Treatment of Variational Crimes in Discretizations of Three–Dimensional Navier–Stokes Equations

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Abstract. We prove optimal convergence results for a finite element discretization of three–dimensional stationary incompressible Navier–Stokes equations with nostandard boundary conditions in case of a tetrahedral triangulation of a computational domain with piecewise smooth boundary.

1 Formulation of the Problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary consisting of the sets Γ^D and Γ^N. We consider the following stationary Navier–Stokes equations:

$$-\nu\,\Delta\,\boldsymbol{u} + (\nabla\,\boldsymbol{u})\boldsymbol{u} + \nabla\,p = \boldsymbol{f}\,, \qquad \operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\boldsymbol{u} = \boldsymbol{u}_b \qquad \text{on } \Gamma^D, \tag{1.2}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0\,, \qquad (\boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n})\,(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}})\,\boldsymbol{n} = \boldsymbol{\varphi} \qquad \text{on } \Gamma^N, \tag{1.3}$$

where $\boldsymbol{u}$ denotes the velocity, p denotes the pressure, ν is a constant and positive kinematic viscosity, $\boldsymbol{f}$ is an outer volume force, $\boldsymbol{\varphi}$ is a surface force, $\boldsymbol{I}$ is the identity tensor and $\boldsymbol{n}$ is the unit outward normal vector to the boundary $\partial\Omega$ of Ω. We assume that $\partial\Omega$ is Lipschitz–continuous and that the sets Γ^D and Γ^N are open in $\partial\Omega$, disjoint and such that $\mathrm{meas}_2(\Gamma^D) > 0$, $\mathrm{meas}_2(\Gamma^N) \geq 0$ and $\partial\Omega = \overline{\Gamma^D \cup \Gamma^N}$. Further we assume that

$$\boldsymbol{f} \in L^2(\Omega)^3\,, \qquad \boldsymbol{\varphi} \in H^{\frac12}(\partial\Omega)^3\,, \qquad \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma^N,$$
$$\boldsymbol{u}_b \in H^{\frac12}(\partial\Omega)^3\,, \qquad \boldsymbol{u}_b\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma^N, \qquad \int_{\partial\Omega} \boldsymbol{u}_b\cdot\boldsymbol{n}\,\mathrm{d}\sigma = 0\,.$$

For investigating effects caused by an approximation of Ω by a polyhedral domain, we need some further assumptions on the regularity of $\partial\Omega$. We suppose that there exists an extension $\boldsymbol{m} \in W^{2,\infty}(\mathbb{R}^3)^3$ of the unit outward normal vector to Γ^N, i.e., $\boldsymbol{m}|_{\Gamma^N} = \boldsymbol{n}|_{\Gamma^N}$. The boundary of Ω is assumed to consist of disjoint relatively open C^2 surfaces $\Gamma_i \subset \partial\Omega$, $i = 1, \dots, K$, such that $\Gamma_i \subset \Gamma^D$ for $i = 1, \dots, K^D$, $\Gamma_i \subset \Gamma^N$ for $i = K^D + 1, \dots, K$, the boundaries $\partial\Gamma_i$ are piecewise C^2 and, in each point without this C^2

regularity, the angle between the respective parts of $\partial\Gamma_i$ is positive. In addition, we assume that each surface Γ_i can be extended to a C^2 surface $\widetilde{\Gamma}_i$ satisfying $\mathrm{dist}(\partial\widetilde{\Gamma}_i, \partial\Gamma_i) > 0$. Finally, we shall assume that $\overline{\mathbf{V} \cap H^l(\Omega)^3} = \mathbf{V}$, where $\mathbf{V} = \{\boldsymbol{v} \in H^1(\Omega)^3\,;\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma^D\,,\ \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma^N\,\}$ and $l \in \{2,3\}$ is an integer for which the assumption A1 in Section 3 is satisfied.

The plan of the paper is as follows. In Section 2, we introduce a weak formulation of (1.1)–(1.3) and mention some related results. In Section 3, we define a finite element discretization of (1.1)–(1.3) and give all necessary assumptions on the triangulation and the discrete spaces. In Section 4, we present some auxiliary results and, particularly, we investigate convergence of integrals over a part of the boundary of the approximating domain Ω_h. In Section 5, we introduce operator formulations of both the weak formulation and the discrete problem which are based on auxiliary problems obtained by replacing the nonlinear terms by linear functionals. Finally, in Section 6, we investigate convergence properties of the auxiliary problems and present optimal convergence results for the discretization of (1.1)–(1.3) which follow using the theory of approximation of branches of nonsingular solutions. Many of the techniques we use are new and most of them can be applied in both 2D and 3D. In addition, we only use the above realistic assumptions on Ω whereas most authors make restrictive additional assumptions such as convexity of Ω and C^2 regularity of $\partial\Omega$. Unfortunately, many details of the proofs cannot be mentioned here and we refer to [4] and [5]. Throughout the paper we use a standard notation which can be found e.g. in [2].

2 Weak Formulation

Denoting $a(\boldsymbol{u},\boldsymbol{v}) = \frac{\nu}{2}\int_\Omega (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}})\cdot(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^{\mathrm{T}})\,\mathrm{d}x$, $n(\boldsymbol{u},\widetilde{\boldsymbol{u}},\boldsymbol{v}) = \int_\Omega \boldsymbol{v}\cdot(\nabla\widetilde{\boldsymbol{u}})\boldsymbol{u}\,\mathrm{d}x$, $b(\boldsymbol{v},p) = -\int_\Omega p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x$ and $<\boldsymbol{g},\boldsymbol{v}> = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \nu\int_{\Gamma^N}\boldsymbol{\varphi}\cdot\boldsymbol{v}\,\mathrm{d}\sigma$, we introduce the following weak formulation which can be obtained in a standard way.

Definition 2.1 Let $\widetilde{\boldsymbol{u}}_b \in H^1(\Omega)^3$ be any function satisfying $\widetilde{\boldsymbol{u}}_b|_{\partial\Omega} = \boldsymbol{u}_b$. Then the functions $\boldsymbol{u}$, p are a *weak solution* of the problem (1.1)–(1.3) if

$$\boldsymbol{u} - \widetilde{\boldsymbol{u}}_b \in \mathbf{V}\,, \qquad p \in L^2_0(\Omega)\,, \tag{2.1}$$

$$a(\boldsymbol{u},\boldsymbol{v}) + n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = <\boldsymbol{g},\boldsymbol{v}> \qquad \forall\, \boldsymbol{v} \in \mathbf{V}\,, \tag{2.2}$$

$$b(\boldsymbol{u},q) = 0 \qquad \forall\, q \in L^2_0(\Omega)\,. \tag{2.3}$$

It can be shown that any classical solution $\boldsymbol{u} \in C^2(\overline{\Omega})^3$, $p \in C^1(\overline{\Omega}) \cap L^2_0(\Omega)$ of (1.1)–(1.3) is a weak solution and that any weak solution satisfying $\boldsymbol{u} \in C^2(\overline{\Omega})^3$, $p \in C^1(\overline{\Omega})$ is a classical solution. Under the assumption

$$\exists\, \varepsilon \in (0,1) \quad \forall\, \boldsymbol{v} \in \mathbf{V}\,,\ \operatorname{div} \boldsymbol{v} = 0 \quad \exists\, \widetilde{\boldsymbol{u}}_b \in H^1(\Omega)^3 : \quad \operatorname{div} \widetilde{\boldsymbol{u}}_b = 0\,,$$
$$\widetilde{\boldsymbol{u}}_b|_{\Gamma^D} = \boldsymbol{u}_b\,, \quad \widetilde{\boldsymbol{u}}_b \cdot \boldsymbol{n}|_{\Gamma^N} = 0\,, \qquad n(\boldsymbol{v}, \boldsymbol{v}, \widetilde{\boldsymbol{u}}_b) \le \varepsilon\, a(\boldsymbol{v}, \boldsymbol{v})\,,$$

one can prove using the Leray–Schauder principle that the weak formulation is solvable. Further, the problem (1.1)–(1.3) has exactly one weak solution if $\|\boldsymbol{f}\|_{0,\Omega}/\nu^2$, $\|\boldsymbol{u}_b\|_{\frac{1}{2},\partial\Omega}/\nu$ and $\|\boldsymbol{\varphi}\|_{0,\Gamma^N}/\nu$ are sufficiently small.

3 The Discrete Problem

We assume that we are given a family $\boldsymbol{\mathcal{T}} = \{\mathcal{T}_h\}$ of triangulations of Ω consisting of closed tetrahedra T having standard properties (cf. [1]). We denote $\Omega_h = \operatorname{int} \bigcup_{T \in \mathcal{T}_h} T$ and assume that, for any face $T' \subset \partial\Omega_h$ of some $T \in \mathcal{T}_h$, there exists $i \in \{1, \dots, K\}$ such that all three vertices of T' belong to $\overline{\Gamma_i}$. Further, we assume that the set of the vertices of any $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$ contains all the points, in which some $\partial\Gamma_i$ is not C^2. The family of the triangulations is assumed to be regular, i.e., there exists a number $\sigma > 0$ such that $h_T/\varrho_T \le \sigma$ for any $T \in \mathcal{T}_h$, $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$, where $h_T = \operatorname{diam}(T)$ and ϱ_T is the maximal diameter of balls contained in T. Since the set of the parameters h is bounded, we can introduce a bounded domain $\widetilde{\Omega} \subset \mathbb{R}^3$ with a Lipschitz–continuous boundary such that $\overline{\Omega} \subset \widetilde{\Omega}$ and $\overline{\Omega}_h \subset \widetilde{\Omega}$ for any $h > 0$.

For any $i \in \{1, \dots, K\}$, the set Γ_i will be approximated by a relatively open set $\Gamma_{ih} \subset \partial\Omega_h$ consisting of boundary faces $T' \subset \partial\Omega_h$ such that all vertices of T' belong to $\overline{\Gamma_i}$ and the barycentre $x^c_{T'}$ of T' satisfies $\operatorname{dist}(x^c_{T'}, \Gamma_i) \le C\, h^2_{T'}$, where $h_{T'} = \operatorname{diam}(T')$. The set Γ^D will be approximated by a relatively open set Γ^D_h consisting of Γ_{ih}, $i = 1, \dots, K^D$, and the set Γ^N will be approximated by a relatively open set Γ^N_h consisting of Γ_{ih}, $i = K^D + 1, \dots, K$.

For a given integer $n > 0$, we assume that we are given spaces

$$\mathbf{V}_h \subset \{\boldsymbol{v} \in C(\overline{\Omega}_h)^3\,;\ \boldsymbol{v} = \mathbf{0} \ \text{on}\ \Gamma^D_h\,,\ \boldsymbol{v}|_T \in P_n(T)^3 \ \ \forall\, T \in \mathcal{T}_h\}\,,$$
$$\mathrm{Q}_h \subset \{\, v \in L^2_0(\Omega_h)\,;\ v|_T \in P_n(T) \ \ \forall\, T \in \mathcal{T}_h\,\}$$

approximating the spaces $\mathbf{V}$ and $L^2_0(\Omega)$, respectively:

A1: There exist an integer $l \in \{2,3\}$ and an operator $r_h \in \mathcal{L}(\mathbf{V} \cap H^l(\Omega)^3, \mathbf{V}_h)$ such that

$$\|\boldsymbol{v} - r_h\,\boldsymbol{v}\|_{1,\Omega_h} \leq C\,h^{\frac{l}{2}}\,\|\boldsymbol{v}\|_{l,\widetilde{\Omega}} \qquad \forall\ \boldsymbol{v} \in H^l(\widetilde{\Omega})^3\,,\ \boldsymbol{v}|_\Omega \in \mathbf{V}\,.$$

If $l = 2$, then

$$\|\boldsymbol{v}_h \cdot \boldsymbol{n}_h\|_{0,\frac{4}{3},\Gamma_h^N} \leq C\,h\,\|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\ \boldsymbol{v}_h \in \mathbf{V}_h\,, \tag{3.1}$$

where $\boldsymbol{n}_h$ is the unit outward normal vector to $\partial\Omega_h$. If $l = 3$, then

$$\|\boldsymbol{v}_h \cdot \bar{\boldsymbol{n}}_h\|_{0,1,\Gamma_h^N} \leq C\,h^{\frac{3}{2}}\,\|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\ \boldsymbol{v}_h \in \mathbf{V}_h\,, \tag{3.2}$$

where $\bar{\boldsymbol{n}}_h \in L^2(\partial\Omega_h)^3$ is the piecewise linear interpolate of $\boldsymbol{n}$. Any $\boldsymbol{v}_h \in \mathbf{V}_h$ satisfies $\boldsymbol{v}_h \cdot \boldsymbol{n}_h^* = 0$ in any vertex of $\mathcal{T}_h$ lying in Γ_h^N, where $\boldsymbol{n}_h^*$ satisfies for some fixed $\alpha \in (0,1]$

$$|\boldsymbol{n}_h^*(x) - \boldsymbol{n}(x)| \leq C\,h_T^{1+\alpha} \quad \text{in any vertex } x \in \Gamma_h^N,$$

with $T \in \mathcal{T}_h$ being any element containing the vertex x.

A2: There exists an operator $s_h \in \mathcal{L}(L_0^2(\Omega) \cap H^{l-1}(\Omega), \mathrm{Q}_h)$ such that

$$\|q - s_h\,q\|_{0,\Omega_h} \leq C\,h^{\frac{l}{2}}\,\|q\|_{l-1,\widetilde{\Omega}} \qquad \forall\ q \in H^{l-1}(\widetilde{\Omega})\,,\ q|_\Omega \in L_0^2(\Omega)\,.$$

A3: There exists a constant $\beta > 0$ independent of h such that

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h, \boldsymbol{v}_h \neq \mathbf{0}} \frac{\displaystyle\int_{\Omega_h} q_h\,\mathrm{div}\,\boldsymbol{v}_h\,\mathrm{d}x}{\|\boldsymbol{v}_h\|_{1,\Omega_h}} \geq \beta\,\|q_h\|_{0,\Omega_h} \qquad \forall\ q_h \in \mathrm{Q}_h\,.$$

Examples of finite element spaces satisfying the above assumptions with $l = 2$ were constructed in [4], Section V.7. Finite element spaces satisfying A1–A3 with $l = 3$ can be found in [5].

We replace the forms used in the weak formulation by the forms a_h, n_h, b_h and $\boldsymbol{g}_h$ obtained by changing Ω to Ω_h and Γ^N to Γ_h^N and by extending $\boldsymbol{f}$ and $\boldsymbol{\varphi}$ onto $\widetilde{\Omega}$, i.e., $\boldsymbol{f} \in L^2(\widetilde{\Omega})^3$ and $\boldsymbol{\varphi} \in H^1(\widetilde{\Omega})^3$. Further, we introduce a function $\widetilde{\boldsymbol{u}}_{bh} \in H^1(\Omega_h)^3$ such that $\lim_{h\to 0} \|\mathrm{P}\,\widetilde{\boldsymbol{u}}_b - \widetilde{\boldsymbol{u}}_{bh}\|_{1,\Omega_h} = 0$ for some extension operator $\mathrm{P} : H^1(\Omega)^3 \to H^1(\widetilde{\Omega})^3$.

Definition 3.1 The functions $\boldsymbol{u}_h$, p_h are a *discrete solution* of (1.1)–(1.3) if

$$\begin{aligned} \boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{bh} \in \mathbf{V}_h\,, \qquad p_h \in \mathrm{Q}_h\,, & & (3.3)\\ a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + n_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h,p_h) &= \;<\boldsymbol{g}_h,\boldsymbol{v}_h>\;\; \forall\, \boldsymbol{v}_h \in \mathbf{V}_h\,, & (3.4)\\ b_h(\boldsymbol{u}_h,q_h) &= 0 \qquad\qquad \forall\, q_h \in \mathrm{Q}_h\,. & (3.5)\end{aligned}$$

It is important that the bilinear from a_h is uniformly elliptic on $\mathbf{V}_h$: there exists a constant $C>0$ independent of h such that $C\,\|\boldsymbol{v}_h\|^2_{1,\Omega_h} \le a_h(\boldsymbol{v}_h,\boldsymbol{v}_h)$ for any $\boldsymbol{v}_h \in \mathbf{V}_h$. It is the so–called discrete Korn inequality (cf. [3] or [5]).

4 Auxiliary Results

In view of the above assumptions, we have

$$\mathrm{meas}_3(\Omega\setminus\Omega_h \cup \Omega_h\setminus\Omega) \le C\,h^2 \tag{4.1}$$

and there exist operators $\mathrm{P} \in \mathcal{L}(H^1(\Omega), H^1_0(\widetilde{\Omega}))$ and $\mathrm{P}_h \in \mathcal{L}(H^1(\Omega_h), H^1_0(\widetilde{\Omega}))$ satisfying for any $v \in H^1(\Omega_h)$, with C independent of h,

$$\|\mathrm{P}_h\, v\|_{1,\widetilde{\Omega}} \le C\,\|v\|_{1,\Omega_h}\,, \qquad \lim_{h\to 0} \|\mathrm{P}\, v - \mathrm{P}_h\,(\mathrm{P}\, v)|_{\Omega_h}\|_{1,\widetilde{\Omega}} = 0\,. \tag{4.2}$$

Further, it can be shown that $\|\mathrm{P}_h\,\boldsymbol{v}_h\|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} \le C\,h^{\frac12}\,\|\boldsymbol{v}_h\|_{1,\Omega_h}$ for any $\boldsymbol{v}_h \in \mathbf{V}_h$ and that $\|q_h\|_{0,\Omega_h\setminus\Omega} \le C\,h^{\frac12}\,\|q_h\|_{0,\Omega_h}$ for any $q_h \in \mathrm{Q}_h$.

The sets Γ_i, $i=1,\dots,K$, can be decomposed into disjoint sets $\Gamma_{T'} \subset \Gamma_i$ assigned to the faces $T' \subset \Gamma_{ih}$. We denote by $K(T')$ the circle lying in the plane determined by T' and having a radius $h_{T'}$ and the centre in the barycentre of T'. For small h, each $\Gamma_{T'}$ can be represented by the graph of a function $\varphi_{T'} \in C^2(K(T'))$ satisfying $|\varphi_{T'}(\overline{x})| \le C\,h^2_{T'}$ and $|\nabla\varphi_{T'}(\overline{x})| \le C\,h_{T'}$ for any $\overline{x} \in K(T')$. Further, we can define disjoint sets $G_{T'}$ related to the "space between T' and $\Gamma_{T'}$" and making up the set $\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega$. We denote by $D_{T'}$ the projection of $\Gamma_{T'}$ into $K(T')$ and by $B_{T'}$ the intersection of T' with the projection of $\Gamma_{T'} \cap \overline{G_{T'}}$ into $K(T')$. Then, for estimating the difference $\int_{\Gamma_1}\theta\,v\,\mathrm{d}\sigma - \int_{\Gamma_{1h}}\theta\,v\,\mathrm{d}\sigma$ with $\theta,\ v \in H^1(\widetilde{\Omega})$, it suffices to estimate the terms $\int_{B_{T'}}\theta(\overline{x},\varphi_{T'}(\overline{x}))\,v(\overline{x},\varphi_{T'}(\overline{x}))\,\rho_{T'}(\overline{x}) - \theta(\overline{x},0)\,v(\overline{x},0)\,\mathrm{d}\overline{x}$, $\int_{D_{T'}\setminus B_{T'}}\theta(\overline{x},\varphi_{T'}(\overline{x}))\,v(\overline{x},\varphi_{T'}(\overline{x}))\,\rho_{T'}(\overline{x})\,\mathrm{d}\overline{x}$ and $\int_{T'\setminus B_{T'}}\theta(\overline{x},0)\,v(\overline{x},0)\,\mathrm{d}\overline{x}$, where $\rho_{T'}(\overline{x}) = (1+|\nabla\varphi_{T'}(\overline{x})|^2)^{\frac12}$. Since $\int_{B_{T'}}|v(\overline{x},\varphi_{T'}(\overline{x})) - v(\overline{x},0)|^2\,\mathrm{d}\overline{x} \le C\,h^2_{T'}\,|v|^2_{1,G_{T'}}\ \forall\, v \in H^1(\widetilde{\Omega})$, $|1-\rho_{T'}(\overline{x})| \le C\,h^2_{T'}$ and $\sum_{T'\subset\partial\Omega_h}\mathrm{meas}_2((D_{T'}\setminus$

$B_{T'}) \cup (T' \setminus B_{T'})) \leq C\, h^2$, we infer that

$$\left|\int_{\Gamma_1} \theta\, v \,\mathrm{d}\sigma - \int_{\Gamma_{1h}} \theta\, v \,\mathrm{d}\sigma\right| \leq C\, h\, \|\theta\|_{1,\widetilde{\Omega}}\, \|v\|_{1,\widetilde{\Omega}} \qquad \forall\, \theta, v \in H^1(\widetilde{\Omega})\,. \tag{4.3}$$

Using the above techniques, we can also prove that

$$\left|\int_{\Gamma_1} \theta\, v \,\mathrm{d}\sigma - \int_{\Gamma_{1h}} \theta\, v \,\mathrm{d}\sigma\right| \leq C\, (h^{\frac{3}{2}}\, \|v\|_{1,\widetilde{\Omega}} + h\, |v|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega})\, \|\theta\|_{0,\infty,\widetilde{\Omega}} + \\ + C\, h\, |\theta|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega}\, \|v\|_{1,\widetilde{\Omega}} \qquad \forall\, \theta \in L^\infty(\widetilde{\Omega}) \cap H^1(\widetilde{\Omega}),\ v \in H^1(\widetilde{\Omega})\,.$$

For any $v \in H^1(\widetilde{\Omega})$ with $v|_{\Gamma_{1h}} = 0$, we obtain $\|v\|_{0,\frac{4}{3},\Gamma_1} \leq C\, h\, \|v\|_{1,\widetilde{\Omega}}$ and $\|v\|_{0,1,\Gamma_1} \leq C\, h^{\frac{3}{2}}\, \|v\|_{1,\widetilde{\Omega}} + C\, h\, |v|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega}$. Finally, for $\boldsymbol{v} \in H^1(\widetilde{\Omega})^3$, we have $|\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{0,\frac{4}{3},\Gamma_1} - \|\boldsymbol{v}\cdot\boldsymbol{n}_h\|_{0,\frac{4}{3},\Gamma_{1h}}| \leq C\, h\, \|\boldsymbol{v}\|_{1,\widetilde{\Omega}}$ and, if the extension $\widetilde{\Gamma}_1$ of Γ_1 is a C^3 surface, then $|\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{0,1,\Gamma_1} - \|\boldsymbol{v}\cdot\bar{\boldsymbol{n}}_h\|_{0,1,\Gamma_{1h}}| \leq C\, h^{\frac{3}{2}}\, \|\boldsymbol{v}\|_{1,\widetilde{\Omega}} + C\, h\, |\boldsymbol{v}|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega}$.

5 Operator Formulations

We denote $\widehat{\mathrm{X}} = H^1(\Omega)^3 \times L^2(\Omega)$, $\widehat{\mathrm{X}}_h = H^1(\Omega_h)^3 \times L^2(\Omega_h)$, $\mathrm{X} = H^1_0(\widetilde{\Omega})^3 \times L^2(\widetilde{\Omega})$ and $\mathrm{Y} = H^{-1}(\widetilde{\Omega})^3$ and define an operator $\widetilde{\mathrm{P}} : \widehat{\mathrm{X}} \to \mathrm{X}$ such that, for $\widehat{\mathrm{U}} = (\boldsymbol{v}, q) \in \widehat{\mathrm{X}}$, it holds $\widetilde{\mathrm{P}}\, \widehat{\mathrm{U}} = (\mathrm{P}\, \boldsymbol{v}, q)$, where P is the operator from Section 4 and q is considered as extended by zero outside Ω. Analogously, we define an operator $\widetilde{\mathrm{P}}_h : \widehat{\mathrm{X}}_h \to \mathrm{X}$ by $\widetilde{\mathrm{P}}_h\, (\boldsymbol{v}_h, q_h) = (\mathrm{P}_h\, \boldsymbol{v}_h, q_h)$. Further, we define an operator $\mathrm{R}_h : \mathrm{X} \to \widehat{\mathrm{X}}_h$ as restriction from $\widetilde{\Omega}$ onto Ω_h. Finally, we introduce arbitrary extension operators $\mathrm{P}_k : H^k(\Omega) \to H^k(\widetilde{\Omega})$, $k \geq 0$, and define operators $\overline{\mathrm{P}}_k : H^k(\Omega)^3 \times H^{k-1}(\Omega) \to H^k(\widetilde{\Omega})^3 \times H^{k-1}(\widetilde{\Omega})$, $k \geq 1$, by $\overline{\mathrm{P}}_k(\boldsymbol{v}, q) = (\mathrm{P}_k\, \boldsymbol{v}, \mathrm{P}_{k-1}\, q)$ for any $(\boldsymbol{v}, q) \in H^k(\Omega)^3 \times H^{k-1}(\Omega)$.

To define an operator formulation of (3.3)–(3.5), we first introduce the following auxiliary problem: Given $\varPhi \in \mathrm{Y}$, find $\boldsymbol{u}_h$, p_h such that

$$\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{bh} \in \mathbf{V}_h\,, \qquad p_h \in \mathrm{Q}_h\,, \tag{5.1}$$

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = \langle \varPhi, \mathrm{P}_h\, \boldsymbol{v}_h \rangle \qquad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h\,, \tag{5.2}$$

$$b_h(\boldsymbol{u}_h, q_h) = 0 \qquad \forall\, q_h \in \mathrm{Q}_h\,. \tag{5.3}$$

This problem is uniquely solvable and we can therefore define an operator $\mathrm{T}_h : \mathrm{Y} \to \widehat{\mathrm{X}}_h$ such that $\widehat{\mathrm{U}}_h = \mathrm{T}_h\, \varPhi$ is the solution of (5.1)–(5.3). Defining a nonlinear operator $\mathrm{G}_h : \mathrm{X} \to \mathrm{Y}$ by

$$\langle \mathrm{G}_h(\mathrm{U}), \boldsymbol{v} \rangle = \langle \boldsymbol{g}_h, \boldsymbol{v} \rangle - n_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})\,, \quad \mathrm{U} = (\boldsymbol{u}, p) \in \mathrm{X},\ \boldsymbol{v} \in H^1_0(\widetilde{\Omega})^3\,,$$

we can introduce an operator $\mathrm{F}_h : \mathrm{X} \to \mathrm{X}$ defined by

$$\mathrm{F}_h(\mathrm{U}) = \widetilde{\mathrm{P}}_h \, \mathrm{T}_h \, \mathrm{G}_h(\mathrm{U}) - \mathrm{U} \,, \qquad \mathrm{U} \in \mathrm{X} \,.$$

Then any solution $\widehat{\mathrm{U}}_h$ of (3.3)–(3.5) satisfies $\mathrm{F}_h(\widetilde{\mathrm{P}}_h \, \widehat{\mathrm{U}}_h) = 0$ and, for any $\mathrm{U} \in \mathrm{X}$ satisfying $\mathrm{F}_h(\mathrm{U}) = 0$, the restriction $\mathrm{R}_h \, \mathrm{U}$ is a solution of (3.3)–(3.5).

Analogously, we can define an operator formulation of (2.1)–(2.3). For this, we define an operator $\mathrm{F} : \mathrm{X} \to \mathrm{X}$ by

$$\mathrm{F}(\mathrm{U}) = \widetilde{\mathrm{P}} \, \mathrm{T} \, \mathrm{G}(\mathrm{U}) - \mathrm{U} \,, \qquad \mathrm{U} \in \mathrm{X} \,,$$

where $\mathrm{T} : \mathrm{Y} \to \widehat{\mathrm{X}}$ corresponds to an auxiliary problem of (2.1)–(2.3) and

$$<\mathrm{G}(\mathrm{U}), \boldsymbol{v}> = <\boldsymbol{g}, \boldsymbol{v}> - n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) \,, \quad \mathrm{U} = (\boldsymbol{u}, p) \in \mathrm{X}, \;\; \boldsymbol{v} \in H_0^1(\widetilde{\Omega})^3 \,.$$

Theorem 5.1 *The operators* F, $\mathrm{F}_h : \mathrm{X} \to \mathrm{X}$ *are* C^1 *mappings and we have*

$$\|\mathrm{DF}_h(\mathrm{U}) - \mathrm{DF}_h(\widehat{\mathrm{U}})\|_{\mathcal{L}(\mathrm{X},\mathrm{X})} \le C \, \|\mathrm{U} - \widehat{\mathrm{U}}\|_{\mathrm{X}} \qquad \forall \; \mathrm{U}, \widehat{\mathrm{U}} \in \mathrm{X} \,,$$

where the constant C *is independent of* h*,* U *and* $\widehat{\mathrm{U}}$*.*

Proof. The proof of the C^1 continuity is easy and the Lipschitz–continuity of DF_h follows from the Lipschitz–continuity of DG_h. □

6 Convergence Results

Theorem 6.1 *The operators* T *and* T_h *satisfy*

$$\lim_{h \to 0} \|\mathrm{R}_h \, \widetilde{\mathrm{P}} \, \mathrm{T} \, \mathrm{G}(\widetilde{\mathrm{U}}) - \mathrm{T}_h \, \mathrm{G}(\widetilde{\mathrm{U}})\|_{\widehat{\mathrm{X}}_h} = 0 \qquad \forall \; \widetilde{\mathrm{U}} \in \mathrm{X} \,. \tag{6.1}$$

Proof. Choose an arbitrary $\widetilde{\mathrm{U}} = (\widetilde{\boldsymbol{u}}, \widetilde{p}) \in \mathrm{X}$ and denote $(\boldsymbol{u}, p) = \widetilde{\mathrm{P}} \, \mathrm{T} \, \mathrm{G}(\widetilde{\mathrm{U}})$ and $(\boldsymbol{u}_h, p_h) = \mathrm{T}_h \, \mathrm{G}(\widetilde{\mathrm{U}})$. For any $\boldsymbol{v}_h \in \mathbf{V}_h$, there exists functions $\bar{\boldsymbol{v}}_h^0$, $\bar{\boldsymbol{v}}_h^b$, $\bar{\boldsymbol{v}}_h \in H_0^1(\widetilde{\Omega})^3$ and $u_h^b \in H_0^1(\widetilde{\Omega})$ satisfying

$$\begin{aligned}
&\mathrm{P}_h \, \boldsymbol{v}_h = \bar{\boldsymbol{v}}_h^0 + \bar{\boldsymbol{v}}_h + \bar{\boldsymbol{v}}_h^b + u_h^b \, \boldsymbol{m} \,, \qquad \bar{\boldsymbol{v}}_h^0|_\Omega \in \mathbf{V} \,, \\
&\|\bar{\boldsymbol{v}}_h^0\|_{1,\widetilde{\Omega}} + \|\bar{\boldsymbol{v}}_h\|_{1,\widetilde{\Omega}} + \|\bar{\boldsymbol{v}}_h^b\|_{1,\widetilde{\Omega}} + \|u_h^b\|_{1,\widetilde{\Omega}} \le C \, \|\boldsymbol{v}_h\|_{1,\Omega_h} \,, \\
&\|\bar{\boldsymbol{v}}_h\|_{1,\Omega_h} \le C \, h^\alpha \, \|\boldsymbol{v}_h\|_{1,\Omega_h} \,, \qquad \|\bar{\boldsymbol{v}}_h\|_{0,4,\Omega_h} \le C \, h^{1+\alpha} \, \|\boldsymbol{v}_h\|_{1,\Omega_h} \,, \\
&\operatorname{supp} \bar{\boldsymbol{v}}_h^b \subset U_h(\Gamma^D) \,, \qquad \operatorname{supp} u_h^b \subset U_h(\partial\Omega) \,,
\end{aligned}$$

where $U_h(\Gamma) = \{x \in \mathbb{R}^3\,;\ \mathrm{dist}(x,\Gamma) \le Ch\}$. Then

$$|a(\boldsymbol{u}, \bar{\boldsymbol{v}}_h^0) - a_h(\boldsymbol{u}, \boldsymbol{v}_h)| \le C\,(|\boldsymbol{u}|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega\cup U_h(\partial\Omega)} + h^\alpha\,|\boldsymbol{u}|_{1,\Omega})\,\|\boldsymbol{v}_h\|_{1,\Omega_h}\,,$$
$$|b(\bar{\boldsymbol{v}}_h^0, p) - b_h(\boldsymbol{v}_h, p)| \le C\,(\|p\|_{0,\Omega\setminus\Omega_h\cup U_h(\partial\Omega)} + h^\alpha\,\|p\|_{0,\Omega})\,\|\boldsymbol{v}_h\|_{1,\Omega_h}\,,$$
$$|n(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}, \bar{\boldsymbol{v}}_h^0 - \mathrm{P}_h\,\boldsymbol{v}_h)| \le C\,(|\widetilde{\boldsymbol{u}}|_{1,\Omega\setminus\Omega_h\cup U_h(\partial\Omega)}\,\|\widetilde{\boldsymbol{u}}\|_{1,\Omega} + h^{1+\alpha}\,\|\widetilde{\boldsymbol{u}}\|_{1,\Omega}^2)\,\|\boldsymbol{v}_h\|_{1,\Omega_h}\,,$$
$$\left|\int_\Omega \boldsymbol{f}\cdot(\bar{\boldsymbol{v}}_h^0 - \mathrm{P}_h\,\boldsymbol{v}_h)\,\mathrm{d}x\right| \le C\,(\|\boldsymbol{f}\|_{0,\Omega\setminus\Omega_h\cup U_h(\partial\Omega)} + h^{1+\alpha}\,\|\boldsymbol{f}\|_{0,\Omega})\,\|\boldsymbol{v}_h\|_{1,\Omega_h}\,,$$
$$\left|\int_{\Gamma^N} \boldsymbol{\varphi}\cdot(\bar{\boldsymbol{v}}_h^0 - \mathrm{P}_h\,\boldsymbol{v}_h)\,\mathrm{d}x\right| \le C\,(\|\boldsymbol{\varphi}\|_{0,\Gamma^N\cap U_h(\Gamma^D)} + h^\alpha\,\|\boldsymbol{\varphi}\|_{1,\widetilde{\Omega}})\,\|\boldsymbol{v}_h\|_{1,\Omega_h}\,.$$

Since $\bar{\boldsymbol{v}}_h^0|_\Omega$ can be used as a test function in the auxiliary problem represented by the operator T, we obtain

$$|a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h - p)| \le K_h\,\|\boldsymbol{v}_h\|_{1,\Omega_h}\,,$$

where $K_h \to 0$ for $h \to 0$. Now the proof can be completed in a similar way as for $\Omega_h = \Omega$ (cf. [2]). □

Corollary 6.1 *The operators* F *and* F_h *satisfy for any* $\mathrm{U} \in \mathrm{X}$

$$\lim_{h\to 0} \|\mathrm{F}(\mathrm{U}) - \mathrm{F}_h(\mathrm{U})\|_{\mathrm{X}} = 0\,, \tag{6.2}$$
$$\lim_{h\to 0} \|\mathrm{DF}(\mathrm{U}) - \mathrm{DF}_h(\mathrm{U})\|_{\mathcal{L}(\mathrm{X},\mathrm{X})} = 0\,. \tag{6.3}$$

Proof. The first relation follows from $\lim_{h\to 0} \|\mathrm{G}(\mathrm{U}) - \mathrm{G}_h(\mathrm{U})\|_{\mathrm{Y}} = 0$, which is a consequence of (4.3) and (4.1), and from (4.2) and (6.1). The relation (6.3) follows using $\lim_{h\to 0} \|\mathrm{DG}(\mathrm{U}) - \mathrm{DG}_h(\mathrm{U})\|_{\mathcal{L}(\mathrm{X},\mathrm{Y})} = 0$, (6.1) and the compactness of $\mathrm{DG}(\mathrm{U})$. □

Theorem 6.2 *Let $k \in \{2,3\}$ be a given integer and let A1 and A2 hold with $l = k$. If $k = 2$, let $\boldsymbol{f} \in L^3(\widetilde{\Omega})^3$. If $k = 3$, let $\boldsymbol{f} \in L^{12}(\widetilde{\Omega})^3$, $\boldsymbol{\varphi} \in W^{1,\infty}(\widetilde{\Omega})^3$ and let the extensions $\widetilde{\Gamma}_i$ of Γ_i, $i = K^D + 1, \ldots, K$, be C^3 surfaces. Let $\widetilde{\mathrm{U}} = (\widetilde{\boldsymbol{u}}, \widetilde{p}) \in \mathrm{X}$ be given and let $\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}_b \in H^k(\widetilde{\Omega})^3$, $\mathrm{T}\,\mathrm{G}(\widetilde{\mathrm{U}}) \in H^k(\Omega)^3 \times H^{k-1}(\Omega)$ and $\|\mathrm{P}_k\,\widetilde{\boldsymbol{u}}_b - \widetilde{\boldsymbol{u}}_{bh}\|_{1,\Omega_h} \le C\,h^{\frac{k}{2}}$. Then*

$$\|\mathrm{R}_h\,\overline{\mathrm{P}}_k\,\mathrm{T}\,\mathrm{G}(\widetilde{\mathrm{U}}) - \mathrm{T}_h\,\mathrm{G}_h(\widetilde{\mathrm{U}})\|_{\widehat{\mathrm{X}}_h} \le C\,h^{\frac{k}{2}}\,. \tag{6.4}$$

Proof. Denote $(\boldsymbol{u}, p) = \overline{\mathrm{P}}_k \,\mathrm{T}\,\mathrm{G}(\widetilde{\mathrm{U}})$ and $(\boldsymbol{u}_h, p_h) = \mathrm{T}_h\,\mathrm{G}_h(\widetilde{\mathrm{U}})$. The functions $\boldsymbol{u}$, p satisfy (1.1)–(1.3) with $(\nabla\,\boldsymbol{u})\boldsymbol{u}$ replaced by $(\nabla\,\widetilde{\boldsymbol{u}})\widetilde{\boldsymbol{u}}$ and, in the same way as used for deriving the weak formulation, we can obtain an integral form of the first equation valid for any test function $\boldsymbol{v} \in H^1(\widetilde{\Omega})^3$. Subtracting this equation with $\boldsymbol{v} = \mathrm{P}_h\,\boldsymbol{v}_h$ from (5.2) with $\Phi = \mathrm{G}_h(\widetilde{\mathrm{U}})$, we get

$$
\begin{aligned}
& a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - a(\boldsymbol{u}, \mathrm{P}_h\,\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) - b(\mathrm{P}_h\,\boldsymbol{v}_h, p) = \\
&\quad = <\mathrm{G}_h(\widetilde{\mathrm{U}}) - \mathrm{G}(\widetilde{\mathrm{U}}), \mathrm{P}_h\,\boldsymbol{v}_h> - \nu \int_{\Gamma^D} (\mathrm{P}_h\,\boldsymbol{v}_h)\cdot(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}})\,\boldsymbol{n}\,\mathrm{d}\sigma - \\
&\qquad - \nu \int_{\Gamma^N} (\boldsymbol{n}\cdot\mathrm{P}_h\,\boldsymbol{v}_h)\boldsymbol{n}\cdot(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}})\,\boldsymbol{n}\,\mathrm{d}\sigma + \int_{\partial\Omega} p\,\boldsymbol{n}\cdot(\mathrm{P}_h\,\boldsymbol{v}_h)\,\mathrm{d}\sigma\,.
\end{aligned}
$$

In view of (3.1), (3.2) and the results of Section 4, the terms on the right–hand side are bounded by $C\,h^{\frac{k}{2}}\,\|\boldsymbol{v}_h\|_{1,\Omega_h}$. Thus, we obtain

$$
|a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h - p)| \le C\,h^{\frac{k}{2}}\,\|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\,\boldsymbol{v}_h \in \mathbf{V}_h
$$

and the proof can be completed in a similar way as for $\Omega_h = \Omega$ (cf. [2]). $\square$

Theorem 6.3 *Let the assumptions given in Sections 1 and 3 be satisfied. Let* $\widehat{\mathrm{U}} = (\boldsymbol{u}, p) \in \widehat{\mathrm{X}}$ *be a nonsingular weak solution of the problem (1.1)–(1.3). Then there exist constants* $h_0 > 0$ *and* $R > 0$ *such that, for* $h \in (0, h_0)$*, the discrete problem (3.3)–(3.5) has a solution which is unique in the ball*

$$
\widehat{\mathcal{B}}_h(\widehat{\mathrm{U}}, R) = \{\mathrm{V} \in \widehat{\mathrm{X}}_h\,;\ \|\mathrm{R}_h\,\widetilde{\mathrm{P}}\,\widehat{\mathrm{U}} - \mathrm{V}\|_{\widehat{\mathrm{X}}_h} \le R\}\,.
$$

Moreover, this unique solution $\widehat{\mathrm{U}}_h = (\boldsymbol{u}_h, p_h) \in \widehat{\mathcal{B}}_h(\widehat{\mathrm{U}}, R)$ *is nonsingular and satisfies*

$$
\lim_{h\to 0}\{\,\|\mathrm{P}_1\,\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega_h} + \|\mathrm{P}_0\,p - p_h\|_{0,\Omega_h}\} = 0\,. \tag{6.5}
$$

If, in addition, $\boldsymbol{u} \in H^k(\Omega)^3$, $p \in H^{k-1}(\Omega)$ *for some* $k \in \{2, 3\}$*, the assumptions A1 and A2 hold with* $l = k$ *and* $\widetilde{\boldsymbol{u}}_b$, $\boldsymbol{f}$, $\boldsymbol{\varphi}$ *and* Ω *are like in Theorem 6.2, then*

$$
\|\mathrm{P}_k\,\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega_h} + \|\mathrm{P}_{k-1}\,p - p_h\|_{0,\Omega_h} \le C\,h^{\frac{k}{2}}\,. \tag{6.6}
$$

Proof. The properties of the operators F and F_h given in Theorem 5.1 and Corollary 6.1 make it possible to apply results of the theory of approximation of branches of nonsingular solutions stated in [2], pp. 301–302, immediately proving the locally unique solvability and the convergence (6.5). The estimate (6.6) follows using (6.4). $\square$

References

[1] Ciarlet, P. G.: The Finite Element Method for Elliptic Problems, North–Holland, Amsterdam, 1978

[2] Girault, V., Raviart, P.–A.: Finite Element Methods for Navier–Stokes Equations, Springer–Verlag, Berlin, 1986

[3] Knobloch, P.: Discrete Friedrichs' and Korn's inequalities in two and three dimensions, East–West J. Numer. Math. 4 (1996), 35–51

[4] Knobloch, P.: Solvability and Finite Element Discretization of a Mathematical Model Related to Czochralski Crystal Growth, PhD Thesis, Preprint MBI–96–5, Otto–von–Guericke–Universität, Magdeburg, 1996

[5] Knobloch, P.: Variational crimes in a finite element discretization of 3D Stokes equations with nonstandard boundary conditions, in preparation

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