Comments beginning with “Note:” aren’t meant to correct anything; they help me understand faster where I struggled.

p 18 l 8 “obtains;” should be “obtains”

p 24 l -11 The definition of NTime($f$) says “accepted by a machine with time complexity” but according to the definition of time complexity, the existence of any $x \in \Sigma^n \setminus L$ means $\mathrm{time}_M(x) = \infty$ and hence $t_M(n) = \infty$, and so any machine accepting such a nontrivial language $L$ (a language satisfying $\forall n \exists x, x \in \Sigma^n \setminus L$) has $t_M(n) = \infty$ for all $n$. This is not a definition of NTime($f$) we want. An additional definition of e.g. “machine $M$ *accepts* $L$ *in time* $t(n)$” would solve this.

p 25 l -12 In the definition of $\mathrm{space}_M(w)$, the “minimum” should be “maximum”. (Otherwise all languages are in NSpace(2).)

p 34 l -4 The lower indices $P$ and $Q$ should be interchanged in the displayed expression. (See Definition 14.1.1 in the 1995 book by Krajíček.)

p 41 l -3 To prove the treelike version of Lemma 2.1.5, it seems that we need that the Claim inside the proof of the lemma ensures for each $F$-rule $\frac{E_1,\ldots,E_\ell}{E_0}$ a tree-like $F$-proof $\eta : q \to E_1, \ldots, q \to E_\ell \vdash q \to E_0$ that, moreover, uses each of $q \to E_1, \ldots, q \to E_\ell$ *at most once as a hypothesis*. This “moreover” part seems to be an additional requirement on $F$, and it is needed to keep the number of steps of the new proof linear. It would help to have e.g. modus ponens. Then we could derive the tautology $(q \to E_1) \to ((q \to E_2) \to \ldots((q \to E_\ell) \to (q \to E_0))\ldots)$ from the empty set of formulas (i.e., the leaves are 0-ary rules of $F$, also called axiom schemes) using a treelike proof, and apply modus ponens.

p 43 l 12 In the proof of the theorem, especially in Claim 2, we need (like in the remark p 41 l -3) that each auxiliary constant-size proof $\pi'$ (substituting into which we obtain statements like Claim 2) has to be tree-like and each of the formulas from the set the proof $\pi'$ is a proof from can appear in at most one leaf of $\pi'$. This is an additional requirement on $F$, and again, having modus ponens would help.

p 43 l -5 Note: The idea behind Claim 2 is (similarly to the Claim in proof of the deduction lemma) to have for each $F$-rule $\frac{E_1,\ldots,E_c}{E_0}$ a *single* tree-like proof of the tautology $(\ldots(p \wedge E_1) \wedge E_2) \wedge \ldots) \wedge E_c) \to (p \wedge E_c)$, which we can instantiate by substitution at any step $i$ of $\pi$ that uses that rule, obtaining $(\ldots(D_i \wedge C_{j_1}) \wedge C_{j_2}) \wedge \ldots) \wedge C_{j_c}) \to (D_i \wedge C_{i+1})$, which by modus ponens and rearranging the brackets gives $D_{i+1}$.

p 50 l -9 The estimate on logical depth should respect the side of the displayed equivalence $b[G(\overline{p}, q_1/H_1, \ldots, q_u/H_u] \equiv b[G](\overline{p}, q_1/b[H_1], \ldots, q_u/b[H_u])$ that has higher logical depth, which is the right hand side, not the left hand side. So the estimate should be $\ell\mathrm{dp}(b[G]) + \max_{v \le u}\{\ell\mathrm{dp}(b[H_v]\} + O(1)$.

p 56 l 16 Rather than “Lemma 2.1.4 then gives us short proofs” of the displayed formula, such proofs are similar to those in the comment about equality axioms right below Claim 1 in the proof of Lemma 2.3.3.

p 56 l -9 “$d_j$ is an axiom” should be “$D_j$ is an axiom”.

p 56 l -8 “$D_u, D_v, u, v \le j-1$” should be “$D_{u_t}, t = 1, \ldots, \ell$, with $u_t \le j-1$”.

p 56 l 7 The notation for the substitution is not very clear. What we want is $D_i(p_1/B_1^j, \ldots, p_n/B_n^j) = D_j(p_1/q_1^j, \ldots, p_n/q_n^j) = E_j$, in particular, we need to make it clear that $B_u^j$ is in the variables $\overline{q}^j$.

p 58 l 11 "$\mathbf{s}_{EF}(A) = O(\mathbf{k}_{EF}(A) + |A|)$" should be "$\mathbf{s}_{EF}(A) = O(\mathbf{k}_{EF}(A) + |A|^2)$".

p 59 l -6 "$\ell\mathrm{dp}_F(A) \leq O(\log \mathbf{s}_F(A))$" should be "$\ell\mathrm{dp}_F(A) \leq O(\log \mathbf{s}_F(A)) + \ell\mathrm{dp}(A)$".

p 66 l 21 Also the cut rule has to be adjusted.

p 68 l -8 The linear bounds on the number of steps and on size don't seem to be correct: it is not enough to suitably substitute the fixed proofs (3.2.2); their leaves — substituted initial sequents of LK — also need to be derived. Polynomial bounds can be achieved, though.

p 72 l 17 Note: As for proving (3.3.2) in case (3.3.3), we apply $\wedge$:right to the sequents $\Gamma, Def_{C_U} \longrightarrow y_U$ and $\Gamma, Def_{C_V} \longrightarrow y_V$ one obtains $\Gamma, Def_{C_U}, Def_{C_V} \longrightarrow y_U \wedge y_V$, which after a cut with the sequent $Def_{y_Z = y_U \wedge y_V}, y_U \wedge y_V \longrightarrow y_Z$, which has a constant size proof (and where $Def_{y_Z = y_U \wedge y_V}$ is $(\neg y_Z \vee y_U) \wedge (\neg y_Z \vee y_U) \wedge (y_Z \vee \neg y_U \vee \neg y_V))$, gives $\Gamma, Def_{C_Z} \longrightarrow y_Z$.

For the other sequent in (3.3.2), cut $Def_{C_U}, y_U \longrightarrow \Delta, A$ with $Def_{y_Z = y_U \wedge y_V}, y_Z \longrightarrow y_U$ to obtain $Def_{C_U}, Def_{y_Z = y_U \wedge y_V}, y_Z \longrightarrow \Delta, A$. Similarly, cutting $Def_{C_V}, y_V \longrightarrow \Delta, B$ with $Def_{y_Z = y_U \wedge y_V}, y_Z \longrightarrow y_V$ one obtains $Def_{C_V}, Def_{y_Z = y_U \wedge y_V}, y_Z \longrightarrow \Delta, B$. Applying $\wedge$:right yields $Def_{C_Z}, y_Z \longrightarrow \Delta, A \wedge B$.

p 83 l 7 Note: Fist the simulated $SF$-proof needs to be made tree-like, but that can be achieved in the same way as in Claim 2 in the proof of Theorem 2.2.1.

p 88 l -16 "$f_B(\overline{p}) := C_{Sk}(\overline{p}, C_{Sk}(\overline{p}, 1))$" instead of "$f_B(\overline{p}) := C(\overline{p}, C(\overline{p}, 1))$", see the definition of a Boolean program.

p 89 l 13 Tree-likeness claimed here for $\sigma_V$ is missing in the statement of induction hypothesis; it should be added there. Lemma 4.2.4, used in the current $\exists$:right case, can produce tree-like proofs.

p 89 l -16 It is not "$C$ replaced by $C(\overline{p}, \neg C_{Sk}(\overline{p}, 1))$", the last line of $\sigma_V$ contains $C_{Sk}(\overline{p}, q)$, which after the substitution becomes $C_{Sk}(\overline{p}, \neg C_{Sk}(\overline{p}, 1))$.

p 141 l -10 $j = (j_1, \ldots, j_k)$ instead of $i = (j_1, \ldots, j_k)$.

p 171 l 15 I suggest to get rid of item 2. completely: 1) it makes the claim "The hypothesis and the compactness theorem imply that T is consistent" unjustified, because if $T$ were inconsistent, compactness only yields $I\Delta_0 \vdash e \leq s_n \vee \exists y \leq t'(e)A(e, y)$ for some $n$ and $t'$, and it still takes some work (using $\Sigma^0_1$-completeness $(n+1)$-times to account for the finitely many cases $e \leq s_n$) to argue that this contradicts the assumption that for any term $t$, $I\Delta_0 \not\vdash \forall x \exists y \leq t(x)A(x, y)$, and 2) by removing item 2. the rest of the proof stays correct (for nothing special about the standard cut is used there).

p 174 l 8 $\beta_j \wedge \beta_{j+1}$ should be $\beta_j \wedge \neg\beta_{j+1}$. The same on line 23.

p 173 l 17 With conditions 1. - 7. as stated the Claim cannot hold, because there is no relation whatsoever between atomic $L_{PA}(I_e)$-sentences in $G$ and atomic $L_{PA}(I_e)$-sentences true in $I_e$. Everything will work if one requires of $G$ besides 1. - 7. also the following: For each atomic $L_{PA}(I_e)$-sentence $B$, $B \in G$ if and only if $I_e \models B$.

p 173 l -14 The atoms mentioned in (i) are not enough; what works is: "(i) $\alpha$ is built from atoms $r_{a,b}$ with $a, b \in I_e$."

p 173 l -11 $\Delta_0(R, I_e)$ instead of $\Delta(R, I_e)$.

p 174 l 10 Note: For this and the following two paragraphs, it is useful to employ the deduction lemma 2.1.5 (adapted to our non-standard setting in a straightforward way).

p 174 l -10 Note: It is not completely trivial to show that $G$ just defined satisfies the requirements 1. - 7. For example, to show that requirement 6. is satisfied, we need the following of the conditions on $S_i$'s: 2., 3. together with 4. and, crucially, 6. (in order to satisfy the "if" from the "if and only if" in requirement 6.).

p 186 l -12 The first axiom in 4. is redundant: it follows from the second axiom in 4. and the second axiom in 3.

p 187 l 12 It is not true that the encoding construction just described satisfies item 8. The one-element sequence encoding a number $a$ has bit-length $|a| \cdot (|a| - 1)/2$ and hence its value is $2^{\Omega(|a|^2)}$, which violates the bound $w' \leq (wa)^{10}$ in 8. (just consider $w$ to be a constant). Moreover, what is worse is that the bound $(wa)^{10}$ in 8. is not good enough for proving basic properties, like Lemma 9.3.2. about coding. A natural bound on $w'$ in 8., which would work in the proof of the lemma about coding, has the form $w' \leq c_1 w a^{c_2}$ for some $c_1, c_2 > 1$. It is trivial to verify that such a bound with $c_1 = 1000$ and $c_2 = 2$ is satisfied e.g. by the construction of encoding in $I\Delta_0$ in the book by Hájek and Pudlák, which writes the binary expansions of the numbers to be encoded one after another into one binary string, and pairs this number with another of the same bit-length and such that 1's in its binary expansion serve as pointers to the first bit of each of the encoded numbers.

p 187 l 19 has been added $\rightarrow$ have been added, and the same on the following line.

p 190 l 4 The lemma does not seem to hold because BASIC(#) is not enough to show that $\mathbf{M}_1(\mathbf{K})$ is downward closed. The simplest way to fix this is to change the definition of $\mathbf{M}_1(\mathbf{K})$ by replacing $a = |b|$ in it with $a \leq |b|$.

p 190 l 11 The displayed formula is a direct consequence of the last axiom in item 5. of BASIC(#); no other axiom is needed.

p 190 l 23 Instead of "and $B := A \setminus \{\max(A)\}$" should be "where $B$ is such that $\forall c(c \in B \equiv c + 1 \in A)$".

p 190 l 25 Instead of "$C := \{\max(A) \cdot \max(B)\}$" should be "$C := \{(\max(A) + 1) \cdot (\max(B) + 1)\}$".