (m, f)) \cdot_M a = \alpha \otimes (m, f \cdot a) \in \bigwedge^{\bullet} T^*M \otimes \underline{H}$ and $(\alpha \otimes (m, f), \beta \otimes (m, h))_M = g(\alpha, \beta)(f, h) \in CH$, where $m \in M$, $\alpha, \beta \in \bigwedge^{\bullet} T_m^*M$, $f, h \in H$ and $a \in CH$. For the smooth Hilbert C^*-bundle structure on $\bigwedge^{\bullet} T^*M \otimes \underline{H}$, we take the product of the maximal atlas of $\bigwedge^{\bullet} T^*M$ with the atlas $\{(U, \mu \mathrm{Id}_{U \times H}) \,|\, \mu \in \mathbb{C}^{\times}, U \text{ open in } M\}$ of $\underline{H}$.

Lemma 11: Let (M, ω) be a symplectic manifold admitting a metaplectic structure (π_P, Λ). For any compatible positive almost complex structure and a Kuiper structure J, the Kuiper map extends to a unitary isomorphism $\widetilde{J} : \mathcal{H}_R^{\bullet} \to \bigwedge^{\bullet} T^*M \otimes \underline{H}$ of Hilbert CH-bundles.

Proof. Obviously, $\mathcal{H}_R^k \simeq \bigwedge^k T^*M \otimes \mathcal{H}_R^0$. We set $\widetilde{J}(\alpha \otimes h) = \alpha \otimes J(h)$, $\alpha \in \bigwedge^k T^*M$ and $h \in \mathcal{H}_R^0$, and extend it linearly. Since CH does not act on the form part, it is sufficient to verify that $J(h \cdot a) = J(h) \cdot_M a$ for any $h \in \mathcal{H}_R^0$ and $a \in CH$. Representing $h = J^{-1}(m, f)$ and using Lemma 9

$$
\begin{aligned}
J(h \cdot a) &= J(J^{-1}(m, f) \cdot_{\mathcal{H}} J^{-1}(m, a)) = J(J^{-1}(m, h \circ a))\\
&= (m, h \circ a) = (m, h) \cdot_M a.
\end{aligned}
$$

It is easy to see

$$
(f, h)_{\mathcal{CH}} = f^{\sharp} \otimes_{\mathcal{CH}} h \tag{3}
$$

for $f, h \in \mathcal{H}_R^0$. Indeed, representing f, h by equivalence classes, we obtain $([(q, v)], [(q, w)])_{\mathcal{CH}} = [(q, v^{\sharp} \otimes w)] = f^{\sharp} \otimes_{\mathcal{CH}} h$.

When proving the unitarity of $\widetilde{J}$, we choose J-preimages of $f, h \in \mathcal{H}^0_R$ and use (3), proceeding in a similar way as in the case of CH-equivariance. The unitarity of $\widetilde{J}$ follows then from the fact that CH does not act on the wedge form part of $\bigwedge^\bullet T^*M \otimes \underline{H}$. $\square$

4.3 Associated elliptic complex to a Kuiper connection

We introduce the complex and prove that it is a complex of C^*-operators. Further we investigate its cohomology groups. For that we focus to the topology of section spaces of the higher oscillator bundles and the topology of their tensor products and quotients.

Let (M, ω) be a symplectic manifold admitting a metaplectic structure (π_P, Λ) and $J : \mathcal{H} \to M \times H$ be a normalized Kuiper map. The $T(M \times H)$-valued 1-form Φ^H defining the canonical product connection on $M \times H \to M$ is given by

$$\Phi^H(t_m, v_h) = (0_m, v_h)$$

where $m \in M$, $t_m \in T_mM$, $h \in H$, $v_h \in T_hH$, and 0_m is the zero tangent vector in T_mM. Note that we identify $T(M \times H) \simeq TM \times TH$. The Kuiper map gives rise to a connection Φ in $\mathcal{H}$ given by the prescription $\Phi = TJ^{-1} \circ \Phi^H \circ TJ$ which we call the *Kuiper connection induced by* J, expressing its roots.

Let us notice that a different description of a connection on symplectic spinors is chosen for contact manifolds by Herczeg, Waldron [23] in the setting of quantum physics, and for the contact projective ones by Krýsl [33].

Remark: The distribution

$$\mathcal{H} \ni f = [(r, h)] \mapsto \mathcal{L}_f = \left(T_{(\pi_P(r),h)} J^{-1}\right)(T_mM, 0) \subseteq T_f\mathcal{H}$$

where $r \in \mathcal{P}, h \in H, m = \pi_P(r)$ and $0 \in T_hH$ (the zero vector of T_hH) is the horizontal distribution defined by Φ. In this way, we get a splitting of the vector bundle $T\mathcal{H} = \mathcal{L} \oplus \mathrm{Ker}\,(Tp_0)$. The vertical part of the bundle $\mathcal{H}$ is denoted by $\mathcal{V}$.

Gateaux and Fréchet differentials in geometric setting: We introduce a notation and recall some facts from analysis on Banach spaces. Let $(Df)(h, v)$ denote the Gateaux derivative of f at h in direction v. For each $h \in H$, $T_hH = \{v_h, v \in H\}$, where $v_hf = (Df)(h, v)$ for any smooth function f defined in a neighborhood of $h \in H$. For any point $h \in H$, let us define $i_h : T_hH \to H$ by $i_h(v_h) = v$, $v \in H$. It is easy to see that i_h is an isomorphism of vector spaces.

Vertical lifts, connectors and induced connections: We recall basic principles for handling of associated connections. It is based on the approach described in Michor [47] and Kolář, Michor, Slovák [31]. For a vector field $X \in \mathfrak{X}(M)(= \Gamma(TM))$, the connection on $\mathcal{H}$ induces a covariant derivative $\nabla_X : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$ defined by $\nabla_X s = \kappa \circ Ts \circ X$, where $X : M \to TM$ is a vector field on M, $Ts : TM \to T\mathcal{H}$ is the tangent map of a bundle section $s : M \to \mathcal{H}$, and $\kappa : T\mathcal{H} \to \mathcal{H}$ is the connector of Φ. We have also a covariant derivative ∇_X^H acting on sections of $M \times H \to M$ which is given by $\nabla_X^H s = \kappa^H \circ Ts \circ X$, where $s : M \to M \times H$ and κ^H is the connector of Φ^H. Let us denote the vertical lifts of $(H \times M \to M, \Phi^H)$ and $(\mathcal{H} \to M, \Phi)$ by vl^H and vl, respectively. Recall that $vl : \mathcal{H} \times_M \mathcal{H} \to \mathcal{V}$ and similarly $vl^H : (M \times H) \times_M (M \times H) \to TM \times H$. We denote the projections $\mathrm{pr}_2 : \mathcal{H} \times_M \mathcal{H} \to \mathcal{H}$ and $\mathrm{pr}_2^H : (M \times H) \times_M (M \times H) \to M \times H$ onto the respective second factors. With this notation, the connector κ is the map $\mathrm{pr}_2 \circ vl^{-1} \circ \Phi$ and similarly for κ^H.

Lemma 12: In the situation described in the previous paragraph, we have for any $m \in M$, and $h, v \in H$

1) $(vl^H)^{-1}(0_m, v_h) = ((m,h),(m,v)) \in (M \times H) \times_M (M \times H)$
2) $vl = TJ^{-1} \circ vl^H \circ (J \times_M J)$
3) $\mathrm{pr}_2^H \circ (J \times_M J) = J \circ \mathrm{pr}_2$

Proof. For $m \in M$ and $h, v \in (M \times H)_m$, $vl^H\left((m,h),(m,v)\right) =$
$= \frac{d}{dt}_{|t=0}\left((m,h) +_M t \odot_M (m,v)\right)$, where $+_M$ and $\odot_M$ denote the addition and the multiplication by scalars in $M \times H \to M$, respectively, as introduced in 3.3. We have $\frac{d}{dt}_{|t=0}((m,h) +_M t \odot_M (m,v)) = \frac{d}{dt}_{|t=0}(m, h + tv) = (0_m, v_h)$.

For $x, y \in \mathcal{H}_m$, $vl(x,y) = \frac{d}{dt}_{|t=0}(x + ty)$. Because J is fiber-wise linear, $vl(x,y) = \frac{d}{dt}_{|t=0} J^{-1}J(x+ty) = TJ^{-1}\frac{d}{dt}_{|t=0}(h +_M t \odot_M v) = TJ^{-1}vl^H(h,v)$, where $h = J(x)$ and $v = J(y)$.

The third equality is obvious. □

Lemma 13: For $X \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{H})$

$$J \circ \nabla_X s = \nabla_X^H (J \circ s)$$

Proof. We use Lemma 12 to prove the identity. For $X \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{H})$

$$\begin{aligned}
\nabla_X^H(J \circ s) &= \kappa^H \circ TJ \circ Ts \circ X \\
&= \mathrm{pr}_2^H \circ vl^{H^{-1}} \circ \Phi^H \circ TJ \circ Ts \circ X \\
&= \mathrm{pr}_2^H \circ (J \times_M J) \circ vl^{-1} \circ TJ^{-1} \circ \Phi^H \circ TJ \circ Ts \circ X \\
&= \mathrm{pr}_2^H \circ (J \times_M J) \circ vl^{-1} \circ \Phi \circ Ts \circ X \\
&= J \circ \mathrm{pr}_2 \circ vl^{-1} \circ \Phi \circ Ts \circ X = J \circ \nabla_X s
\end{aligned}$$

□

Remark: Since we shall compute derivations of the CH-action on $\underline{H}$ (by a fixed element), we use a notation which is more usual for maps. For any $a \in CH$ and $(m,v) \in M \times H$, we set $G_a(m,v) = (m, v \circ a)$ and $g_a(v) = v \circ a$, defining maps $G_a : M \times H \to M \times H$ and $g_a : H \to H$. Note that $G_a = \mathrm{Id}_M \times g_a$.

Lemma 14: For any $X \in \mathfrak{X}(M)$, $s \in \Gamma(\underline{H})$ and $a \in CH$

$$\nabla^H_X(G_a \circ s) = G_a \circ \nabla^H_X s$$

Proof. For $v, h \in H$ and $a \in CH$, $(Dg_a)(h,v) = \lim_{t\to 0}[t^{-1}(h \circ a + tv \circ a - h \circ a)] = v \circ a$. For $f \in C^\infty(H)$, $(Tg_a)(v_h)f = v_h(f \circ g_a) = D(f \circ g_a)(h,v) = (Df)(h \circ a, (Dg_a)(h,v)) = (Df)(h \circ a, v \circ a)$ by the chain rule. The last expression equals to $(v \circ a)_{h\circ a} f$. Thus $(Tg_a)v_h = (v \circ a)_{h \circ a}$, which implies

$$TG_a(t_m, v_h) = (t_m, (Tg_a)(v_h)) = (t_m, (v \circ a)_{h\circ a}) \tag{4}$$

where $m \in M$ and $t_m \in T_m M$.

By Lemma 12

$$\begin{aligned}(G_a \circ \kappa^H)(t_m, v_h) &= (G_a \circ \mathrm{pr}_2^H \circ {vl^H}^{-1} \circ \Phi^H)(t_m, v_h)\\ &= (G_a \circ \mathrm{pr}_2^H \circ {vl^H}^{-1})(0_m, v_h)\\ &= (G_a \circ \mathrm{pr}_2^H)((m,h),(m,v)) = G_a(m,v) = (m, v \circ a)\end{aligned}$$

Further, we get by Lemma 12 and (4) that

$$\begin{aligned}(\kappa^H \circ TG_a)(t_m, v_h) &= (\mathrm{pr}_2^H \circ {vl^H}^{-1} \circ \Phi^H \circ TG_a)(t_m, v_h)\\ &= (\mathrm{pr}_2^H \circ {vl^H}^{-1} \circ \Phi^H)(t_m, i^{-1}_{h\circ a}(v \circ a))\\ &= (\mathrm{pr}_2^H \circ {vl^H}^{-1})(0_m, i^{-1}_{h\circ a}(v \circ a)) = (m, v \circ a)\end{aligned}$$

Summing-up, $G_a \circ \kappa^H = \kappa^H \circ TG_a$. The theorem follows by $\nabla_X(G_a \circ s) = \kappa^H \circ TG_a \circ Ts \circ X = G_a \circ \kappa^H \circ Ts \circ X = G_a \circ \nabla^H_X s$ for any $s \in \Gamma(\underline{H})$ and $X \in \mathfrak{X}(M)$. $\square$

Hilbert C^*-module structure on section spaces: Notice that CH acts not only on the higher oscillator bundle but also on its smooth sections space $\Gamma(\mathcal{H}^\bullet)$. Namely, $(s \cdot a)_m = s_m \cdot a$ for any $a \in CH$, $s \in \Gamma(\mathcal{H}^\bullet)$ and $m \in M$.[5] In this way, $\Gamma(\mathcal{H}^\bullet)$ is a right CH-module. The space of smooth sections of the reduced higher oscillator bundle $\mathcal{H}^\bullet_R$ is also a pre-Hilbert CH-module if (M, ω) is compact. The CH-valued product is given by

$$(s, s') = \int_{m\in M} (s_m, s'_m)\mathrm{vol}_m$$

where $s, s' \in \Gamma(\mathcal{H}^\bullet)$ and vol_m denotes the volume form in m induced by ω_m. The integral is the Pettis integral (for the measure on M induced by the volume form vol_m) of Pettis CH-valued integrable function on M. (See Ryan [60] or Grothendieck [18].) Similarly, we consider the space $\Gamma(\underline{H})$ of smooth sections of the product bundle $\underline{H} \to M$ as a pre-Hilbert CH-module as well. Completions of this pre-Hilbert module (see Fomenko, Mishchenko [14]) with respect to the induced norm make it a Hilbert C^*-module. (Let us notice that for reasons of

[5] At the left-hand side a new action is defined.

analysis, several further C^*-valued products of 'Sobolev type' are introduced in [14] besides the CH-product given above. However, the integral seems to be rather not specified there.)

Corollary 15: For a symplectic manifold (M,ω) admitting a metaplectic structure $(\mathcal{P},\pi_P)$, a Kuiper map J and a vector field X on M, the operator

$$\nabla_X : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$$

is a first order CH-equivariant differential operator.

Proof. The map ∇_X is a first order differential operator for any $X \in \mathfrak{X}(M)$ (all operations involved in its definition are of zero order, except of $s \mapsto Ts$). Using Lemmas 9, 13 and 14 for $a \in CH$, $X \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{H})$

$$\begin{aligned}\nabla_X(s\cdot a) &= J^{-1}\circ\nabla^H_X(J\circ(s\cdot a)) = J^{-1}\circ\nabla^H_X(J\circ J^{-1}\circ G_a\circ J\circ s)\\ &= J^{-1}\circ\nabla^H_X(G_a\circ J\circ s) = (J^{-1}\circ G_a)\circ\nabla^H_X(J\circ s)\\ &= (J^{-1}\circ G_a\circ J)\circ\nabla_X s = (\nabla_X s)\cdot a\end{aligned}$$

□

Remark (Connection to non-commutative geometry): It is easy to see that ∇_X is CH-hermitian in the sense that $X(s,s') = (\nabla_X s, s') + (s, \nabla_X s')$ where $s, s' \in \Gamma(\mathcal{H}^\bullet_R)$. In Dubois-Violette, Michor [11], central A-bimodules and connections for them are investigated. To some extent, our structures fit into the specific *non-commutative geometry concept* described there. Namely, for the not necessarily normed algebra A, considered in [11], we take the smooth sections of the bundle of measurements, $A = \Gamma(\mathcal{CH})$, and for the module, we take the smooth sections $\Gamma(\mathcal{H}^\bullet)$. The A-valued metric h (p. 229 in [11]) is then the 'sectionised' $\mathcal{CH}$-product, i.e., $h : \Gamma(\mathcal{H}^\bullet_R)\times\Gamma(\mathcal{H}^\bullet_R) \to \Gamma(\mathcal{CH})$, $h(s,s')(m) = (s_m, s'_m)_{\mathcal{CH}}$ where $s, s' \in \Gamma(\mathcal{H}^\bullet)$ and $m \in M$. However, the vector fields X in our paper are generically not all derivations of $A = \Gamma(\mathcal{CH})$. (They coincide iff the manifold is discrete as a topological space.) The module $\Gamma(\mathcal{H}^\bullet_R)$ with the $\Gamma(\mathcal{CH})$-hermitian connection may be considered as a modification of a non-commutative hermitian manifold as considered in the program of Connes [8]. They have a global realization by topological and differential structures.

Let $(e_i)_{i=1}^{2n}$ and $(\epsilon^i)_{i=1}^{2n}$ be dual local frames of $TM \to M$ and $T^*M \to M$ respectively. Recall that the exterior covariant derivatives $d^\Psi_k : \Gamma(\bigwedge^k T^*M \otimes \mathcal{F}) \to \Gamma(\bigwedge^{k+1} T^*M \otimes \mathcal{F})$ induced by a vector bundle connection Ψ, defined on a vector bundle $\mathcal{F} \to M$, are given by

$$d^\Psi_k : \alpha\otimes s \mapsto d_k\alpha\otimes s + \sum_{i=1}^{2n}\epsilon^i\wedge\alpha\otimes\nabla^\Psi_{e_i}s$$

where ∇^Ψ denotes the covariant derivative associated to Ψ and d_k denotes the de Rham differential. They are extended linearly to non-homogeneous elements.

Theorem 16: The exterior covariant derivatives induced by the Kuiper connection Φ form a cochain complex in the category of right-modules over the ring CH of compact operators, i.e.,

$$d_k^\Phi(s \cdot a) = (d_k^\Phi s) \cdot a \text{ and}$$
$$d_{k+1}^\Phi \circ d_k^\Phi = 0$$

for any $a \in CH$ and $s \in \Gamma(\mathcal{H}_R^\bullet)$.

Proof. The second formula follows by applying the definition of the exterior covariant derivatives and using the fact that Φ is flat.

Using Corollary 15, for any $\phi = \alpha \otimes s \in \Gamma(\mathcal{H}^k)$ and $a \in CH$

$$\begin{aligned}
d_k^\Phi\left((\alpha \otimes s) \cdot a\right) &= d_k^\Phi\left(\alpha \otimes (s \cdot a)\right) \\
&= d_k\alpha \otimes (s \cdot a) + \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes \nabla_{e^i}^\Phi (s \cdot a) \\
&= (d_k\alpha \otimes s) \cdot a + \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes (\nabla_{e_i}^\Phi s) \cdot a \\
&= \left(d_k\alpha \otimes s + \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes \nabla_{e_i}^\Phi s\right) \cdot a \\
&= \left(d_k^\Phi(\alpha \otimes s)\right) \cdot a
\end{aligned}$$

□

Definition: For a symplectic manifold admitting a metaplectic structure (π_P, Λ), a compatible positive almost complex structure and a normalized Kuiper map J, we call the cochain complex $d_\bullet^\Phi = (\Gamma(\mathcal{H}_R^k), d_k^\Phi)_{k\in\mathbb{Z}}$ in the category of pre-Hilbert CH-modules the *Kuiper complex induced by J*.

Remark: Let A be a C^*-algebra. In categories of pre-Hilbert and Hilbert A-modules and adjointable maps between them, the images of morphisms need not have closed range. (See Solovyev, Troitsky [65].) Consequently, the quotient topology on the cohomology groups of complexes in these categories may be non-Hausdorff and thus, the quotients may not be Hilbert A-modules in the quotient topology. (See [36] for simple examples.)

Convention on the cohomology groups and tensor products: For a compact manifold M, we denote the space of k-cocycles and the k-th cohomology group of a cochain complex $D_\bullet$ on $\mathcal{H}_R^\bullet$ by $Z^k(D_\bullet)$ and $H^k(D_\bullet)$, respectively. The topology on $\Gamma(\mathcal{H}_R^\bullet)$ is determined by the norm induced from the pre-Hilbert C^*-module structure of $\Gamma(\mathcal{H}_R^\bullet)$. For the topology on the cohomology groups, we take the quotient topology. In the case of the de Rham complex on M, we denote the space of k-cocycles by $Z_{dR}^k(M)$, the space of k-coboundaries by $B_{dR}^k(M)$ and the cohomology group by $H_{dR}^k(M)$, and consider the quotient topology on it as well.

On the space of exterior differential k-forms $\Omega^k(M)$ on the manifold M, we take the norm topology given by the Hodge scalar product. When we form a tensor product of pre-Hilbert spaces, we mean the algebraic tensor product equipped with the scalar product induced by the scalar products on the individual factors (especially, no completions involved).

Lemma 17: Let $(h_i)_{i\in\mathbb{N}}$ be a complete orthonormal system on H and, for $i \in \mathbb{N}$, let $\underline{h}_i$ be the constant extension of h_i to a section of $\Gamma(\underline{H})$. If $h = \sum_{i=1}^{\infty} c_i \underline{h}_i$ is a C^1-section of $\underline{H}$, then for any $i \in \mathbb{N}$, c_i are C^1-functions and for any $v \in TM$

$$\nabla^H_v h = \sum_{i=1}^{\infty} v(c_i)\underline{h}_i$$

Proof. Since the pointwise scalar product $(,)$ and h are continuous, we see that c_j is continuous for any $j \in \mathbb{N}$ by taking the scalar product of h and $\underline{h}_j$. Let us choose a map (U, ϕ), $x \in U$, $v \in T_xM$ and compute 'in coordinates'

$$\begin{aligned}\nabla_v h &= \lim_{t\to 0} \frac{1}{t} \sum_{i=1}^{\infty} (c_i(x+tv)\underline{h}_i(x+tv) - c_i(x)\underline{h}_i(x)) \\ &= \lim_{t\to 0} \sum_{i=1}^{\infty} \frac{1}{t}(c_i(x+tv) - c_i(x))\underline{h}_i(x).\end{aligned}$$

Taking the inner product with $\underline{h}_j(x)$, we get

$$(\nabla_v h, \underline{h}_j(x)) = (\lim_{t\to 0} \sum_{i=1}^{\infty} \frac{1}{t}(c_i(x+tv) - c_i(x))\underline{h}_i(x), \underline{h}_j(x))$$

Since the pointwise inner product $(,)$, c_i's and $\underline{h}_i$'s are continuous, we can take the limit out of the bracket getting

$$\begin{aligned}&\lim_{t\to 0} \frac{1}{t}(\sum_{i=1}^{\infty}(c_i(x+tv) - c_i(x))\underline{h}_i(x), \underline{h}_j(x)) = \lim_{t\to 0} \frac{1}{t}(c_j(x+tv) - c_j(x)) \\ &= v(c_j)\end{aligned}$$

Especially, the derivation of c_j in direction v exists and it is continuous since the scalar product and $\underline{h}_j$ are continuous and since $\nabla_v h$ is continuous by the assumption. From the uniqueness of coordinates in a Hilbert space, we get that $\nabla_v h = \sum_{i=1}^{\infty} v(c_i)\underline{h}_i$. □

Theorem 18: Let (M, ω) be a compact symplectic manifold of dimension $2n$ admitting a metaplectic structure (π_P, Λ) and J be a normalized Kuiper map. Then for any $k \in \mathbb{Z}$ and any compatible positive almost complex structure, the cohomology groups $H^k(d^{\Phi}_{\bullet})$ of the Kuiper complex induced by J have a structure of finitely generated projective Hilbert CH-modules whose norm topology coincides with the quotient topology.

Proof. 1) By Lemma 11, the bundles $\mathcal{H}^k_R$ and $\bigwedge^k T^*M \otimes \underline{H}$ are isomorphic as smooth Hilbert C^*-bundles. The map

$$j : \Gamma(\mathcal{H}^\bullet_R) \to \Gamma(\bigwedge^\bullet T^*M \otimes \underline{H}) \text{ given by } j(s) = \widetilde{J} \circ s$$

where $s \in \Gamma(\mathcal{H}^\bullet_R)$, preserves the CH-products and it is a CH-homomorphism as follows from Lemma 11 and the linearity of the Pettis integral with respect to continuous maps (see, e.g., [60]), respectively. The same is true for the inverse $j^{-1}(s) = \widetilde{J}^{-1} \circ s$. In particular, $\Gamma(\bigwedge^k T^*M \otimes \underline{H})$ and $\Gamma(\mathcal{H}^\bullet_R)$ are isomorphic as pre-Hilbert CH-modules.

2) For $\alpha \otimes s \in \Gamma(\mathcal{H}^k_R)$, we have using Lemma 13

$$d_k^{\Phi^H}(j(\alpha \otimes s)) = d_k^{\Phi^H}(\alpha \otimes (J \circ s)) = d_k\alpha \otimes J \circ s + \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes \nabla^H_{e_i}(J \circ s)$$

$$= d_k\alpha \otimes J \circ s + \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes J \circ \nabla_{e_i} s = \widetilde{J} \circ d_k^\Phi(\alpha \otimes s) = j\left(d_k^\Phi(\alpha \otimes s)\right)$$

where $(e_i)_{i=1}^{2n}$ and $(\epsilon)_{i=1}^{2n}$ are dual local frames of M. Thus j is a cochain map. Because j is a pre-Hilbert CH-isomorphism (item 1), the corresponding cohomology groups $H^k(d^\Phi_\bullet)$ and $H^k(d^{\Phi^H}_\bullet)$ of $(\Gamma(\mathcal{H}^k_R), d^\Phi_k)_k$ and $(\Gamma(\bigwedge^k T^*M \otimes \underline{H}), d_k^{\Phi^H})_k$ are isomorphic as CH-modules and topological vector spaces when considered with the quotient topologies.

3) Note that $H^k_{dR}(M) \otimes H$ is a CH-module by defining $([\alpha] \otimes h) \cdot a = [\alpha] \otimes (h \cdot a)$ for $\alpha \in \Omega^k(M)$, $h \in H$ and $a \in CH$. We show that the cohomology groups of $(\Gamma(\bigwedge^k T^*M \otimes \underline{H}), d_k^{\Phi^H})_k$ are isomorphic to $H^k_{dR}(M) \otimes H$ as CH-modules. Let us consider the linear extension, denoted by $[\gamma]$, of the map $H^k_{dR}(M) \otimes H \ni [\alpha] \otimes h \mapsto [\alpha \otimes \underline{h}] \in H^k(d^{\Phi^H}_\bullet)$ where $\underline{h}$ is the constant extension of an element $h \in H$. It is easy to check that the resulting map is well defined (maps cocycles into cocycles and vanishes on coboundaries). Let us analyze the continuity of $[\gamma]$. For it, we denote the extension map $h \mapsto \underline{h}$ by ext. Then $[\gamma]$ fits into the following commutative diagram

$$\begin{array}{ccc} Z^k_{dR}(M) \otimes H & \xrightarrow{\mathrm{Id} \otimes ext} & Z^k(d^{\Phi^H}_\bullet) \\ \big\downarrow {\scriptstyle q_1 \otimes \mathrm{Id}_H} & & \big\downarrow {\scriptstyle q_2} \\ H^k_{dR}(M) \otimes H & \xrightarrow{[\gamma]} & H^k(d^{\Phi^H}_\bullet) \end{array}$$

where q_i, $i = 1, 2$, denote the appropriate quotient projections and Id denotes $\mathrm{Id}_{Z^k_{dR}(M)}$. Since q_2, Id and ext are continuous and $q_1 \otimes \mathrm{Id}_H$ is open, $[\gamma]$ is continuous.

For to prove that $[\gamma]$ is bijective, let us choose a complete orthonormal system $(h_i)_{i \in \mathbb{N}}$ of H and let us consider an element $f = [\sum_{j=1}^r \alpha_j \otimes (\sum_{i=1}^\infty c_{ij} \underline{h_i})]$ of $H^k(d^{\Phi^H}_\bullet)$, where c_{ij} are functions on M. The element $f' = \sum_{i=1}^\infty [\sum_{j=1}^r c_{ij}\alpha_j] \otimes$

$h_i \in H^k_{dR}(M) \otimes H$ is a well defined element of the tensor product, as can be seen by the following consideration. Since c_{ij} are smooth (by an iterative use of Lemma 17), $\sum_{j=1}^r c_{ij}\alpha_j$ is smooth for each $i \in \mathbb{N}$. Further, the sum $\sum_{i=1}^\infty \sum_{j=1}^r c_{ij}\alpha_j \otimes h_i$, denoted by f'', converges pointwise on the manifold to $\sum_{j=1}^r \alpha_j \otimes \sum_{i=1}^\infty c_{ij}\underline{h_i}$, that converges pointwise in each point on the manifold by the choice of the representation for f. Let us consider the operator $T : Z^k_{dR}(M)\otimes H \to H^k_{dR}(M)\otimes H$ given as the linear extension of $T(\alpha\otimes h) = [\alpha]\otimes h$. Let us see that this operator is continuous. Since M is compact, $H^k_{dR}(M)$ is finite dimensional and $B^k_{dR}(M)$ is orthogonally complemented by the usual Hodge theory. Therefore, we can define the orthogonal projection pr of $Z^k_{dR}(M)$ onto the complement, which is the space of harmonic forms. The quotient norm of an element $[\alpha]$ in the de Rham cohomology group equals to the norm of $\mathrm{pr}(\alpha)$ in the cocycle space. Consequently, the norm of T is bounded by the norm of pr which is one. Now, f' is the T-image of f'' by the continuity of T. Especially, f' is well defined (from the point of view of convergence). The facts that $\sum_{j=1}^r \alpha_j \otimes \sum_{i=1}^\infty c_{ij}\underline{h_i}$ is closed and that $\underline{h}_i$'s are constant imply

$$\begin{aligned}
0 &= d_k^{\Phi^H}\Big(\sum_{j=1}^r \alpha_j \otimes \sum_{i=1}^\infty c_{ij}\underline{h_i}\Big)\\
&= \sum_{j=1}^r \left(d_k\alpha_j \otimes \sum_{i=1}^\infty c_{ij}\underline{h}_i + \sum_{p=1}^{2n} \epsilon^p \wedge \alpha_j \otimes \sum_{i=1}^\infty e_p(c_{ij})\underline{h}_i \right)\\
&= \sum_{j=1}^r \left(d_k\alpha_j \otimes \sum_{i=1}^\infty c_{ij}\underline{h}_i + \sum_{i=1}^\infty (d_0 c_{ij}) \wedge \alpha_j \otimes \underline{h}_i \right)\\
&= \sum_{i=1}^\infty \sum_{j=1}^r d_k(c_{ij}\alpha_j) \otimes \underline{h}_i
\end{aligned}$$

Since $(h_i)_{i\in\mathbb{N}}$ is linearly independent, $\sum_{j=1}^r c_{ij}\alpha_j$ is closed for each $i \in \mathbb{N}$. Especially, $f' \in H^k_{dR}(M) \otimes H$. Since $[\gamma]$ is continuous, $f' = \sum_{i=1}^\infty [\sum_{j=1}^r c_{ij}\alpha_j] \otimes h_i$ is a preimage of f. Summing-up, $[\gamma]$ is surjective.

For the injectivity, let us suppose that there is an element $\sum_{l=1}^s \sum_{i=1}^\infty [\alpha_l] \otimes c_{li}h_i$ mapped to $[\sum_{l=1}^s \sum_{i=1}^\infty \alpha_l \otimes c_{li}\underline{h}_i] = 0$ by $[\gamma]$, where c_{li} are complex numbers. Thus, there exist smooth exterior differential $(k-1)$-forms $\widetilde{\alpha}_j$, smooth functions $\widetilde{c}_j$ and smooth sections $v_j \in \Gamma(\underline{H})$, $j = 1, \ldots, r$, such that

$$\sum_{l=1}^s \sum_{i=1}^\infty c_{li}\alpha_l \otimes \underline{h}_i = d_{k-1}^{\Phi^H} \left(\sum_{j=1}^r \widetilde{c}_j \widetilde{\alpha}_j \otimes v_j \right)$$

For any v_j, there are smooth functions $\widetilde{c}_{ij}$ on M (Lemma 17) for which $v_j =$

$\sum_{i=1}^{\infty} \widetilde{c}_{ij}\underline{h}_i$. Using the Leibniz rule, we get

$$\sum_{l=1}^{s}\sum_{i=1}^{\infty} c_{li}\alpha_l \otimes \underline{h}_i = d_{k-1}^{\Phi^H}\left(\sum_{j=1}^{r}\widetilde{c}_j\widetilde{\alpha}_j \otimes \sum_{i=1}^{\infty}\widetilde{c}_{ij}\underline{h}_i\right) =$$

$$d_{k-1}\left(\sum_{j=1}^{r}\widetilde{c}_j\widetilde{\alpha}_j\right) \otimes \sum_{i=1}^{\infty}\widetilde{c}_{ij}\underline{h}_i + \sum_{p=1}^{2n}\sum_{j=1}^{r}\widetilde{c}_j\epsilon^p \wedge \widetilde{\alpha}_j \otimes \nabla^H_{e_p}(\sum_{i=1}^{\infty}\widetilde{c}_{ij}\underline{h}_i) =: (*)$$

Using Lemma 17, we can take the derivative behind the infinite sum, getting

$$\begin{aligned}
(*) &= \sum_{j=1}^{r}\left(d_{k-1}(c_j\widetilde{\alpha}_j) \otimes \sum_{i=1}^{\infty}\widetilde{c}_{ij}\underline{h}_i\right) + \sum_{j=1}^{r}\left(\sum_{p=1}^{2n}\widetilde{c}_j\epsilon^p \wedge \widetilde{\alpha}_j \otimes \sum_{i=1}^{\infty} e_p(\widetilde{c}_{ij})\underline{h}_i\right)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{r}\left(\widetilde{c}_{ij}d_{k-1}(\widetilde{c}_j\widetilde{\alpha}_j) + \sum_{p=1}^{2n} e_p(\widetilde{c}_{ij})\epsilon^p \wedge \widetilde{c}_j\widetilde{\alpha}_j\right) \otimes \underline{h}_i\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{r}(d_{k-1}(\widetilde{c}_{ij}\widetilde{c}_j\widetilde{\alpha}_j)) \otimes \underline{h}_i
\end{aligned}$$

since $d_0\widetilde{c}_{ij} = \sum_{p=1}^{2n} e_p(\widetilde{c}_{ij})\epsilon^p$. Summing-up, $\sum_{l=1}^{s} c_{li}\alpha_l = \sum_{j=1}^{r} d_{k-1}(\widetilde{c}_{ij}\widetilde{c}_j\alpha_j)$ for each $i \in \mathbb{N}$. In particular, $\sum_{l=1}^{s} c_{li}\alpha_l$ is exact for each $i \in \mathbb{N}$. Thus, $\sum_{i=1}^{\infty}\sum_{l=1}^{s}[\alpha_l] \otimes c_{li}h_i = \sum_{i=1}^{\infty}[\sum_{l=1}^{s} c_{li}\alpha_l] \otimes h_i = 0$ and $[\gamma]$ is injective. The verification of the CH-equivariance of $[\gamma]$ is straightforward.

4) The inverse to $[\gamma]$ which we denote by $[\delta]$, is defined by $[\delta]([\sum_{j=1}^{r}\alpha_j \otimes \sum_{i=1}^{\infty} c_{ij}\underline{h}_i]) = \sum_{i=1}^{\infty}[\sum_{j=1}^{r} c_{ij}\alpha_j] \otimes h_i$ where c_{ij} are smooth functions on M and $\alpha_j \in \Omega^k(M)$, $j = 1, \ldots, r$. The sum at the right-hand side can be shown to converge by undertaking a similar procedure as in the proof of the surjectivity of $[\gamma]$. Further, it is immediate to check that $[\delta]$ maps cocycles into cocycles and vanishes on coboundaries. Summing-up, $[\delta]$ is well defined. Let us define a map $\Xi(\sum_{j=1}^{r}\alpha_j \otimes \sum_{i=1}^{\infty} c_{ij}\underline{h}_i) = \sum_{i=1}^{\infty}\sum_{j=1}^{r} c_{ij}\alpha_j \otimes h_i$. Since M is compact, the norm of Ξ is easily seen to be finite and thus, Ξ is continuous as a linear map between normed spaces. The continuity of $[\delta]$ is established by the use of the commutative diagram ($q_1 \otimes \mathrm{Id}_H$ and Ξ are continuous, and q_2 is open)

$$\begin{array}{ccc}
Z^k_{dR}(M) \otimes H & \xleftarrow{\ \Xi\ } & Z^k(d_\bullet^{\Phi^H})\\
\big\downarrow{\scriptstyle q_1\otimes \mathrm{Id}_H} & & \big\downarrow{\scriptstyle q_2}\\
H^k_{dR}(M) \otimes H & \xleftarrow{\ [\delta]\ } & H^k(d_\bullet^{\Phi^H})
\end{array}$$

Summing-up, we have $H^k(d_\bullet^{\Phi^H}) \simeq H^k_{dR}(M) \otimes H$ as CH-modules (by item 3) and as topological spaces by this item.

5) Notice that we already proved that $H^k(d_\bullet^{\Phi})$ and $H^k(d_\bullet^{\Phi^H})$ are isomorphic as CH-modules and topological spaces (item 2). Since $H^k_{dR}(M)$ is finite dimensional, we may use the projection pr defined in item 3. The quotient topology on the cohomology group is generated by the scalar product

$([\alpha],[\beta])_{H^k_{dR}} = (\mathrm{pr}(\alpha),\mathrm{pr}(\beta))_{\Omega^k}$ where at the right-hand side, the Hodge scalar product on differential forms is meant. Consequently, $H^k_{dR}(M)\otimes H$ possesses a pre-Hilbert C^*-product given by $([\alpha]\otimes h,[\beta]\otimes h')_\otimes = ([\alpha],[\beta])_{H^k_{dR}}(h,h')$ where at the right-hand side, (h,h') denotes the Hilbert CH-product on H. The norm induced by this pre-Hilbert CH-product coincides with the norm induced by the scalar product considered on the tensor product (Lemma 1). By the finite dimensionality of the de Rham cohomology groups, $H^k_{dR}(M)\otimes H$ is a complete inner product space. In particular, the pre-Hilbert CH-product on the tensor product of the de Rham cohomology group and H is a Hilbert CH-product.

Since $H^k(d_\bullet^{\Phi^H})$ is homeomorphic and CH-isomorphic to $H^k_{dR}(M)\otimes H$ (item 4) we can equip it with the inner product $([s_1],[s_2]) = ([\delta]([s_1]),[\delta]([s_2]))_\otimes$ where s_1, s_2 are cocycles for $d_\bullet^{\Phi^H}$. Because $[\delta]$ is an isomorphism of CH-modules, the resulting map is a well defined Hilbert C^*-product. Since $[\delta]$ is a homeomorphism, the topology on $H^k(d_\bullet^{\Phi^H})$ generated by the norm induced by the scalar product coincides with the quotient topology on $H^k(d_\bullet^{\Phi^H})$. Similarly, by the use of the map j (item 1 and item 2), we induce the Hilbert CH-product from $H^k(d_\bullet^{\Phi^H})$ to $H^k(d_\bullet^{\Phi})$ making it a Hilbert CH-module isomorphic to $H^k_{dR}(M)\otimes H$ as well.

Because $H^k_{dR}(M)$ is finite dimensional and H is finitely generated (Lemma 2), $H^k_{dR}(M)\otimes H$ is also finitely generated. Consequently, it is projective by the Magajna theorem. Summing-up, $H^k(d_\bullet^{\Phi})$ are finitely generated projective Hilbert CH-modules for each k. □

We give a much shorter proof of Thm. 18 based on the Hodge theory for CH-bundles.

Second proof of Theorem 18. We use the Mishchenko–Fomenko theory elaborated for complexes in [35] and [36]. It is immediate to compute that the symbols s_k of d_k^Φ are given by $s_k(\alpha\otimes s,\tau) = (\tau\wedge\alpha)\otimes s$, $\tau\in T^*_mM$, $\alpha\otimes s\in(\mathcal{H}^k_R)_m$ (see [34]), and a matter of multilinear algebra to see that they form an exact sequence in places $k=0,\ldots,2n$ for any $\tau\neq 0$. In the places $k=0,\ldots,2n$, the Kuiper complex is thus elliptic. Due to Lemma 2, it is a complex on finitely generated projective Hilbert CH-bundles. Consequently (compactness and ellipticity), its cohomology groups are finitely generated and projective Hilbert CH-modules by Thm. 9 in Krýsl [36]. In places $\mathbb{Z}\setminus\{0,\ldots,2n\}$, the cohomology groups are zero. □

Remark: 1) Since the cohomology groups are Hilbert C^*-modules, the images of $d_\bullet^\Phi$ and $d_\bullet^{\Phi^H}$ are necessarily closed.

2) The Hodge theory holds for the Kuiper complex by Thm. 9 in [36] as well. From the proof of Theorem 18 (and the proof of Lemma 2), we see that the CH-rank of $H^k(d_\bullet^\Phi)$ equals to the k-th Betti number of M. (See Bakić, Guljaš [2] for a definition of the CH-rank if needed.)

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