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INFLUENCE OF MESH–DEPENDENT KORN'S INEQUALITY ON THE CONVERGENCE OF NONCONFORMING FINITE ELEMENT SCHEMES *

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Abstract. We discuss the validity of a discrete analogue of Korn's first inequality for two–dimensional nonconforming finite elements. For the Crouzeix–Raviart element and the rotated bilinear element, the constant in this inequality is mesh–dependent and we investigate its influence on the convergence properties of finite element discretizations of the Stokes equations involving deformation tensor formulation of the Laplace operator. Whereas for the rotated bilinear element convergence results can be proved, no convergence of the standard discret ization can be expected if the Crouzeix–Raviart element is applied.

Key words. nonconforming finite elements, Korn's inequality, Stokes equations, error estimates

AMS subject classifications. 65N12, 65N30, 65N15

1. Introduction. The aim of this paper is to discuss the validity of a discrete analogue of Korn's first inequality for nonconforming finite elements and, in cases when the constant in this inequality is mesh–dependent, to investigate its influence on the convergence properties of finite element discretizations.

To fix the ideas, let us introduce a simple model problem. We denote by Ω a bounded domain in $\mathbb{R}^2$ having a polygonal boundary $\partial\Omega$ and by Γ^D a measurable subset of $\partial\Omega$ with a positive one–dimensional measure. In the domain Ω, we consider the Stokes equations

$$-\Delta\,\boldsymbol{u} + \nabla\, p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma^D, \tag{1.3}$$

$$\boldsymbol{t}\cdot\sigma(\boldsymbol{u},p)\,\boldsymbol{n} = 0 \qquad \text{on } \Gamma^N \equiv \partial\Omega\setminus\Gamma^D, \tag{1.4}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma^N. \tag{1.5}$$

Here $\boldsymbol{u}$ and p are the unknown velocity and pressure, respectively, $\boldsymbol{f} \in L^2(\Omega)^2$ is an outer volume force, $\boldsymbol{n}$ is the outer unit normal vector to $\partial\Omega$, $\boldsymbol{t}$ is a tangent vector to $\partial\Omega$ and $\sigma(\boldsymbol{u}, p)$ is the stress tensor defined by

$$\sigma(\boldsymbol{u},p) = -p\,I + 2\,D(\boldsymbol{u})\,, \qquad D(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\right)$$

with I being the identity tensor. The homogenous Dirichlet boundary condition (1.3) is considered for simplicity only; all the below results remain valid for the non–

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homegenous case as well. Boundary conditions of the form (1.3)–(1.5) appear when a part of the boundary of Ω represents a free surface.

Denoting

$$V = \{\boldsymbol{v} \in H^1(\Omega)^2;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma^D\}\,, \qquad W = \{\boldsymbol{v} \in V;\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma^N\}\,,$$

the standard weak formulation of (1.1)–(1.5) reads: Find $\boldsymbol{u} \in W$ and $p \in L^2_0(\Omega)$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall\ \boldsymbol{v} \in W,\ q \in L^2_0(\Omega)\,.$$

Here, $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\Omega)^2$ and

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2\,(D(\boldsymbol{u}), D(\boldsymbol{v}))\,, \qquad b(\boldsymbol{v}, p) = -(p, \operatorname{div} \boldsymbol{v})\,.$$

Since the Korn first inequality (cf. e.g. [17]) states that there exists a positive constant C such that

$$|\boldsymbol{v}|_{1,\Omega} \le C\,\|D(\boldsymbol{v})\|_{0,\Omega} \qquad \forall\ \boldsymbol{v} \in V\,, \tag{1.6}$$

the bilinear form a is W–elliptic. Moreover, the spaces W and $L^2_0(\Omega)$ satisfy an inf–sup condition (see [10]) and hence it can be proved that there always exists a unique weak solution of (1.1)–(1.5) (cf. e.g. [3] or [10]). In what follows, we shall still assume that this solution possesses at least the regularity $\boldsymbol{u} \in H^2(\Omega)^2$, $p \in H^1(\Omega)$, which implies that the functions $\boldsymbol{u}$, p satisfy the equations (1.1)–(1.5) almost everywhere.

For approximating the velocity in incompressible flow problems like (1.1)–(1.5), nonconforming finite elements are often used. One advantage of nonconforming finite elements in comparison to conforming ones is that they usually satisfy inf–sup conditions with more convenient pressure spaces and that discretely divergence–free bases can often be more easily constructed for this type of finite elements. Another reason for the application of nonconforming finite elements may be that they are more suitable for a parallel implementation since their degrees of freedom are associated with edges (or with interior points of the elements of the triangulation), which leads to a cheap local communication between processors. In addition, nonconforming finite elements often show nice stability properties and lead to very efficient finite element solvers. We refer to [11, 19] for more details on the properties of nonconforming finite elements applied to incompressible flow problems.

Since nonconforming finite element spaces approximating V (or W) are not contained in V, the validity of a discrete analogue of the Korn inequality (1.6) is usually not obvious. Moreover, it is often not clear how the constant in the discrete Korn inequality depends on the discretization parameter, which is important for deriving error estimates. Both these questions will be discussed in the present paper.

First, in the next section, we describe the nonconforming finite element spaces considered in this paper, introduce a finite element discretization of (1.1)–(1.5) and mention a standard error estimate. Section 3 is devoted to the validity of the discrete Korn inequality for higher order nonconforming finite elements. Then, in Section 4, we discuss the validity of the discrete Korn inequality for the Crouzeix–Raviart element and the rotated bilinear element. Finally, in Section 5, we mention which error estimates can be derived for these two finite elements.

Throughout the paper we use a standard notation (cf. e.g. [5]). Particularly, we denote by $\|\cdot\|_{0,G}$ the norm in the space $L^2(G)$ and by $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$ the norm and seminorm, respectively, in the Sobolev space $H^k(G) \equiv W^{k,2}(G)$, $k \ge 1$. The notation $L^2_0(G)$ is used for the space of those functions from $L^2(G)$ which have zero mean value over G. As usual, we shall denote by C a generic positive constant independent of h.

2. Nonconforming finite element discretization of (1.1)–(1.5). We assume that we are given a family $\{\mathcal{T}_h\}$ of triangulations of the domain Ω consisting of triangular and/or quadrilateral elements K having the usual compatibility properties (see e.g. [5]) and satisfying $h_K \equiv \operatorname{diam}(K) \le h$ for any $K \in \mathcal{T}_h$. We require that any edge of $\mathcal{T}_h$ lying on $\partial\Omega$ belongs either to Γ^D or to Γ^N. The triangulations are assumed to be shape–regular in the sense that there exists a constant $\bar\sigma$ such that

$$\frac{h_K}{\varrho_K} \le \bar\sigma \qquad \forall\, K \in \mathcal{T}_h\,,\ h > 0\,, \tag{2.1}$$

where ϱ_K is the maximum diameter of circles inscribed into K. Moreover, if K is a quadrilateral, we further assume that $h_{\widetilde K}/\varrho_{\widetilde K} \le \bar\sigma$ for any triangle $\widetilde K$ sharing its vertices with K.

We denote by $\mathcal{E}_h$ the set of all edges E of $\mathcal{T}_h$, by $\mathcal{E}_h^{in}$ the set of the inner edges (i.e., $E \in \mathcal{E}_h^{in} \Leftrightarrow E \not\subset \partial\Omega$), by $\mathcal{E}_h^D$ the set of the edges lying on Γ^D, by $\mathcal{E}_h^N$ the set of the edges lying on Γ^N, and by $\boldsymbol{n}_E$ a fixed unit normal vector to E which corresponds to the outer normal vector $\boldsymbol{n}$ for $E \subset \partial\Omega$. Further, for any inner edge $E \in \mathcal{E}_h^{in}$, we define the jump $[|v|]_E$ of a function v across E by

$$[|v|]_E = (v|_K)|_E - (v|_{\widetilde K})|_E\,,$$

where K, $\widetilde K$ are the two elements adjacent to E denoted in such a way that $\boldsymbol{n}_E$ points into $\widetilde K$. For boundary edges, we simply set $[|v|]_E = v|_E$.

The typical feature of nonconforming finite element spaces is that they contain functions which have jumps across the edges of the triangulation. However, these jumps cannot be arbitrary but they have either to vanish at certain points on the edges or to be L^2 orthogonal to some spaces of polynomials defined on the edges. Often (but not always, see e.g. [18]) these two requirements are equivalent. To simplify the exposition, we shall consider the latter case only. Thus, to approximate the space V, we consider nonconforming spaces of the type

$$\begin{aligned} V_h = \{\boldsymbol{v}_h \in L^2(\Omega)^2;\ & \boldsymbol{v}_h|_K \in P(K)^2\ \ \forall\, K \in \mathcal{T}_h, \\ & \int_E [|\boldsymbol{v}_h|]_E\, q \,\mathrm{d}\gamma = \mathbf{0}\ \ \forall\, q \in P_k(E),\ E \in \mathcal{E}_h^{in} \cup \mathcal{E}_h^D\}\,, \end{aligned} \tag{2.2}$$

where $P(K) \subset H^1(K)$ are some finite–dimensional local spaces and $k \ge 0$ is a given integer. Examples of such spaces V_h can be found in [4, 6, 7, 13, 14, 18]. The spaces $P(K)$ should be chosen in such a way that, for an integer $l \ge 1$,

$$\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,\Omega} \le C\, h^m\, \|\boldsymbol{v}\|_{m+1,\Omega} \qquad \forall\, \boldsymbol{v} \in V \cap H^{m+1}(\Omega)^2\,,\ m = 1, \dots, l\,, \tag{2.3}$$

see [5]. The usual choice is $l = k + 1$ but other possibilities can also be found in the literature (cf. e.g. [13, 14]). In what follows, we shall assume that $l \le k + 1$. Since $V_h \not\subset H^1(\Omega)^2$, we define the 'elementwise' differential operators D_h and div_h by

$$D_h(\boldsymbol{v})|_K = \frac{1}{2}\Big(\nabla(\boldsymbol{v}|_K) + (\nabla\boldsymbol{v}|_K)^{\mathsf{T}}\Big)\,, \qquad (\operatorname{div}_h \boldsymbol{v})|_K = \operatorname{div}(\boldsymbol{v}|_K) \qquad \forall\, K \in \mathcal{T}_h$$

and we set

$$a_h(\boldsymbol{u}, \boldsymbol{v}) = 2\,(D_h(\boldsymbol{u}), D_h(\boldsymbol{v}))\,, \qquad b_h(\boldsymbol{v}, p) = -(p, \operatorname{div}_h \boldsymbol{v})\,.$$

Further, we define a discrete analogue of $|\cdot|_{1,\Omega}$ by

$$|\boldsymbol{v}|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} |\boldsymbol{v}|_{1,K}^2 \right)^{1/2} .$$

It is easy to see that $|\cdot|_{1,h}$ is a norm on V_h.

The velocity space W from the weak formulation is approximated by the space

$$W_h = \{\boldsymbol{v}_h \in V_h;\ \int_E \boldsymbol{v}_h \cdot \boldsymbol{n}\, q \,\mathrm{d}\gamma = 0 \quad \forall\, q \in P_k(E),\, E \in \mathcal{E}_h^N\}$$

satisfying an analogue of (2.3) and the pressure space $L_0^2(\Omega)$ by a finite element subspace Q_h such that

$$\text{(2.4)} \qquad \inf_{q_h \in Q_h} \|v - q_h\|_{0,\Omega} \leq C\, h^m\, \|v\|_{m,\Omega} \qquad \forall\, v \in L_0^2(\Omega) \cap H^m(\Omega)\,,\ m = 1, \dots, l\,.$$

We assume that the spaces W_h and Q_h satisfy the inf–sup condition

$$\text{(2.5)} \qquad \sup_{\boldsymbol{v}_h \in W_h \setminus \{\boldsymbol{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geq \beta\, \|q_h\|_{0,\Omega} \qquad \forall\, q_h \in Q_h$$

with a positive constant β independent of h.

It is natural to define the discrete solution of (1.1)–(1.5) as functions $\boldsymbol{u}_h \in W_h$ and $p_h \in Q_h$ such that

$$\text{(2.6)} \quad a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) - b_h(\boldsymbol{u}_h, q_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \qquad \forall\, \boldsymbol{v}_h \in W_h,\ q_h \in Q_h\,.$$

Let us denote $\mathcal{W}_h = \{\boldsymbol{v}_h \in W_h;\ b_h(\boldsymbol{v}_h, q_h) = 0 \ \forall q_h \in Q_h\}$. Obviously, the discrete problem is uniquely solvable if and only if the only function $\boldsymbol{v}_h \in \mathcal{W}_h$ for which $a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) = 0$ is $\boldsymbol{v}_h = \boldsymbol{0}$. This means that $\sqrt{a_h(\boldsymbol{v}_h, \boldsymbol{v}_h)} = \sqrt{2}\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega}$ has to be a norm on $\mathcal{W}_h$. Thus, in view of the equivalence of norms on finite–dimensional spaces, we can say that the discrete problem is uniquely solvable if and only if, for any $h > 0$, there exists a positive constant $\bar{C}_h$ such that $|\boldsymbol{v}_h|_{1,h} \leq \bar{C}_h\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega}$ for any $\boldsymbol{v}_h \in \mathcal{W}_h$. In what follows, we shall consider the more general inequality

$$\text{(2.7)} \qquad |\boldsymbol{v}_h|_{1,h} \leq \bar{C}_h\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega} \qquad \forall\, \boldsymbol{v}_h \in V_h\,.$$

This is a discrete analogue of the Korn first inequality (1.6). In all cases we shall investigate, the depence of $\bar{C}_h$ on h is the same for both V_h and $\mathcal{W}_h$.

Since the space W_h is nonconforming, the exact solution $\boldsymbol{u}$, p does not solve the discrete problem. Indeed, multiplying (1.1) by a function $\boldsymbol{v}_h \in W_h$, integrating by parts over each $K \in \mathcal{T}_h$ and summing up, we obtain

$$\text{(2.8)} \qquad a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p) = (\boldsymbol{f}, \boldsymbol{v}_h) + e_h(\boldsymbol{u}, p; \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in W_h\,,$$

where the consistency error e_h is defined by

$$e_h(\boldsymbol{u}, p; \boldsymbol{v}_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{v}_h|_K \cdot \sigma(\boldsymbol{u}, p)\, \boldsymbol{n}_{\partial K} \,\mathrm{d}\gamma$$

with $\boldsymbol{n}_{\partial K}$ being the outer unit normal vector to the boundary of K. Using the techniques of [7], we derive the estimate

$$\text{(2.9)} \quad e_h(\boldsymbol{u}, p; \boldsymbol{v}_h) \leq C\, h^m\, (|\boldsymbol{u}|_{m+1,\Omega} + |p|_{m,\Omega})\, |\boldsymbol{v}_h|_{1,h} \quad \forall\, \boldsymbol{v}_h \in W_h,\ m = 1, \dots, k+1$$

which holds as far as the weak solution possesses the regularity indicated by the seminorms on the right–hand side of (2.9). If the weak solution of (1.1)–(1.5) satisfies

$\boldsymbol{u} \in H^{m+1}(\Omega)^2$, $p \in H^m(\Omega)$ with some $m \in \{1, \dots, l\}$, then we derive applying standard techniques (cf. [3, 10]) that

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} + \|p - p_h\|_{0,\Omega} \leq C\, \bar{C}_h^2\, h^m \left(\|\boldsymbol{u}\|_{m+1,\Omega} + \|p\|_{m,\Omega}\right). \tag{2.10}$$

As we see, the standard technique leads to an optimal error etimate only if the constant $\bar{C}_h$ from the discrete Korn inequality (2.7) can be bounded independently of h.

3. Validity of the discrete Korn inequality. At the beginning of the nineties, several papers have been published where the uniform validity (i.e., with $\bar{C}_h$ bounded independently of h) of the discrete Korn inequality (2.7) was investigated for particular finite element spaces, see [8, 9, 16]. Later it was proved in [12] that there exists a constant $\bar{C}$ depending only on Ω, Γ^D and $\bar{\sigma}$ from (2.1) such that

$$|\boldsymbol{v}_h|_{1,h} \leq \bar{C}\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega} \qquad \forall\, \boldsymbol{v}_h \in \widetilde{V}_h\,,$$

where

$$\widetilde{V}_h = \{\boldsymbol{v}_h \in L^2(\Omega)^2;\ \boldsymbol{v}_h|_K \in H^1(K)^2\ \ \forall\, K \in \mathcal{T}_h,\\ \int_E [|\boldsymbol{v}_h|]_E\, q\, \mathrm{d}\gamma = \boldsymbol{0}\ \ \forall\, q \in P_1(E),\ E \in \mathcal{E}_h^{in} \cup \mathcal{E}_h^D\}\,.$$

Since the space V_h defined in (2.2) satisfies $V_h \subset \widetilde{V}_h$ for $k \geq 1$, we see that the discrete Korn inequality (2.7) holds uniformly whenever $k \geq 1$. This result also follows from the more general proof recently published in [2]. Consequently, for any nonconforming space V_h with $k \geq 1$, which are particularly all spaces of approximation order at least 2, the error estimate (2.10) guarantees an optimal convergence of the discrete solution.

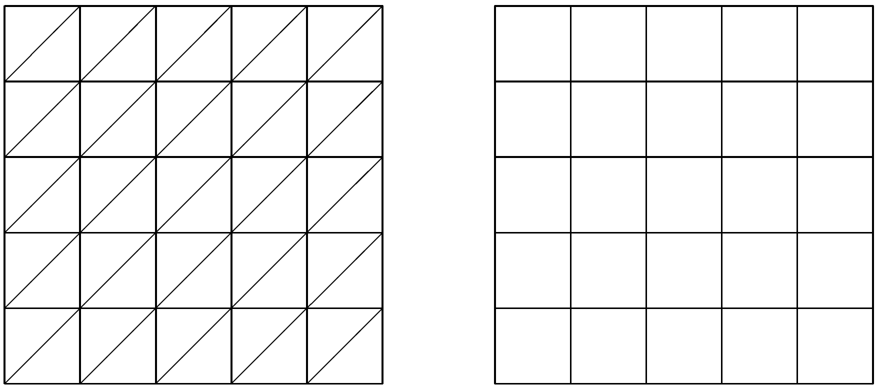
If $k = 0$, then the constant $\bar{C}_h$ cannot be bounded independently of h (cf. [9, 12, 15]) and it may even happen that the discrete Korn inequality (2.7) does not hold at all since the right–hand side of (2.7) vanishes for a non–vanishing $\boldsymbol{v}_h$ (cf. [1, 9]). This behaviour can be observed for the linear triangular Crouzeix–Raviart element [7], for which $P(K) = P_1(K)$, and for the quadrilateral rotated bilinear element [18], for which $P(K) = \mathrm{span}\{1, x, y, x^2 - y^2\}$ if K is a rectangle. If K is a general convex quadrilateral, then $P(K)$ is defined using a bilinear transformation of the reference square onto K, but this possibility will not be considered here.

The remaining part of this paper will be devoted to the Crouzeix–Raviart element and the rotated bilinear element. To simplify our considerations, we shall confine ourselves to

$$\Omega = (0,1)^2\,, \qquad \Gamma^D = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$$

and to triangulations of the type depicted in Fig. 3.1(a) in case of the Crouzeix–Raviart element and of the type from Fig. 3.1(b) in case of the rotated bilinear element. It is easy to verify that, in both cases, $a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \neq 0$ for any $\boldsymbol{v}_h \in V_h \setminus \{\boldsymbol{0}\}$ and hence the discrete Korn inequality (2.7) holds with some constant $\bar{C}_h$. We shall thoroughly discuss the dependece of $\bar{C}_h$ on h and we shall investigate to what extent the h–dependent constant $\bar{C}_h$ influences the convergence behaviour of the discrete solution.

4. Dependence of $\bar{C}_h$ on h for the Crouzeix–Raviart element and the rotated bilinear element. It was shown in [9] and [12] that, for both finite elements and the mentioned types of triangulations, there exists a constant $C > 0$ independent of h such that $\bar{C}_h \geq C\, h^{-1/2}$. For the rotated bilinear element, this result was



(a) Triangulation for the Crouzeix–Raviart element

(b) Triangulation for the rotated bilinear element

FIG. 3.1. *Considered types of triangulations.*

improved to $\bar{C}_h \geq C\,h^{-1}$ in [15]. Let us prove this estimate also for the Crouzeix–Raviart element.

For any edge $E \in \mathcal{E}_h$, we introduce the functional

$$J_E(\boldsymbol{v}) = \frac{1}{|E|} \int_E \boldsymbol{v}\,\mathrm{d}\gamma\,.$$

Let $\widetilde{\boldsymbol{v}}_h$ be a piecewise linear function with respect to $\mathcal{T}_h$ which is continuous at the midpoints of the inner edges of $\mathcal{T}_h$ and satisfies for any $E \in \mathcal{E}_h$

$$\begin{aligned} &J_E(\widetilde{\boldsymbol{v}}_h) = (1,0) && \text{if } E \text{ is parallel to the } x\text{–axis},\\ &J_E(\widetilde{\boldsymbol{v}}_h) = (0,1) && \text{if } E \text{ is parallel to the } y\text{–axis},\\ &J_E(\widetilde{\boldsymbol{v}}_h) = \mathbf{0} && \text{if } E \in \mathcal{E}_h^{diag}, \end{aligned}$$

where $\mathcal{E}_h^{diag}$ is the set of edges having the direction $(1,1)$ (cf. Fig. 4.1(a)). The continuity of $\widetilde{\boldsymbol{v}}_h$ at midpoints of edges $E \in \mathcal{E}_h^{in}$ implies that $J_E(\widetilde{\boldsymbol{v}}_h|_K) = J_E(\widetilde{\boldsymbol{v}}_h|_{\widetilde{K}})$ for the two elements K and $\widetilde{K}$ adjacent to E. Therefore, we may simply write $J_E(\widetilde{\boldsymbol{v}}_h)$.

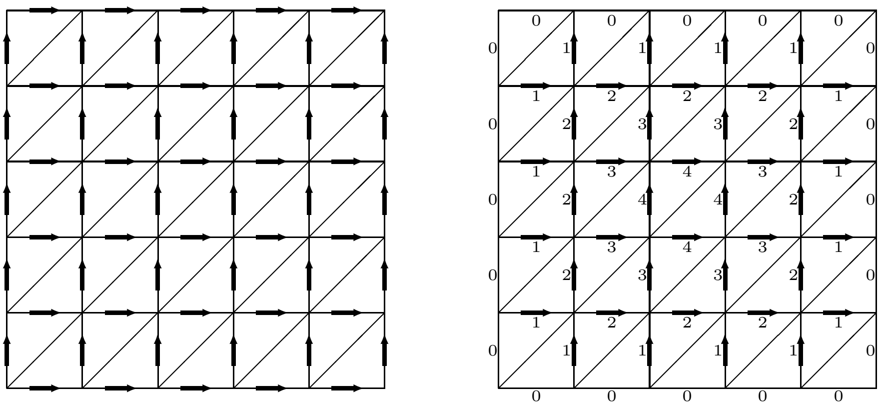
Let n be the number of edges of $\mathcal{T}_h$ on one side of Ω (i.e., $n = 5$ for $\mathcal{T}_h$ from Fig. 3.1(a)) and let $\mathcal{E}_h^* \subset \mathcal{E}_h$ consist of all edges parallel to coordinate axes. We decompose the set $\mathcal{E}_h^*$ into the sets $\mathcal{E}_h^0, \dots, \mathcal{E}_h^{n-1}$ defined in the following way (cf. Fig. 4.1(b)):

$$\begin{aligned} &\mathcal{E}_h^0 = \{E \in \mathcal{E}_h^*;\ E \subset \partial\Omega\}\,,\\ &\mathcal{E}_h^i = \{E \in \mathcal{E}_h^* \setminus \bigcup_{j=0}^{i-1} \mathcal{E}_h^j;\ \exists\, E' \in \mathcal{E}_h^{i-1} : E \cap E' \neq \emptyset \wedge E \perp E'\}\,, \quad i = 1, \dots, n-1\,. \end{aligned}$$

Now we introduce a function $\overline{\boldsymbol{v}}_h \in V_h$ satisfying

$$\begin{aligned} &J_E(\overline{\boldsymbol{v}}_h) = i\,J_E(\widetilde{\boldsymbol{v}}_h) \qquad \forall\ E \in \mathcal{E}_h^i,\ i = 0, \dots, n-1\,,\\ &J_E(\overline{\boldsymbol{v}}_h) = \mathbf{0} \qquad \forall\ E \in \mathcal{E}_h^{diag}\,. \end{aligned}$$

The degrees of freedom of $\overline{\boldsymbol{v}}_h$ have the same directions as those of $\widetilde{\boldsymbol{v}}_h$ but their magnitudes increase towards the centre of Ω as depicted in Fig. 4.1(b). Consider



(a) Degrees of freedom of $\widetilde{\boldsymbol{v}}_h$

(b) Decomposition of $\mathcal{E}_h^*$ and degrees of freedom of $\overline{\boldsymbol{v}}_h$

FIG. 4.1. *Definition of functions* $\widetilde{\boldsymbol{v}}_h$ *and* $\overline{\boldsymbol{v}}_h$.

any $K \in \mathcal{T}_h$. Then there exists $i \equiv i_K \in \{0, \dots, n-2\}$ such that the two edges of K parallel to the coordinate axes belong to $\mathcal{E}_h^i \cup \mathcal{E}_h^{i+1}$. Moreover, at least one of these edges belongs to $\mathcal{E}_h^i$ or they both belong to $\mathcal{E}_h^{n-1}$. Setting $\boldsymbol{u}_K = \overline{\boldsymbol{v}}_h|_K - i_K\, \widetilde{\boldsymbol{v}}_h|_K$, we have $|J_E(\boldsymbol{u}_K)| \leq 1$ for any edge $E \subset \partial K$. In addition, either $J_E(\boldsymbol{u}_K) = \boldsymbol{0}$ for two edges of K or $\boldsymbol{u}_K = \widetilde{\boldsymbol{v}}_h|_K$. Since $D_h(\widetilde{\boldsymbol{v}}_h) = 0$, we obtain

$$(4.1) \qquad \sum_{K \in \mathcal{T}_h} \|\nabla \overline{\boldsymbol{v}}_h + (\nabla \overline{\boldsymbol{v}}_h)^{\mathsf{T}}\|_{0,K}^2 = \sum_{K \in \mathcal{T}_h} \|\nabla \boldsymbol{u}_K + (\nabla \boldsymbol{u}_K)^{\mathsf{T}}\|_{0,K}^2 \leq 16\, n^2 \,.$$

On the other hand, since $\operatorname{card} \mathcal{E}_h^i = 4\,(n-i)$ for $i = 0, \dots, n-1$, we have (for $n \geq 2$)

$$\sum_{K \in \mathcal{T}_h} |\overline{\boldsymbol{v}}_h|_{1,K}^2 = 4 \sum_{E \in \mathcal{E}_h^*} |J_E(\overline{\boldsymbol{v}}_h)|^2 = 16 \sum_{i=1}^{n-1} (n-i)\, i^2 = \frac{4}{3}\,(n^4 - n^2) \geq n^4 \,.$$

This implies that $\bar{C}_h \geq \frac{1}{\sqrt{2}}\, h^{-1}$. The following theorem shows that this estimate corresponds to the real behaviour of $\bar{C}_h$.

THEOREM 1. *For both the Crouzeix–Raviart element and the rotated bilinear element on triangulations of the type depicted in Fig. 3.1, there exist constants* C_0 *and* C_1 *independent of* h *such that*

$$(4.2) \qquad C_0\, \|\boldsymbol{v}_h\|_{0,\Omega} + C_1\, h\, |\boldsymbol{v}_h|_{1,h} \leq \|D_h(\boldsymbol{v}_h)\|_{0,\Omega} \qquad \forall\ \boldsymbol{v}_h \in V_h \,.$$

Proof. For the rotated bilinear element, this theorem was proved in [15]. The proof is based on rewriting the inequality (4.2) using the degrees of freedom of $\boldsymbol{v}_h$ and on manipulations with the resulting sums. To this end, a numbering of the edges of the triangulation from Fig. 3.1(b) is introduced, see Fig. 4.2. First, the square elements are numbered by indices $i, j = 1, \dots, n$ such that the coordinates of the centre of the element K_{ij} are $((i-0.5)/n, (j-0.5)/n)$. Then the indices of the inner edges of the triangulation are defined as the averages of the indices of the two square elements adjacent to the respective edge. For edges lying on the boundary of Ω, the indices are also defined in this way imagining that the triangulation continues outside Ω.

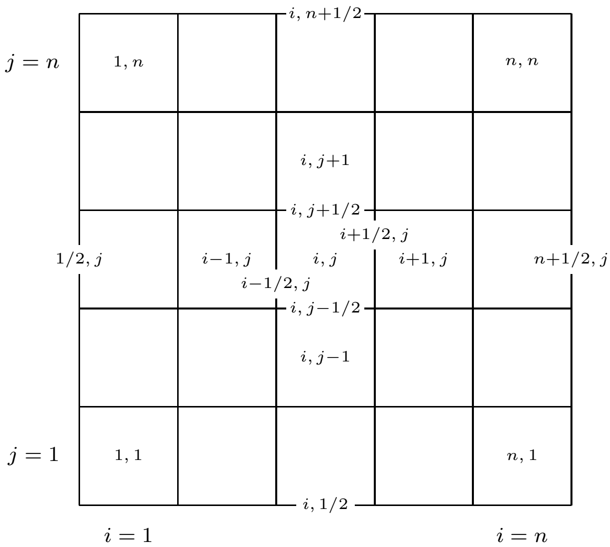


FIG. 4.2. *Numbering of elements and edges for the triangulation from Fig. 3.1(b).*

Here we prove the inequality (4.2) for the Crouzeix–Raviart element and a triangulation of the type depicted in Fig. 3.1(a). The idea of the proof is analogous as for the rotated bilinear element. For edges parallel to the coordinate axes, we shall use the numbering mentioned above and, for edges having the direction $(1,1)$, we shall use the indices of the corresponding square elements (i.e., $((i-0.5)/n, (j-0.5)/n)$ are the coordinates of the midpoint of such a 'diagonal' edge E_{ij}).

Now consider any $\boldsymbol{v}_h \in V_h$. We denote by u, v the components of $\boldsymbol{v}_h$ and, for an edge $E_{\alpha,\beta}$, we set $(u_{\alpha,\beta}, v_{\alpha,\beta}) = J_{E_{\alpha,\beta}}(\boldsymbol{v}_h)$. Note that $(u_{\alpha,\beta}, v_{\alpha,\beta})$ is the value of $\boldsymbol{v}_h$ at the midpoint of $E_{\alpha,\beta}$. Since the degrees of freedom associated with boundary edges lying on Γ^D vanish, we have

$$u_{i,1/2} = v_{i,1/2} = 0\,, \quad i = 1, \dots, n\,, \qquad u_{1/2,j} = v_{1/2,j} = 0\,, \quad j = 1, \dots, n\,.$$

A direct computation gives $\|D_h(\boldsymbol{v}_h)\|_{0,\Omega}^2 = \sum_{i,j=1}^n (2\,r_{ij} + s_{ij})$ with

$$r_{ij} = (u_{i-1/2,j} - u_{ij})^2 + (u_{ij} - u_{i+1/2,j})^2 + (v_{i,j-1/2} - v_{ij})^2 + (v_{ij} - v_{i,j+1/2})^2\,,$$
$$s_{ij} = (u_{ij} - u_{i,j+1/2} + v_{i-1/2,j} - v_{ij})^2 + (u_{i,j-1/2} - u_{ij} + v_{ij} - v_{i+1/2,j})^2\,.$$

Since $a^2 + b^2 + c^2 \geq \frac{1}{3}\,(a+b+c)^2$ for any $a, b, c \in \mathbb{R}$, we deduce that $\|D_h(\boldsymbol{v}_h)\|_{0,\Omega}^2 \geq \sum_{i,j=1}^n (r_{ij} + \frac{1}{3}\,t_{ij})$ with

$$t_{ij} = (f_{i-1/2,j} - f_{i,j+1/2})^2 + (f_{i,j-1/2} - f_{i+1/2,j})^2\,, \qquad f = u + v\,.$$

For any $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $\alpha_0 = 0$, we have

$$\sum_{i=1}^n (\alpha_i - \alpha_{i-1})^2 \geq \frac{1}{n^2} \sum_{i=1}^n \alpha_i^2 \tag{4.3}$$

and hence we immediately get

$$\sum_{i,j=1}^n r_{ij} \geq \frac{1}{4\,n^2} \sum_{i,j=1}^n \{u_{i+1/2,j}^2 + v_{i,j+1/2}^2 + u_{ij}^2 + v_{ij}^2\}\,.$$

Further, applying (4.3) along the diagonals $i - j = const.$ (cf. [15]), we derive

$$\sum_{i,j=1}^{n} t_{ij} \geq \frac{1}{4\,n^2} \sum_{i,j=1}^{n} (f_{i,j+1/2}^2 + f_{i+1/2,j}^2)\,.$$

Since $u_{i+1/2,j}^2 + \frac{1}{3} f_{i+1/2,j}^2 \geq \frac{2}{9}\{u_{i+1/2,j}^2 + (u_{i+1/2,j} - f_{i+1/2,j})^2\} = \frac{2}{9}(u_{i+1/2,j}^2 + v_{i+1/2,j}^2)$ and, analogously, $v_{i,j+1/2}^2 + \frac{1}{3} f_{i,j+1/2}^2 \geq \frac{2}{9}(u_{i,j+1/2}^2 + v_{i,j+1/2}^2)$, we get

$$\|D_h(\boldsymbol{v}_h)\|_{0,\Omega}^2 \geq \frac{1}{4\,n^2} \sum_{i,j=1}^{n} \left(\frac{2}{9}\, z_{ij} + u_{ij}^2 + v_{ij}^2\right)$$

with $z_{ij} = u_{i,j+1/2}^2 + v_{i,j+1/2}^2 + u_{i+1/2,j}^2 + v_{i+1/2,j}^2$. An easy computation gives $\|\boldsymbol{v}_h\|_{0,\Omega}^2 \leq \frac{1}{3\,n^2}\sum_{i,j=1}^{n}(z_{ij} + u_{ij}^2 + v_{ij}^2)$ and $|\boldsymbol{v}_h|_{1,h}^2 \leq 8\sum_{i,j=1}^{n}(z_{ij} + 2\,u_{ij}^2 + 2\,v_{ij}^2)$. Thus, $\|\boldsymbol{v}_h\|_{0,\Omega} \leq \sqrt{6}\,\|D_h(\boldsymbol{v}_h)\|_{0,\Omega}$ and $|\boldsymbol{v}_h|_{1,h} \leq 12\,n\|D_h(\boldsymbol{v}_h)\|_{0,\Omega}$, which completes the proof. $\square$

Theorem 1 implies that, for both the Crouzeix–Raviart element and the rotated bilinear element, the right–hand side of the standard error estimate (2.10) tends to infinity like h^{-1}. In the next section, we shall discuss whether this really means that no convergence of the discrete solutions of the problem (1.1)–(1.5) can be expected.

5. Error estimates for the Crouzeix–Raviart element and the rotated bilinear element. Let us consider the discrete problem (2.6) where the space W_h is defined using the Crouzeix–Raviart element or the rotated bilinear element and using a triangulation of the type depicted in Fig. 3.1(a) or Fig. 3.1(b), respectively. The space Q_h consists of piecewise constant functions from the space $L_0^2(\Omega)$. Then the inf–sup condition (2.5) is satisfied and (2.3) and (2.4) hold with $l = 1$.

According to (2.6) and (2.8), we have for any $\boldsymbol{v}_h \in W_h$

$$\begin{aligned} e_h(\boldsymbol{u}, p; \boldsymbol{v}_h) &= a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p - p_h) \\ &\leq 2\,(\|D_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega})\,\|D_h(\boldsymbol{v}_h)\|_{0,\Omega} \end{aligned} \tag{5.1}$$

and hence a necessary condition for the convergence of the discrete solution $\boldsymbol{u}_h$, p_h to the weak solution $\boldsymbol{u}$, p (with respect to the usual norms) is

$$\lim_{h\to 0} \sup_{\boldsymbol{v}_h \in W_h\setminus\{\boldsymbol{0}\}} \frac{e_h(\boldsymbol{u}, p; \boldsymbol{v}_h)}{\|D_h(\boldsymbol{v}_h)\|_{0,\Omega}} = 0 \qquad \forall\, \boldsymbol{v}_h \in W_h\,. \tag{5.2}$$

Let us first consider the discrete problem defined using the Crouzeix–Raviart element. For any inner edge $E \in \mathcal{E}_h^{in}$, let ζ_E be the standard Crouzeix–Raviart basis function associated with E (i.e., ζ_E is piecewise linear, equals 1 on E and vanishes at the midpoints of all edges different from E). Then, for any $\boldsymbol{\alpha} \in \mathbb{R}^2$, we have

$$e_h(\boldsymbol{0}, x; \boldsymbol{\alpha}\,\zeta_E) = \begin{cases} \frac{1}{6}\,h^2\,(\alpha_2 - \alpha_1) & \text{if } E \text{ is parallel to the } x\text{–axis}, \\ \frac{1}{6}\,h^2\,\alpha_1 & \text{if } E \text{ is parallel to the } y\text{–axis}, \\ -\frac{1}{6}\,h^2\,\alpha_2 & \text{if } E \in \mathcal{E}_h^{diag}. \end{cases}$$

Let $\boldsymbol{u} = \boldsymbol{0}$ and $p = x - \frac{1}{2}$ and let $\overline{\boldsymbol{v}}_h$ be the function defined in Section 4. Then, using (4.1) and the notation of Section 4, we get (for $n \geq 2$)

$$|e_h(\boldsymbol{u}, p; \overline{\boldsymbol{v}}_h)| = \frac{h^2}{6} \sum_{E\in\mathcal{E}_h^*} J_E(\overline{\boldsymbol{v}}_h)\cdot(1,0) = \frac{h^2}{3}\sum_{i=1}^{n-1}(n-i)\,i \geq \frac{n}{18} \geq \frac{1}{36}\,\|D_h(\overline{\boldsymbol{v}}_h)\|_{0,\Omega}\,.$$

Therefore, the necessary condition (5.2) generally does not hold for the Crouzeix–

Raviart element and hence we cannot expect that the Crouzeix–Raviart element will lead to a convergent discrete solution.

For the rotated bilinear element, it was shown in [15] that (5.2) holds if $\boldsymbol{u} \in H^3(\Omega)^2$ and $p \in H^2(\Omega)$. Precisely, it was proved that

$$e_h(\boldsymbol{u}, p; \boldsymbol{v}_h) \le C\, h\, (|\boldsymbol{u}|_{3,\Omega} + |p|_{2,\Omega})\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega} \qquad \forall\, \boldsymbol{v}_h \in W_h\,. \tag{5.3}$$

The proof of (5.3) relies on the discrete Korn inequality (4.2) and the fact that

$$\sum_{E\subset\partial K} \int_E (\boldsymbol{v}_h|_K - J_E(\boldsymbol{v}_h)) \cdot \sigma\, \boldsymbol{n}_{\partial K}\, \mathrm{d}\gamma = 0 \qquad \forall\, \boldsymbol{v}_h \in W_h,\ \sigma \in P_1(K)^{2\times 2},\ K \in \mathcal{T}_h\,.$$

Thus, let us assume that the weak solution of (1.1)–(1.5) satisfies $\boldsymbol{u} \in H^3(\Omega)^2$ and $p \in H^2(\Omega)$. We consider any $\boldsymbol{z}_h \in W_h$ satisfying $b_h(\boldsymbol{z}_h, q_h) = 0\ \forall\, q_h \in Q_h$ and set $\boldsymbol{w}_h = \boldsymbol{u}_h - \boldsymbol{z}_h$. Then, using (5.1), we obtain for any $q_h \in Q_h$

$$2\, \|D_h(\boldsymbol{w}_h)\|_{0,\Omega}^2 = a_h(\boldsymbol{u} - \boldsymbol{z}_h, \boldsymbol{w}_h) + b_h(\boldsymbol{w}_h, p - q_h) - e_h(\boldsymbol{u}, p; \boldsymbol{w}_h)\,.$$

Hence, in view of (5.3), we get

$$2\, \|D_h(\boldsymbol{w}_h)\|_{0,\Omega} \le 2\, \|D_h(\boldsymbol{u}_h - \boldsymbol{z}_h)\|_{0,\Omega} + \sqrt{2}\, \|p - q_h\|_{0,\Omega} + C\, h\, (|\boldsymbol{u}|_{3,\Omega} + |p|_{2,\Omega})\,.$$

This implies by applying standard techniques (cf. e.g. [10]) and using (4.2) that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} + \|D_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \le C\, h\, (\|\boldsymbol{u}\|_{3,\Omega} + \|p\|_{2,\Omega})\,. \tag{5.4}$$

Unfortunately, we were not able to prove the convergence of $\boldsymbol{u}_h$ with respect to the seminorm $|\cdot|_{1,h}$ although numerical experiments show optimal convergence behaviour. The estimate of $\|D_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega}$ and $\|p - p_h\|_{0,\Omega}$ is optimal with respect to the convergence order but the required regularity of the weak solution is higher than usually. However, a counterexample in [15] shows that the usual regularity requirement $\boldsymbol{u} \in H^2(\Omega)^2$, $p \in H^1(\Omega)$ is not sufficient. The suboptimal estimate of the velocity error in the L^2 norm cannot be improved using the Aubin–Nitsche duality technique due to the higher regularity requirement in (5.3). Let us also remark how the results change if we consider the boundary condition $\sigma(\boldsymbol{u}, p)\, \boldsymbol{n} = \boldsymbol{g}$ on Γ^N instead of (1.4) and (1.5), which leads to the additional term $\sum_{E\subset\Gamma^N} \int_E \boldsymbol{g} \cdot (\boldsymbol{v}_h - J_E(\boldsymbol{v}_h))\, \mathrm{d}\gamma$ when estimating the consistency error. For sufficiently regular $\boldsymbol{g}$, this term can be estimated by $C\, h^{1/2}\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega}$, which limits the convergence order in (5.4) to 1/2.

For the Crouzeix–Raviart element, the estimate (2.9) with $m = 1$ and the discrete Korn inequality (4.2) show that $e_h(\boldsymbol{u}, p; \boldsymbol{v}_h) \le C\, (|\boldsymbol{u}|_{2,\Omega} + |p|_{1,\Omega})\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega}$ for any $\boldsymbol{v}_h \in W_h$. As we have seen above, this estimate cannot be improved. Using this estimate together with the techniques just applied for the rotated bilinear element, we obtain error estimates of the type $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} = \mathcal{O}(1)$, $\|D_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} = \mathcal{O}(1)$, $|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} = \mathcal{O}(h^{-1})$ and $\|p - p_h\|_{0,\Omega} = \mathcal{O}(1)$ which were also confirmed by numerical experiments. Some results are presented in Table 5.1 where we compare the convergence behaviour of the Crouzeix–Raviart element (denoted P_1^{nc}) and the rotated bilinear element (denoted Q_1^{rot}) for $\Gamma^N = \emptyset$ and an exact solution $u_1(x, y) = 2\, x^2\, (1-x)^2\, y\, (1-y)\, (1-2\, y)$, $u_2(x, y) = -2\, y^2\, (1-y)^2\, x\, (1-x)\, (1-2\, x)$ and $p(x, y) = x^3 + y^3 - 0.5$. The number n has the meaning introduced in the preceding section, i.e., the respective triangulation contains $(n + 1)^2$ vertices. The convergence orders were always computed using values from triangulations with $n = 32$ and $n = 64$.

Thus, we can conclude that, for the Crouzeix–Raviart element, the mesh–dependent Korn inequality prevents the discrete solutions from converging to the weak solution. On the other hand, for the rotated bilinear element, the negative influence of the mesh–dependent Korn inequality is compensated by a superconvergent behaviour

Table 5.1
Comparison of the convergence for the Crouzeix–Raviart and rotated bilinear elements.

n	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$		$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,h}$		$\|p-p_h\|_{0,\Omega}$	
	P_1^{nc}	Q_1^{rot}	P_1^{nc}	Q_1^{rot}	P_1^{nc}	Q_1^{rot}
8	2.71−2	1.10−3	7.13−1	3.55−2	1.64−1	6.92−2
16	2.71−2	2.81−4	1.44+0	1.80−2	1.56−1	3.44−2
32	2.71−2	7.09−5	2.89+0	9.07−3	1.54−1	1.72−2
64	2.71−2	1.78−5	5.79+0	4.54−3	1.53−1	8.57−3
order	0.00	2.00	−1.00	1.00	0.00	1.00

of the consistency error so that a convergence of the discrete solution is possible, provided that the weak solution is sufficiently regular.

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