Boundary layers in a Riemann-Liouville fractional derivative two-point boundary value problem

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Abstract A two-point boundary value problem whose highest-order term is a Riemann-Liouville fractional derivative of order $\delta \in (1,2)$ is considered on the interval $[0,1]$. It is shown that the solution u of the problem lies in $C[0,1]$ but not in $C^1[0,1]$ because $u'(x)$ blows up at $x \to 0$ for each fixed value of δ. Furthermore, $u'(1)$ blows up as $\delta \to 1^+$ if and only if the constant convection coefficient b satisfies $b \geq 1$.

1 Introduction

Let $\delta \in (1,2)$. Let $g \in C^1(0,1]$ with $g' \in L_1[0,1]$. The Riemann-Liouville fractional derivative D^δ_{RL} of order δ associated with the point $x = 0$ is defined by

$$D^\delta_{RL} g(x) = \frac{d^2}{dx^2}\left[\frac{1}{\Gamma(2-\delta)}\int_{t=0}^{x}(x-t)^{1-\delta} g(t)\,dt\right] \quad \text{for } 0 < x \leq 1;$$

see [6].

In this paper we shall consider the two-point boundary value problem

$$-D^\delta_{RL} u(x) + b u'(x) = f \ \text{ for } x \in (0,1), \tag{1a}$$
$$u(0) = 0, \quad u(1) + \alpha_1 u'(1) = \gamma_1, \tag{1b}$$

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where b, f, α_1, γ_1 are given constants. We assume that $\alpha_1 \geq 0$, as in 2nd-order elliptic problems. Remark 1 below explains the necessity of imposing the Dirichlet condition $u(0) = 0$ at $x = 0$.

Existence and uniqueness of solution to (1) is discussed in [4]. Assume that f and γ_1 are not both zero as otherwise the solution to (1) is $u \equiv 0$ in the set of functions $u \in C[0,1]$, with u' and $D_{RL}^{\delta-1} u$ absolutely continuous on [0,1] (see [4, Theorem 2.8]).

Problem (1) is used to model anomalous diffusion processes; for example, we refer to [3] for a motivation of this model.

In [8] we considered a related problem where the Riemann-Liouville derivative of (1a) is replaced by a Caputo fractional derivative, and discussed under what circumstances one would observe a boundary layer in its solution at $x = 1$ as $\delta \to 1^+$ (with the other data of the problem fixed). Our main aim in the present paper is similar: to determine when $u'(1)$ blows up as $\delta \to 1^+$.

In Section 2 we solve (1) exactly using Laplace transforms. We shall see easily that in general $|u'(x)| \to \infty$ as $x \to 0$ for each fixed value of $\delta \in (1,2)$, so $u \notin C^1[0,1]$. A more demanding investigation in Section 3 exploits properties of Mittag-Leffler functions to show that $u'(1)$ blows up as $\delta \to 1^+$ when $b \geq 1$ but no such singular behaviour is present when $b < 1$.

Notation. We use the "big O" notation in its sharp form. Thus when we write for example $g = O(1/(\delta - 1))$ as $\delta \to 1^+$, we mean that $\lim_{\delta \to 1^+} [(\delta - 1) g]$ exists and is non-zero. Throughout the paper C denotes a generic constant that is independent of δ but may depend on b, f, α_1 and γ_1. Set $\|u\|_\infty = \max_{x \in [0,1]} |u(x)|$.

2 Solution via Laplace transform

We compute the solution of the problem (1) by using the Laplace transform. Our analysis makes heavy use of the two-parameter Mittag-Leffler function (see, for example, [1, 6])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \text{for } \alpha, \beta, z \in \mathbb{R} \text{ with } \alpha > 0, \tag{2}$$

which is an entire function if, furthermore, $\beta > 0$.

The Laplace transform $\mathscr{L}$ of the Riemann-Liouville fractional derivative is [6, (2.248)]

$$\mathscr{L}\{D_{RL}^{\delta} u\} = s^{\delta} \mathscr{L}\{u\} - C_1 - sC_2,$$

with $C_1 = \left[D_{RL}^{\delta-1} u\right](0)$ and $C_2 = \left[D_{RL}^{\delta-2} u\right](0)$. Thus, taking the Laplace transform of (1a), one obtains

$$\frac{f}{s} = -[s^{\delta} \mathscr{L}\{u\} - C_1 - sC_2] + b[s\mathscr{L}\{u\} - u(0)] = -[s^{\delta} \mathscr{L}\{u\} - C_1 - sC_2] + bs\mathscr{L}\{u\}$$

and therefore

$$\mathscr{L}\{u\} = \frac{C_1}{s(s^{\delta-1}-b)} + \frac{C_2}{s^{\delta-1}-b} - \frac{f}{s^2(s^{\delta-1}-b)}. \tag{3}$$

Now the Laplace transform of the Mittag-Leffler function is [6, (1.80)]

$$\mathscr{L}\left\{x^{\beta-1}E_{\alpha,\beta}(\pm\lambda x^{\alpha})\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}\mp\lambda}.$$

Hence one can deduce from (3) that the solution of (1) is

$$u(x) = C_1 x^{\delta-1}E_{\delta-1,\delta}(bx^{\delta-1}) + C_2 x^{\delta-2}E_{\delta-1,\delta-1}(bx^{\delta-1}) - f x^{\delta}E_{\delta-1,\delta+1}(bx^{\delta-1}). \tag{4}$$

In this formula the constants C_1 and C_2 must be chosen to satisfy the boundary conditions (1b). The boundary condition $u(0) = 0$ forces $C_2 = 0$.

Remark 1. Recall that $1 < \delta < 2$. From (4) one sees that to obtain a solution u that lies in $C[0,1]$, the formulation of the problem (1) must include the homogenous Dirichet boundary condition $u(0) = 0$ in order to eliminate the singular component $x^{\delta-2}E_{\delta-1,\delta-1}(bx^{\delta-1})$.

The value of C_1 in (4) will be deduced from the boundary condition (1b) at $x = 1$. By [6, (1.82)] one has

$$D_{RL}^{\gamma}\big(x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha})\big) = x^{\beta-\gamma-1}E_{\alpha,\beta-\gamma}(\lambda x^{\alpha}), \tag{5}$$

for constant α, β, γ and λ. When $\gamma = 1$ one has $D_{RL}^{\gamma} = d/dx$ [1, p.27]; hence (4) yields

$$u'(x) = C_1 x^{\delta-2}E_{\delta-1,\delta-1}(bx^{\delta-1}) - f x^{\delta-1}E_{\delta-1,\delta}(bx^{\delta-1}). \tag{6}$$

Thus, by (1b) one has

$$\begin{aligned}\gamma_1 &= u(1) + \alpha_1 u'(1)\\ &= C_1\left[E_{\delta-1,\delta}(b) + \alpha_1 E_{\delta-1,\delta-1}(b)\right] - f\left[E_{\delta-1,\delta+1}(b) + \alpha_1 E_{\delta-1,\delta}(b)\right]\end{aligned}$$

and consequently

$$C_1 = \frac{\gamma_1 + f\left[E_{\delta-1,\delta+1}(b) + \alpha_1 E_{\delta-1,\delta}(b)\right]}{E_{\delta-1,\delta}(b) + \alpha_1 E_{\delta-1,\delta-1}(b)}. \tag{7}$$

Substituting (7) into (4) and (6) yields closed-form representations of the solution

$$\begin{aligned}u(x) = {}&\gamma_1 x^{\delta-1}\frac{E_{\delta-1,\delta}(bx^{\delta-1})}{E_{\delta-1,\delta}(b) + \alpha_1 E_{\delta-1,\delta-1}(b)}\\ &+ f\left[x^{\delta-1}\frac{E_{\delta-1,\delta+1}(b) + \alpha_1 E_{\delta-1,\delta}(b)}{E_{\delta-1,\delta}(b) + \alpha_1 E_{\delta-1,\delta-1}(b)}E_{\delta-1,\delta}(bx^{\delta-1}) - x^{\delta}E_{\delta-1,\delta+1}(bx^{\delta-1})\right]\end{aligned} \tag{8}$$

and its first-order derivative

$$
\begin{aligned}
u'(x) &= \gamma_1 x^{\delta-2} \frac{E_{\delta-1,\delta-1}(bx^{\delta-1})}{E_{\delta-1,\delta}(b)+\alpha_1 E_{\delta-1,\delta-1}(b)} \\
&+ f\left[x^{\delta-2}\frac{E_{\delta-1,\delta+1}(b)+\alpha_1 E_{\delta-1,\delta}(b)}{E_{\delta-1,\delta}(b)+\alpha_1 E_{\delta-1,\delta-1}(b)} E_{\delta-1,\delta-1}(bx^{\delta-1}) - x^{\delta-1}E_{\delta-1,\delta}(bx^{\delta-1})\right].
\end{aligned}
\tag{9}
$$

Using the elementary identity

$$
E_{\delta-1,i}(z) = zE_{\delta-1,\delta-1+i}(z) + \frac{1}{\Gamma(i)} \quad \text{for } i = 0,1,2 \tag{10}
$$

in (8) and (9), we get

$$
\begin{aligned}
u(x) &= \gamma_1 \frac{E_{\delta-1,1}(bx^{\delta-1})-1}{E_{\delta-1,1}(b)-1+\alpha_1 E_{\delta-1,0}(b)} \\
&\quad + \frac{f}{b}\left\{x - \frac{E_{\delta-1,2}(b)-1+\alpha_1[E_{\delta-1,1}(b)-1]}{E_{\delta-1,1}(b)-1+\alpha_1 E_{\delta-1,0}(b)}\right\} \\
&\quad + \frac{f}{b}\left\{\frac{E_{\delta-1,2}(b)-1+\alpha_1[E_{\delta-1,1}(b)-1]}{E_{\delta-1,1}(b)-1+\alpha_1 E_{\delta-1,0}(b)} E_{\delta-1,1}(bx^{\delta-1}) - xE_{\delta-1,2}(bx^{\delta-1})\right\}
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
u'(x) &= \gamma_1 x^{-1} \frac{E_{\delta-1,0}(bx^{\delta-1})}{E_{\delta-1,1}(b)-1+\alpha_1 E_{\delta-1,0}(b)} + \frac{f}{b} \\
&+ \frac{f}{b}x^{-1}\left\{\frac{E_{\delta-1,2}(b)-1+\alpha_1[E_{\delta-1,1}(b)-1]}{E_{\delta-1,1}(b)-1+\alpha_1 E_{\delta-1,0}(b)} E_{\delta-1,0}(bx^{\delta-1}) - xE_{\delta-1,1}(bx^{\delta-1})\right\}.
\end{aligned}
\tag{12}
$$

Lemma 1. *For $j = 1,2$ the function*

$$
\phi(x) = x^{\delta-j}E_{\delta-1,\delta-j+1}(bx^{\delta-1}), \quad \text{with } x > 0, \tag{13}
$$

is a solution of

$$
-D^{\delta}_{RL}\phi + b\phi' = 0.
$$

Proof. It follows from [6, (1.82)] that

$$
\begin{aligned}
-D^{\delta}_{RL}\phi + b\phi' &= -x^{-j}E_{\delta-1,-j+1}(bx^{\delta-1}) + bx^{\delta-j-1}E_{\delta-1,\delta-j}(bx^{\delta-1}) \\
&= x^{-j}\left[-\sum_{k=0}^{\infty}\frac{(bx^{\delta-1})^k}{\Gamma((\delta-1)k-j+1)} + \sum_{k=0}^{\infty}\frac{(bx^{\delta-1})^{k+1}}{\Gamma((\delta-1)k+\delta-j)}\right] \\
&= x^{-j}\left[-\sum_{k=1}^{\infty}\frac{(bx^{\delta-1})^k}{\Gamma((\delta-1)k-j+1)} + \sum_{k=1}^{\infty}\frac{(bx^{\delta-1})^{k}}{\Gamma((\delta-1)k-j+1)}\right] \\
&= 0,
\end{aligned}
$$

where in the first series we have used $\Gamma(-1) = \Gamma(0) = \infty$.

Lemma 2. *The function*

$$
\psi(x) = x^{\delta}E_{\delta-1,\delta+1}(bx^{\delta-1}), \quad \textit{with } x > 0, \tag{14}
$$

is a solution of

$$
-D^{\delta}_{RL}\psi + b\psi' = -1.
$$

Proof. It follows from [6, (1.82)] that

$$
\begin{aligned}
-D^{\delta}_{RL}\psi + b\psi' &= -E_{\delta-1,1}(bx^{\delta-1}) + bx^{\delta-1}E_{\delta-1,\delta}(bx^{\delta-1}) \\
&= -\sum_{k=0}^{\infty}\frac{(bx^{\delta-1})^k}{\Gamma((\delta-1)k+1)} + \sum_{k=0}^{\infty}\frac{(bx^{\delta-1})^{k+1}}{\Gamma((\delta-1)k+\delta)} \\
&= -\sum_{k=0}^{\infty}\frac{(bx^{\delta-1})^k}{\Gamma((\delta-1)k+1)} + \sum_{k=1}^{\infty}\frac{(bx^{\delta-1})^{k}}{\Gamma((\delta-1)k+1)} \\
&= -1.
\end{aligned}
$$

Observe that the functions $\phi(x)$ and $\psi(x)$ in Lemmas 13 and 14 are infinitely differentiable for $x > 0$.

Using Lemmas 1 and 2 it is straightforward to verify that the function u defined in (8) satisfies (1). In addition, from (9) we have $|u'(x)| \to \infty$ as $x \to 0^+$ for each fixed value of $\delta \in (1,2)$.

3 Boundary layers in the solution

We now discuss the behaviour of $\|u\|_\infty$ and $u'(1)$ when $\delta \to 1^+$.

Note immediately that (9) and the hypothesis $1 < \delta < 2$ imply that $u'(x)$ blows up as $x \to 0^+$. This is a singularity in u, not a boundary layer (in the typical usage of this terminology in singularly perturbed differential equations), and we do not discuss it further.

Thus we investigate the other endpoint $x = 1$. This will involve different cases depending on the value of the convective term b; cf. [8].

Let $\beta > \alpha > 0$ and $y \in \mathbb{R}$. We begin with the useful Mittag-Leffler identity

$$E_{\alpha,\beta}(y) = \frac{1}{\alpha\Gamma(\beta-\alpha)}\int_{t=0}^{1}\left(1-t^{1/\alpha}\right)^{\beta-\alpha-1}E_{\alpha,\alpha}(ty)\,dt \tag{15}$$

of [5, Lemma 2], which is easily proved by expanding $E_{\alpha,\alpha}(ty)$ as an infinite series in powers of ty and then integrating term by term. In [5] this identity is used to prove that $E_{\alpha,\beta}(-y)$ is completely monotonic for $0<\alpha\le 1,\ \beta\ge\alpha$ and $y\ge 0$. Hence in particular

$$E_{\alpha,\beta}(y)\ge 0 \text{ and } (d/dy)E_{\alpha,\beta}(y)\ge 0 \quad \text{for } 0<\alpha\le 1,\ \beta\ge\alpha,\ y\le 0. \tag{16}$$

Of course one has trivially $E_{\alpha,\beta}(y)>0$ for $y\ge 0$ (and any $\alpha\ge 0,\ \beta>0$) from the definition (2). One can sharpen (16) to

$$E_{\alpha,\beta}(y)>0 \quad \text{for } 0<\alpha\le 1,\ \beta>\alpha,\ y\le 0 \tag{17}$$

because in (15) the integrand is continuous and non-negative with $E_{\alpha,\alpha}(0)=1/\Gamma(\alpha)>0$.

Furthermore, the identity (15) and the properties $E_{\alpha,\alpha}(0)=1/\Gamma(\alpha)$ and $E_{\alpha,\alpha}(s)\ge 0$ for all $s\in\mathbb{R}$ imply that for $i=0,1$ one has

$$0<E_{\delta-1,\delta+1+i}(y)<E_{\delta-1,\delta+i}(y) \quad \text{for all } y\in\mathbb{R}. \tag{18}$$

Thus for the quotients appearing in (8) and (9) it follows that

$$0<\frac{E_{\delta-1,\delta+1}(b)+\alpha_1E_{\delta-1,\delta}(b)}{E_{\delta-1,\delta}(b)+\alpha_1E_{\delta-1,\delta-1}(b)}\le\frac{E_{\delta-1,\delta+1}(b)}{E_{\delta-1,\delta}(b)}+\alpha_1<1+\alpha_1\,. \tag{19}$$

3.1 Case $b\le 0$

In this subsection assume that $b\le 0$. By (16) and (18), for $0\le x\le 1$ and $i=0,1$ one has

$$0<E_{\delta-1,\delta+i}(bx^{\delta-1})\le E_{\delta-1,\delta+i}(0)\le 1/\theta,$$

where $\theta:=\min\{\Gamma(x):1\le x\le 2\}\approx 0.8856$. Invoking this inequality and (19) in (8) yields $\|u\|_\infty\le C$ (for some constant C) for $1<\delta<2$.

Similarly, (16) implies that $E_{\delta-1,\delta-1}(bx^{\delta-1})\le E_{\delta-1,\delta-1}(0)\le 1$; combining this inequality and (19) with (9) yields $|u'(1)|\le C$ (for some constant C) for $1<\delta<2$, so there is no boundary layer at $x=1$ as $\delta\to 1^+$ when $b\le 0$.

3.2 Case $0<b<1$

In this subsection assume that $0<b<1$.

The definition (2) yields $0 \le E_{\delta-1,i}(bx^{\delta-1}) \le E_{\delta-1,i}(b)$ for $i = 1,2$ and $0 \le x \le 1$. The analysis in [8, Subsection 2.2.3] shows that

$$\frac{1}{4(1-b)} \le E_{\delta-1,2}(b) \le \frac{1}{1-b}, \tag{20}$$

$$\frac{1-b^{1+\lfloor 1/(\delta-1)\rfloor}}{1-b} \le E_{\delta-1,1}(b) \le \frac{1}{\theta(1-b)}, \tag{21}$$

where $\theta \approx 0.8856$ was defined earlier and $\lfloor n \rfloor$ denotes the greatest integer satisfying $\lfloor n \rfloor \le n$. Similarly one has

$$0 \le E_{\delta-1,0}(bx^{\delta-1}) \le E_{\delta-1,0}(b) = \sum_{k=1}^{\infty} \frac{b^k}{\Gamma(k(\delta-1))} \le \frac{1}{\theta} \sum_{k=1}^{\infty} b^k = \frac{b}{\theta(1-b)}. \tag{22}$$

It follows from (11) and (20)–(22) that

$$\|u\|_\infty \le C$$

for some constant C whose value depends on b but is independent of δ.

By (12) and (20)–(22) we get $|u'(x)| \le C$ (where C depends on b but not on δ) for $x > c > 0$ where $c \in (0,1)$ is any fixed constant. It follows that u does not have a boundary layer at $x = 1$ as $\delta \to 1^+$ when $0 < b < 1$.

3.3 Case $b = 1$

In this subsection assume that $b = 1$.

For any constant $r \ge 0$ we have

$$\begin{aligned}
\int_{x=r}^{\infty} \frac{dx}{\Gamma(x)} = \int_{x=0}^{\infty} \frac{dx}{\Gamma(x+r)} &= \sum_{k=0}^{\infty} \int_{x=k(\delta-1)}^{(k+1)(\delta-1)} \frac{dx}{\Gamma(x+r)} \\
&= \lim_{\delta\to 1^+} \sum_{k=0}^{\infty} \frac{\delta-1}{\Gamma(k(\delta-1)+r)} \\
&= \lim_{\delta\to 1^+} (\delta-1) E_{\delta-1,r}(1),
\end{aligned}$$

where the penultimate equality holds true by the theory of Riemann sums in integration. Now Table VI of [2] gives the numerical values

$$\int_{x=0}^{\infty} \frac{dx}{\Gamma(x)} \approx 2.808, \quad \int_{x=0}^{1} \frac{dx}{\Gamma(x)} \approx 0.541, \quad \int_{x=1}^{2} \frac{dx}{\Gamma(x)} \approx 1.085,$$

so

$$\int_{x=1}^{\infty} \frac{dx}{\Gamma(x)} \approx 2.267, \quad \int_{x=2}^{\infty} \frac{dx}{\Gamma(x)} \approx 1.182.$$

Thus

$$\lim_{\delta\to 1^+}(\delta-1)E_{\delta-1,i}(1) \approx \begin{cases} 2.808 & \text{if } i=0,\\ 2.267 & \text{if } i=1,\\ 1.181 & \text{if } i=2. \end{cases} \tag{23}$$

We first deduce bounds for $\|u\|_\infty$. Observe that the first two terms in (11), viz.,

$$\gamma_1 \frac{E_{\delta-1,1}(bx^{\delta-1})-1}{E_{\delta-1,1}(b)-1+\alpha_1 E_{\delta-1,0}(b)}, \quad \frac{f}{b}\left\{x - \frac{E_{\delta-1,2}(b)-1+\alpha_1[E_{\delta-1,1}(b)-1]}{E_{\delta-1,1}(b)-1+\alpha_1 E_{\delta-1,0}(b)}\right\}$$

are bounded by some constant C, so we analyse the third term $(f/b)\{\dots\}$ with $b=1$. Invoking (23) we obtain

$$\lim_{\delta\to 1^+}\frac{E_{\delta-1,2}(1)-1+\alpha_1[E_{\delta-1,1}(1)-1]}{E_{\delta-1,1}(1)-1+\alpha_1 E_{\delta-1,0}(1)} \approx \frac{1.181+2.267\alpha_1}{2.267+2.808\alpha_1} \ge \frac{1.181}{2.808} > 0.42. \tag{24}$$

If δ is sufficiently close to 1 and $0<x<0.42$, by (24) and the trivial inequality $E_{\delta-1,1}(x^{\delta-1}) \ge E_{\delta-1,2}(x^{\delta-1})$ we have

$$\begin{aligned} &\frac{E_{\delta-1,2}(1)-1+\alpha_1[E_{\delta-1,1}(1)-1]}{E_{\delta-1,1}(1)-1+\alpha_1 E_{\delta-1,0}(1)}E_{\delta-1,1}(x^{\delta-1}) - xE_{\delta-1,2}(x^{\delta-1})\\ &\quad \ge (0.42-x)\,E_{\delta-1,2}(x^{\delta-1})\\ &\quad \ge (0.42-x)\sum_{k=0}^{\lfloor 1/(\delta-1)\rfloor}\frac{(x^{\delta-1})^k}{\Gamma(3)}\\ &\quad = (0.42-x)\,\frac{1-(x^{\delta-1})^{1+\lfloor 1/(\delta-1)\rfloor}}{2(1-x^{\delta-1})}\\ &\quad > (0.42-x)\,\frac{1-x}{2(1-x^{\delta-1})} \end{aligned}$$

because $(x^{\delta-1})^{1+\lfloor 1/(\delta-1)\rfloor} < (x^{\delta-1})^{1/(\delta-1)} = x$. But $(1-x)/(1-x^{\delta-1}) \to \infty$ as $\delta \to 1^+$ since $x>0$. Consequently $\lim_{\delta\to 1^+}\|u\|_\infty = \infty$.

We now show that $u'(1)$ blows up as $\delta \to 1^+$. Set $x=1$ in (12), multiply by $\delta-1$ then take the limit as $\delta\to 1^+$, and appeal to (23): this yields

$$\lim_{\delta\to 1^+}(\delta-1)u'(1) \approx \left[\frac{2.808(1.181+2.267\alpha_1)}{2.267+2.808\alpha_1} - 2.267\right] f\,.$$

That is, $|u'(1)| = O(1/(\delta-1))$ as $\delta \to 1^+$. Thus, the derivative of u at $x=1$ blows up as δ tends to 1^+ when $b=1$ and $f\neq 0$.

Figure 1 displays the exact solution for two values of δ when $b=1$ and δ equals 1.01 and 1.0001. Note that the scales on the vertical axes in the two plots are different and a typical boundary layer is not observed at $x=1$ although $u'(1)$ is large. In [7] a related problem (where the Riemann-Liouville derivative is replaced by a Caputo derivative) is analysed in detail.

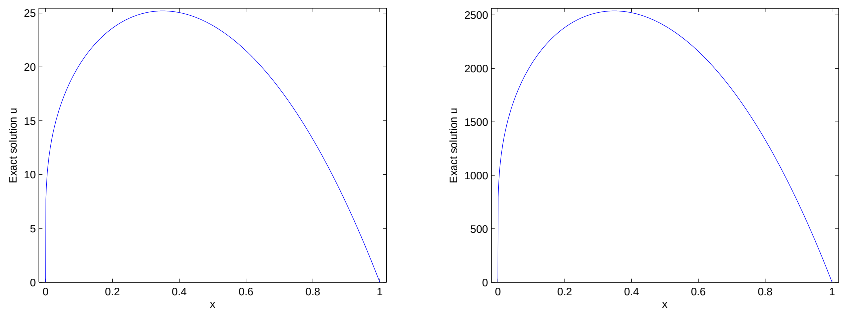

Fig. 1 Exact solution of (1) for $b = 1, f = 1, \alpha_1 = 0, \gamma_1 = 0$ and $\delta = 1.01$ (left figure) and $\delta = 1.0001$ (right figure).

3.4 Case $b > 1$

In this subsection assume that $b > 1$. We begin with a technical lemma. Recall the asymptotic relation

$$E_{\delta-1,n}(b) = \frac{1}{\delta-1} b^{(1-n)/(\delta-1)} \exp\left(b^{1/(\delta-1)}\right) + O\left(\frac{1}{(\delta-1)^2}\right) \quad \text{as } \delta \to 1^+ \tag{25}$$

of [8, (2.19)]; in this formula the index n must be fixed independently of δ.

Lemma 3.

$$\begin{aligned} &E_{\delta-1,\delta+1}(b)E_{\delta-1,\delta-1}(b) - \left[E_{\delta-1,\delta}(b)\right]^2 \\ &\qquad = O\left(\frac{1}{(\delta-1)^3} b^{1/(\delta-1)} \exp\left(b^{1/(\delta-1)}\right)\right) \quad \textit{as } \delta \to 1^+. \end{aligned} \tag{26}$$

Proof. By (10) we have

$$\begin{aligned} &E_{\delta-1,\delta+1}(b)E_{\delta-1,\delta-1}(b) - \left[E_{\delta-1,\delta}(b)\right]^2 \\ &= \frac{1}{b^2}\left[E_{\delta-1,2}(b) - 1\right] E_{\delta-1,0}(b) - \frac{1}{b^2}\left[E_{\delta-1,1}(b) - 1\right]^2 \\ &= \frac{1}{b^2}\left\{E_{\delta-1,2}(b)E_{\delta-1,0}(b) - \left[E_{\delta-1,1}(b)\right]^2 - E_{\delta-1,0}(b) + 2E_{\delta-1,1}(b) - 1\right\} \\ &= O\left(\frac{1}{(\delta-1)^3} b^{1/(\delta-1)} \exp\left(b^{1/(\delta-1)}\right)\right) \quad \text{as } \delta \to 1^+, \end{aligned} \tag{27}$$

on invoking (25), because the highest-order terms cancel and the expression in (27) is then the dominant term among those remaining.

We use Lemma 3 to analyse the behaviour of u near $x = 1$. From (9) one has

$$
\begin{aligned}
u'(1) &= \gamma_1 \frac{E_{\delta-1,\delta-1}(b)}{E_{\delta-1,\delta}(b)+\alpha_1 E_{\delta-1,\delta-1}(b)} \\
&\qquad + f\left[\frac{E_{\delta-1,\delta+1}(b)+\alpha_1 E_{\delta-1,\delta}(b)}{E_{\delta-1,\delta}(b)+\alpha_1 E_{\delta-1,\delta-1}(b)} E_{\delta-1,\delta-1}(b) - E_{\delta-1,\delta}(b)\right] \\
&= \frac{1}{E_{\delta-1,\delta}(b)+\alpha_1 E_{\delta-1,\delta-1}(b)} \times \\
&\qquad \left\{\gamma_1 E_{\delta-1,\delta-1}(b) + f E_{\delta-1,\delta+1}(b) E_{\delta-1,\delta-1}(b) - f\left[E_{\delta-1,\delta}(b)\right]^2\right\}.
\end{aligned}
$$

Here

$$
\left\{\dots\right\} = O\left(\frac{1}{(\delta-1)^3} b^{1/(\delta-1)} \exp\left(b^{1/(\delta-1)}\right)\right)
$$

by Lemma 3 and (25). From (25) we also get

$$
E_{\delta-1,\delta}(b)+\alpha_1 E_{\delta-1,\delta-1}(b) = \frac{1+\alpha_1 b^{1/(\delta-1)}}{b(\delta-1)} \exp\left(b^{1/(\delta-1)}\right) + O\left(\frac{1}{(\delta-1)^2}\right).
$$

Consequently

$$
|u'(1)| = O\left(\frac{b^{1/(\delta-1)}}{(\delta-1)^2}\left(1+\alpha_1 b^{-1/(\delta-1)}\right)\right) \quad \text{as } \delta \to 1^+. \tag{28}
$$

Thus $u'(1)$ blows up when $\delta \to 1^+$ and $b > 1$, and this behaviour is more extreme if $\alpha_1 = 0$, i.e., if there is a Dirichlet boundary condition at the endpoint $x = 1$. Figure 2 indicates that a layer appears at $x = 1$ when δ is near 1. One can prove analytically that this layer is present, but this derivation is too long to include here.

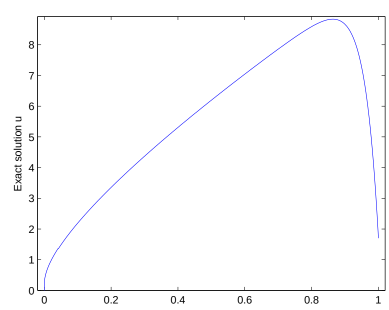

Fig. 2 Exact solution of (1) for $b = 1.1, f = 1, \alpha_1 = 0, \gamma_1 = 1.7$ and $\delta = 1.03$.

In closing, we mention that when $b > 1$, $\|u\|_\infty$ is unbounded as $\delta \to 1^+$. The analysis needed to show this resembles the analysis given above for $u'(1)$.

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