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MODAL SET THEORY

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In [8] we proposed a set theory MST formalized in modal logic. The aim of this lecture is to announce new consistency result concerning MST (these form Part II). For completeness of a presentation we recapitulate the motivation of MST and some definitions and results (without proofs) from [8] (these form Part I).

We do not mention here connections with other related systems (see [1],...,[6]); this is done, in detail, in [8].

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Part I, §0. Introduction

Cantor's comprehension (CC):

$$\exists y \forall t; \varphi(t) \equiv t \in y$$

where φ is any property, is a very elegant principle. Its substance describes Cantor's naive set-universe. Unfortunately, in the most customary formalization, where any formula of the set-theoretical language is accepted as a property, is CC controversial.

In most set-theories the motivation lies in Cantor's universe. They replace CC by a list of weaker axioms (e.g.ZF) or restrict it (e.g.NF).

At the same time they lose important features of CC: homogeneity, simplicity and elegance or apparent intuitive picture.

We think that these features of CC justify the search for other possible reformulations of CC.

The modification of CC which is formalized by MST covers some mathematics (it interprets PA) and, on the other side, some of its considerable fragments are proved to be consistent.

§1. Theory MST

Let us imagine the following situation. There exists some Set-universe which is the object of our consideration. The only atomic predicates are "to be equal" and "to be element of". Each atomic sentence and hence each sentence is true or false in the Set-universe.

Our wish is to recognize the truth, i.e. the sentences true in the Set-universe. So some true sentences are known to us, are in our knowledge.

For formalizing the modal operator "to be known" we extend the usual classical set-theoretical language by adopting a new unary logical connective $\Box$ which should be an epistemic modality. Thus our language (the modal set-theoretical language) is the modal predicate calculus with identity (see [7]) with a binary predicate $\in$ as the only non-logical symbol.

1.1 When we decide to try to understand the Set-universe we can, already, take the fact of looking for the knowledge as a part of the knowledge. Put otherwise, we may accept assumptions which manifest the principles and the correctness of our knowledge. Hence the following two axiom schemas and one deduction rule should be accepted:

$$\left.\begin{array}{ll}\text{(i)} & \Box\varphi \rightarrow \varphi \\ \text{(ii)} & \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)\end{array}\right\} \text{T-axioms}$$

(iii) $\dfrac{\varphi}{\Box\varphi}$ necessitation rule (N-rule)

This extension of the classical predicate calculus is called T in [7].

1.2 The main idea of MST is that CC does not refer to the whole Set-universe but only to its known part, to our "universe of discourse". That means: it seems to us from the point of view of our knowledge that the Set-universe behaves as if CC were sound.

In the chosen language this modification of CC (Modal CC or shortly MCC) can be described as follows:

MCC : for $\varphi(t,a_1,\ldots,a_k)$ any formula of the modal set-theoretical language with free variables among $t,a_1,\ldots,a_k$ the universal closure of the following formula holds:

$$\exists y\,\forall t;\ (\Box\varphi(t,a_1,\ldots,a_k)\equiv\Box t\in y)\ \&$$
$$\&\ (\Box\neg\varphi(t,a_1,\ldots,a_k)\equiv\Box t\notin y)\quad.$$

The a_i's are called parameters and will be omitted further.

1.3 The last but one principle we adopt is the extensionality in the usual formalization:

$$(\forall t;\, t\in x\equiv t\in y)\rightarrow x=y\quad.$$

One reason for it is simply our usage in thinking about Set-universe. It also helps to prove various properties of a given set a: it suffices to define, in some useful way, a set b with the same extension as a (see 3.7). On the other side, many concepts and results using extensionality can be interpreted without it (see [3]).

1.4 So far we have not accepted any concrete "theory of knowledge", any non-logical epistemic assumptions. The last principle of MST is of this kind. As usual $\Diamond\varphi$ abbreviates $\neg\Box\neg\varphi$. The principle is:

LP :
$$\Diamond x=y\rightarrow\Box x=y$$

or equivalently

(i) $(x=y\rightarrow\Box x=y)\ \&\ (x\neq y\rightarrow\Box x\neq y)$ or (ii) $\Box x=y\vee\Box x\neq y$.

We leave out the question whether LP has logical or empirical character.

It is a kind of a finitistic assumption. There are also close connections to Leibniz's principle: "No two monads are exactly alike", from his theory of monads (see [9]). This may be formalized as $\Diamond x=y\rightarrow x=y$ (where $\Diamond x=y$ simulates indistinguishability) or equivalently $x\neq y\rightarrow\Box x\neq y$. Thus Leibniz's principle coincides with the second conjunct of (i). Surely, the first is a trivial consequence of the substitution properties of identity. Hence the name LP seems to be justified for this axiom.

We have to remark that LP is also discussed in [7]; the first conjunct of (i) is called LI and the second LNI there.

1.5 Let us summarize the definitions. The formal theory MST is formalized within modal predicate calculus with identity. The axioms of identity are assumed. The underlying logical system is T. The only non-logical assumptions are: extensionality, MCC and LP.

Let us stress explicitly that N-rule is applicable generally (in contrast to theories of [2] and [3]). In particular, all instances of MCC are "known". This gives to the whole system features of a "logical calculus".

We will use freely various results about modal logics which are proved in [7].

In the whole text we do not discuss possible extensions of MST (see [8]). Also in consistency results, by way, stronger theories are proved to be consistent then is explicitly stated, but these are irrelevant to our discussion.

§2. Russell's paradox

Let us discuss Russell's paradox formally. Applying MCC to Russell's formula $t \notin t$ we obtain :

$$\exists y \forall t ; \Box t \notin t \equiv \Box t \in y \;\&\; \Box t \in t \equiv \Box t \notin y$$

and hence :

$$\exists y ; \Box y \in y \equiv \Box y \notin y \quad .$$

Now surely :

$$(\Box y \in y \rightarrow y \in y) \;\&\; (\Box y \notin y \rightarrow y \notin y)$$

and the only escape from the contradiction gives :

$$\exists y ; (\Box y \in y \equiv \Box y \notin y) \;\&\; \Diamond y \in y \;\&\; \Diamond y \notin y \quad .$$

Fortunately, this situation does not lead to inconsistency because $\Box y \in y \vee \Box y \notin y$ is not a theorem of T.

We even profit by this trivial but important corollary:

2.1 Corollary: $\exists y ; \Diamond y \in y \;\&\; \Diamond y \notin y$.

The reader could calculate himself that other modifications of Russell's paradox (e.g.Curry's) also fails for MST.

§3. Decidable and small sets

In this Chapter we develop two important notions of MST. It is not surprising that "known formulas" or "known sets" will have pleasant properties. This stands behind the following definitions and results.

3.1 Metadefinition : Call a formula φ $\Box$-decidable iff

$$\Box\varphi \vee \Box\neg\varphi \text{ holds.}$$

Other equivalent conditions are : $\neg\Box\varphi \to \Box\neg\varphi$, $\Diamond\varphi \to \Box\varphi$ or $(\varphi \to \Box\varphi) \& (\neg\varphi \to \Box\neg\varphi)$. Note that decidability of φ (i.e. MST $\vdash \varphi$ or MST $\vdash \neg\varphi$) implies (by N-rule) $\Box$-decidability of it but the converse does not generally holds.

Some of the following results are proved using additional assumptions; namely: Barcan's formula $\forall x \Box\varphi(x) \to \Box\forall x \varphi(x)$ (BF) and Brouwer's axiom $\Diamond\Box\varphi \to \varphi$ (B) (for details see [8]). This will always be indicated in brackets before a statement.

3.2 Theorem: (i) Boolean combination of $\Box$-decidable formulas is $\Box$-decidable.

(ii) (BF) All formulas built up from $\Box$-decidable ones are $\Box$-decidable.

3.3 Corollary: Boolean combination of equalities is $\Box$-decidable.

3.4 Definition: Call a set y decidable $(D(y))$ iff

$$\forall t; \Box t \in y \vee \Box t \notin y \quad \text{holds.}$$

Next results show that the domain of decidable sets is rich and behaves reasonably.

3.5 Theorem: If $\varphi(t)$ is $\Box$-decidable then there exists a decidable set y s.t. $\forall t; t \in y \equiv \varphi(t)$.

3.6 Corollary: (i) $\exists y \forall t; t \notin y$ (ii) $\exists y \forall t; t \in y$

(iii) $\exists y \forall t; t \in y \equiv (t=a_1 \vee \ldots \vee t=a_k)$

(iv) $\exists y \forall t; t \in y \equiv (t \neq a_1 \& \ldots \& t \neq a_k)$

and for a, b decidable:

(v) $\exists y \forall t; t \in y \equiv (t \in a \& t \in b)$

(vi) $\exists y \forall t; t \in y \equiv (t \in a \vee t \in b)$

(vii) $\exists y \forall t; t \in y \equiv t \notin a$

(viii) $\exists y \forall t; t \in y \equiv (t \in a \vee t=c)$.

Moreover, y is decidable in each case.

<u>3.7 Theorem</u> (BF) : Let <u>a</u> be a decidable set. Then :

(i) (the union) $(\forall b \in a; D(b)) \to \exists c \forall t; t \in c \equiv (\exists b \in a; t \in b)$

(ii) (the power) $\exists c \forall t; D(t) \to (t \in c \equiv t \subseteq a)$

(iii) (the replacement) If $\varphi(x,t)$ is 0-decidable then $\exists b \forall t; t \in b \equiv (\exists x \in a; \varphi(x,t))$.

The result 3.6 (iii) (and extensionality) implies that all sets finite from outside of Set-universe are decidable. Hence the following

<u>3.8 Definition</u>: Call a set y <u>small</u> (S(y)) iff $\forall x; x \subseteq y \to D(x)$, should substitute finiteness.

Observe that 2.1 implies existence of a set which is not small.

<u>3.9 Theorem</u> (the comprehension): Let <u>a</u> be a small set and $\varphi(t)$ any formula. Then: $\exists y \forall t; t \in y \equiv (t \in a \,\&\, \varphi(t))$ (and y is small).

<u>3.10 Theorem</u>: (i) (BF) The union of a small set of small sets is a small set.

(ii) (BF,B) A small set has a power set.

<u>§4. Arithmetic</u>

Using familiar von Neumann's definition we may introduce ordinals. Then the "natural" candidates for natural numbers are the small ordinals. Since it is not evident why there could not be a limit small ordinal, we use the following

<u>4.1 Definition:</u> <u>a</u> is a <u>natural number</u> (N(a)) iff a conjunction of the following holds:

(i) <u>a</u> is transitive

(ii) <u>a</u> is strictly well-ordered by $\in$

(iii) <u>a</u> is small

(iv) $\forall b \trianglelefteq a \exists c;\ c = \max_{\in} b$

(where $b \trianglelefteq a$ and $c = \max_{\in} b$ are obvious abbreviations).

Then, trivially, empty set is natural number, an element and a successor of a natural number is a natural number and natural numbers are strictly ordered by $\in$.

<u>4.2 Metadefinition:</u> A formula $\varphi(t)$ is called a <u>cut</u> in N iff a conjunction of the following holds:

(i) $\forall t;\ \varphi(t) \to N(t)$
(ii) $(\varphi(a) \,\&\, b<a) \to \varphi(b)$
(iii) $(\varphi(a) \,\&$ "b is a successor of a"$) \to$
$\to \varphi(b)$.

A cut $\varphi(t)$ is called a <u>nontrivial</u> iff also:
(iv) $\exists a,b;\ \varphi(a) \,\&\, \neg\varphi(b) \,\&\, N(b)$.

Now we are ready to state

<u>4.3 Theorem</u> (the induction) : There are no nontrivial cuts in N.

Let us now sketch how to introduce the arithmetical structure on N. Define in some reasonable way addition. For example: $a + b = c$ iff "there exists a small sequence $s_0,\dots,s_b$ s.t. $s_0=a$, $s_1=a+1,\dots,s_b=c$ (u+1 abbreviates a successor of u) " (the "sequence" will be defined as usual). So we easily prove $a+0=a$ and $a+(b+1)=(a+b)+1$. Then, using 4.3, we prove that + is defined for any two natural numbers. The same can be done for multiplication and 4.3 will guarantee the wanted arithmetical properties of + and . . Observe that 4.3 also implies that this can be done uniqueli.

<u>4.4</u> Through arithmetic and through the notion of "small set" theory MST formalizes some properties of finiteness. On the other side, result 2.1 suggests that it is also possible to introduce some infinity in MST. Let us sketch one approach.

The "infinite" sets will be "non-small" ones but the different "degree" of infinity of two infinite sets a,b will lay rather in their different "complexity" (in the sense of knowledge) than in the different cardinality. Thus we may define $a \sim b$ iff "there exists a decidable bijection f such that $\forall x;\ \Box x \in a \equiv \Box f(x) \in b$ " (there is, clearly, a number of other possibilities). It is easy to prove, for example, that there are infinite sets of different "degree" (e.g.Russell's and universal).

§5. Two consistent fragments

In this Chapter we present two consistency results which are both proved by construction of an appropriate Kripke-style model.

5.1 Theorem: The theory MST without extensionality and with MCC restricted only to nonmodal formulas is consistent.

(a formula is nonmodal iff it does not contain $\Box$)

5.2 Theorem: The theory MST with MCC replaced by a scheme:
$$\exists y \forall t; \left(\Box t \in y \rightarrow \Box \varphi(t,\bar{z})\right) \& \left(\Box t \notin y \rightarrow \Box \neg \varphi(t,\bar{z})\right)$$
is consistent.

Various consistency results concerning fragments of MST with restricted applicability of N-rule can be proved, in particular using interpretation into the Feferman's theory from [3] (see [8]).

Part II, §6. New consistency result

In this Chapter we prove consistency of the following subtheory S of MST. Theory S has N-rule generally applicable and the following axiom schemas over those of predicate calculus :

(1) $\Box\varphi \rightarrow \varphi$

(2) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

$\Diamond x=y \rightarrow \Box x=y$

(4) extensionality

(5) for $\varphi(t,\bar{z})$ any formula:

$$\forall\bar{z}\,\exists y\,\forall t;\ (\Box\varphi(t,\bar{z}) \rightarrow t\in y)\ \&$$
$$\&\ (\Box\neg\varphi(t,z) \rightarrow t\notin y)$$

The proof will be done by interpreting S in Peano's arithmetic PA.

6.1 Lemma : There exists a provability predicate $P(x)$ in PA s.t. for any arithmetical formulas φ,ψ holds:

(i) if $PA \vdash \varphi$ then $PA \vdash P(\bar{\varphi})$

(ii) $PA \vdash P(\overline{\varphi\rightarrow\psi}) \rightarrow (P(\bar{\varphi}) \rightarrow P(\bar{\psi}))$

(iii) $PA \vdash (x=y \rightarrow P(\overline{\underset{\sim}{x}=\underset{\sim}{y}}))\ \&\ (x\neq y \rightarrow P(\overline{\underset{\sim}{x}\neq\underset{\sim}{y}}))$

(iv) $PA \vdash \neg P(\bar{\varphi}) \vee \neg P(\overline{\neg\varphi})$

(Remark: we use usual formalization of syntax in PA; thus $\bar{\varphi}$ denote a code of a formula φ , $Prf(d,\bar{\varphi})$ means "d is a proof of φ in PA", for b a number $\underset{\sim}{b}$ denotes an appropriate numeral and so on.)

Proof (in PA) : A proof d is a sequence of formulas which will be called steps of d. Letters d, d_o, d_1,.... are reserved for proofs. Define in PA :

a) $d \succ\!- \bar{\varphi} \overset{df}{\equiv}$ " φ is a step in d"

b) $d \oplus \bar{\varphi} \overset{df}{=}$ "the sequence which is a prologation of d by adding φ as a new step"

c) "d is inconsistent" $\overset{df}{\equiv}$ "from set $\{\varphi \mid d \succ\!- \bar{\varphi}\}$ is the formula $x\neq x$ derivable in predicate calculus only"

d) $d^+ \overset{df}{=}$ "min X_d" , where X_d is a set of all $d\oplus\bar{\varphi}$ which satisfies a conjunction of:

(i) $d\oplus\overline{\varphi}$ is a proof in PA
(ii) $d\oplus\overline{\varphi}$ is not inconsistent
(iii) $\neg d \succ \overline{\varphi}$

e) "d is good" $\overset{df}{\equiv}$ "there exists a sequence $(s_0, s_1, \dots, s_u)$ s.t. $s_0 = \overline{x=x}$, $\forall i<u; s_{i+1} = s_i^+$ and $s_u = d$ "

Now we are ready to define provability predicate P(x):

$$P(\overline{\varphi}) \overset{df}{\equiv} \exists d; \; Prf(d, \overline{\varphi}) \;\&\; \text{"d is good"} \quad .$$

Sublemma (i): Predicate P(x) satisfies 6.1(i).

Proof: Let φ be any formula s.t. $PA \vdash \varphi$. We prove $PA \vdash P(\overline{\varphi})$ by metainduction on the number of steps in the shortest proof of φ. Thus suppose that for all steps ψ_i's before φ in the shortest proof of φ holds $PA \vdash P(\overline{\psi_i})$. Let d_0 be a good proof s.t. $d_0 \succ \overline{\psi_i}$'s (such clearly exists). If also $d_0 \succ \overline{\varphi}$ we are done so suppose $\neg d_0 \succ \overline{\varphi}$. Consider good proofs $d \geqslant d_0$ s.t. $d^+ < d\oplus\overline{\varphi}$. There is only finitely many formulas bellow $\overline{\varphi}$ hence we may choose d_1 the least good proof $\geqslant d_0$ s.t. $d_1^+ \geqslant d_1 \oplus \overline{\varphi}$. Now, by choice of d_0 (and by consistency of PA), $d_0 \oplus \overline{\varphi} \in X_{d_0}$ thus, by definition of d_1, also $d_1 \oplus \overline{\varphi} \in X_{d_1}$. Hence $d_1^+ = d_1 \oplus \overline{\varphi}$, i.e. $PA \vdash P(\overline{\varphi})$ and we are done.

Sublemma(ii): Predicate P(x) satisfies 6.1(ii).

Proof(in PA): Let d_0 be the greatest of good proofs of $\varphi \to \psi$ and φ. Thus $d_0 \succ \overline{\varphi \to \psi}$ and $d_0 \succ \overline{\varphi}$. Consider d_1 the least good proof s.t. $d_1 \geqslant d_0$ and $d_1^+ \geqslant d_1 \oplus \overline{\psi}$. If not $d_1 \succ \overline{\psi}$ then clearly $d_1 \oplus \overline{\psi} \in X_{d_1}$ hence $d_1^+ = d_1 \oplus \overline{\psi}$ and $P(\overline{\psi})$ follows. We are done.

Sublemma(iii): Predicate P(x) satisfies 6.1(iii).

Proof(in PA) : For any numbers a,b: $a=b \leftrightarrow \underset{\sim}{a} = \underset{\sim}{b}$. Clearly all recursive properties of + and $\cdot$ which are needed for proving true equality (or inequality) have good proofs. Then the statement follows by induction on max(a,b). We are done.

Sublemma iv : Predicate P(x) satisfies 6.1(iv).

Proof (in PA): Let $P(\overline{\varphi})$ and $P(\overline{\neg\varphi})$ and choose d_0 to be the greatest of good proofs of $\varphi, \neg\varphi$. Then d_0 is inconsistent--contradiction. We are done.

This completes also the proof of 6.1.

6.2 Definition (in PA):

(i) $Fle(y) \overset{df}{\equiv}$ "y is a code of some Σ_2-formula $\varphi(t)$ (possibly with parameters) with one free variable t (thus also $y=\overline{\varphi(t)}$)"

($P(x)$ is a Σ_2-formula)

(ii) $y \simeq z \overset{df}{\equiv} Fle(y) \,\&\, Fle(z) \,\&$ "if $y=\overline{\varphi(t)}$ and $z=\overline{\psi(t)}$ then $\forall t; \varphi(t) \equiv \psi(t)$"

(This is possible to define in PA since the formulas φ, ψ have bounded complexity.)

(iii) $S(u,v) \overset{df}{\equiv}$ "there exists a sequence $(s_0,\dots,s_u)$ s.t. $\forall i \leq u; Fle(s_i)$ and $\forall w \leq v; Fle(w) \rightarrow \exists i \leq u; s_i \simeq w$ and $\forall i \neq j \leq u; \neg\, s_i \simeq s_j$ and $s_u \simeq v$"

(iv) $x \in y \overset{df}{\equiv} \exists v;\ S(y,v) \,\&$ "if $v=\overline{\varphi(t)}$ then $\varphi(x)$"

6.3 Lemma: $PA \vdash (\forall t; t \in x \equiv t \in y) \rightarrow x=y$.

6.4 Theorem: The theory S is consistent relative to PA.

Proof: Define the interpretation φ^I of any modal set-theoretical formula φ as follows:

a) $\in$ interpret according to Definition 6.2 (iv)

b) $=$ interpret absolutely

c) $(\Box\varphi)^I \overset{df}{\equiv} [P(\overline{\varphi^I}) \,\&\, \varphi^I]$

d) I commutes with $\neg$, $\&$ and $\forall$.

Now we claim: If $S \vdash \varphi$ then $PA \vdash \varphi^I$.

By 6.1 and 6.3 this is clear for axiom schemas (1),...,(4). For (5) let $\varphi(t,\bar z)$ be any modal set-theoretical formula and $\bar a$ any parameters (i.e. numbers). Then choose b, s.t. $S(b,P(\overline{\varphi(t,\bar a)^I}))$. By definition 6.2: $\forall t; t \in b \equiv P(\overline{\varphi(t,\bar a)^I})$ and hence: $\forall t; P(\overline{\varphi(t,\bar a)^I}) \,\&\, \varphi(t,\bar a)^I \rightarrow t \in b$.

By 6.1(iv): $P(\overline{\neg\varphi(t,\bar a)^I}) \rightarrow \neg P(\overline{\varphi(t,\bar a)^I})$, so also : $P(\overline{\neg\varphi(t,\bar a)^I}) \rightarrow t \notin b$. Hence $P(\overline{\neg\varphi(t,\bar a)^I}) \,\&\, \neg\varphi(t,\bar a)^I \rightarrow t \notin b$.

Thus I-interpretation of any instance of (5) is provable in PA. The proof of the claim is completed by showing that $PA \vdash \varphi^I$ implies $PA \vdash (\Box\varphi)^I$. But this is immediate by 6.1(i).

Since $(x \neq x)^I = (x \neq x)$ this also completes the proof of theorem.

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