On the delay and inviscid nature of turbulent break-away separation in the high-*Re* limit

Bernhard Scheichl

Abstract We complement the recently achieved status quo of a self-consistent asymptotic theory: incompressible-flow separation from the perfectly smooth surface of a bluff rigid obstacle that perturbs an otherwise uniform flow in an unbounded domain. Here the globally formed Reynolds number, *Re*, takes on arbitrarily large values, and we are concerned with a long-standing challenge in boundary layer theory. Specifically, the external flow is sought in the class of potential flows with free streamlines, and the level of turbulence intensity, concentrated in the boundary layer undergoing separation, is measured in terms of distinguished limits. Their particular choices categorise the type of the viscous-inviscid interaction mechanism governing local separation and the strength of its downstream delay when compared with laminar-flow separation. In the case of extreme retardation, this implies the selection of a fully attached potential flow around a closed body, the singular member of the family of free-streamline flows. In turn, the asymptotic theory predicts the distance of the separation from the thus emerging rear stagnation point or trailing edge of the body to vanish at a rate much weaker than that given by $1/\ln Re$, which plays a crucial role in the scaling of firmly attached turbulent boundary layers. Notably, the overall theory only resorts to specific turbulence closures when it comes to numerical studies.

1 Motivation, global potential and attached shear flow

Gross separation of a nominally two-dimensional (most developed) turbulent boundary layer (BL) in subsonic flow around a bluff body in the limit of large values of

Bernhard Scheichl
Vienna University of Technology, Institute of Fluid Mechanics and Heat Transfer, Tower BA/E322, Getreidemarkt 9, 1060 Vienna, Austria, e-mail: bernhard.scheichl@tuwien.ac.at
AC²T research GmbH (Austrian Center of Competence for Tribology), Viktor-Kaplan-Straße 2/C, 2700 Wiener Neustadt, Austria, e-mail: scheichl@ac2t.at

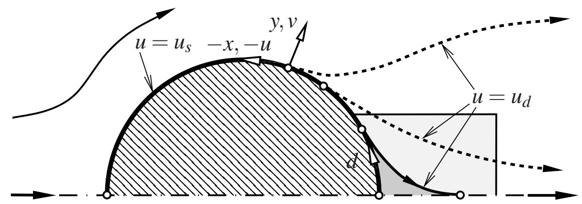

Fig. 1 Structure of external HK flow for k increasing from moderate to large values (open cavity: free streamlines *dashed*, finite cusp-edged cavity: *light-shaded*), subregion of extent d: *very light-shaded*, not to scale.

the globally formed Reynolds number, Re, has regained awareness in the last years, after different questions have attracted research in the asymptotic description of high-Re flows for more than two decades — not unlikely owing to the difficulty of that particular subject. According references are [4, 6–9]; also note citations therein.

This contribution forges a bridge from reappraising the theory as available in the above references to scrutinising some subtle (open) questions of the flow structure in more depths and breadths and, finally, addressing new, hitherto unpublished results. Let us first give a brief overview on the central findings and the current status of the theory. The situation outlined is sketched in Fig. 1, referred to tacitly in the following and with the notations introduced in the subsequent excursus on the external flow.

The asymptotic concept ties in with the well-established theory of laminar separation, cf. [10], as the intensity of turbulence, concentrated in the BL, relative to that characterising a (hypothetical) fully developed BL is measured by some gauge factor, T: $0 \leq T \leq 1$. Its dependence on Re and the closely associated question of the correct scaling of the attached flow that allows for a self-consistent description of the separation process further downstream have posed major challenges in the establishment of the present theory; tackling this was inspired by the scenario of a strictly laminar flow ($T = 0$).

Overall flow structure

Hence, the initial degree of freedom introduced by the quantity T is equivalent to parametrise the imposed potential flow of Helmholtz–Kirchhoff (HK) type by the two positions where the free streamlines, confining a stagnating-fluid cavity, depart from the body surface and which collapse with the separation points as $Re \to \infty$. As our interest here is with the local picture of separation, it is sufficient to consider symmetric (circulation-free) flows, thus around symmetric bodies. Alternatively, HK flows then are (uniquely) described by a single parameter, k (> 0), that controls the strength of the singularity encountered by the surface pressure [10] and increases monotonically for increasing distance between the points of separation and front stagnation point along the surface. Simultaneously, the turbulence intensity increases, i.e. retards separation. Finally, the separation points merge at a rear stagnation point formed in the singular limit $k \to \infty$. Here the HK flow represents a perturbation of the fully attached potential flow and a near-surface subregion having an extent of $O(1/k^2)$ and enclosing a correspondingly small, cuspidal cavity [6, 11]. It is this flow regime of excessive delay and the highest turbulence intensities possible where formulating a proper distinguished (least-degenerate) limit that captures the essential features of the flow not only involves Re and T but also k. Due to the emergence of a further stagnation point, the situation of ultimate delay is interpreted as the (symmetric) collision of two turbulent BLs, forming a slender jet breaking off the surface and two tiny recirculation bubbles [7].

One must concede that, for the canonical case of a circular cylinder in cross-flow, the comparison of the theoretical predictions with those extracted from experiments and/or simulation of the full Navier–Stokes equations still suffers from the rather moderate values of Re employed in the latter activities. Specifically, in experiments Re can still hardly exceed a long-standing threshold of 8.89×10^6 [6], and the data are rather scarce in the regime $Re \gtrapprox 10^6$ where the attached BL can be considered as turbulent almost from stagnation upstream on. However, this on the other hand renders the asymptotic theory attractive in the light of engineering applications.

We start the analysis with the Reynolds- or time-averaged Navier–Stokes equations for incompressible flow of uniform fluid density and viscosity, most concisely written in Einstein notation making use of covariant derivatives [5]:

$$u^i|_i = 0, \quad u^j u^i|_j = -p|_i - \langle u_f^i u_f^j \rangle|_j + Re^{-1}\, u^i|_{jj} \quad (i,j = 1,2)\,. \tag{1a,b}$$

Herein x^i, u^i, u_f^i, p denote natural (contravariant) coordinates along ($i = 1$) and perpendicular from ($i = 2$) the closed body contour, the corresponding (contravariant) components of the nominal flow velocity, those of the corresponding turbulent fluctuations, and the pressure difference with respect to potential-flow detachment, respectively. All lengths are non-dimensional with a typical body dimension (a radius of surface curvature), the flow speed with that of the unperturbed parallel flow, and p and the Reynolds stress tensor $-\langle u_f^i u_f^j \rangle$ with this speed squared times the density; those reference values together with the kinematic viscosity define Re. Equations (1) are supplemented with the usual adhesion condition $u^i = u_f^i = 0$ for $x^2 = 0$. We next revisit the external and the attached BL flow that arise in the singular limit $Re \to \infty$.

External potential flow

With the small parameter $\varepsilon \ll 1$, defined in Sect. 2.1, measuring the magnitude of u^1 inside the BL and the value δ_d of the BL thickness, $\delta (\ll 1)$, at separation, we expand

$$[u^i, p] \sim [u_0^i, p_0](x^1, x^2; k) + \varepsilon \delta_d\, [u_1^i, p_1](x^1, x^2; k) + O(\varepsilon^2 \delta_d)\,. \tag{2}$$

Here the first two terms represent the imposed HK flow and its irrotational perturbation induced by the BL displacement. The HK flow exerts a surface slip on the BL: $u_s(x;k) := u_0^1(x^1, 0; k)\ (> 0)$; the local variable $x := x^1 - x_d(k)$ means the streamwise distance from potential-flow detachment at $x^1 = x_d(k)$, say. With $u_d(k) := u_s(0;k)$ (≤ 1), the associated singularity assumes the well-known canonical form [10]

$$u_s(x;k) \sim u_d(k) \left[1 + 2k\sqrt{-x} + 10k^2(-x)/3 + O(x^{3/2})\right] \quad (x \to 0_-)\,, \tag{3}$$

and the flow speed equals u_d along a free streamline. Denotes $d(k)$ the streamwise distance from flow detachment to stagnation at the trailing edge existing for $k = \infty$,

$$u_d = O(d)\,, \quad d \sim 1/(6k^2) \quad (k \to \infty) \tag{4}$$

expresses the related retardation and a rebirth of expansion (3) in the aforementioned subregion where x, $y := x^2$ are of $O(d)$ and two free streamlines encompass a tiny cavity [6].

As their effects on (3) and the BL flow proves negligibly small, we disregard surface curvature and (initially) surface roughness when it comes to the flow description on the BL scale for sufficiently small values of y. Therefore, x and y are taken as Cartesian coordinates and $[u,v] := [u^1,u^2]$ as the associated velocity components for $y \ll 1$. We thus write $u_{0,1} - \mathrm{i}v_{0,1} = u_d\, w'_{0,1}(z;k)$, $z = x + \mathrm{i}y$ with complex flow potentials $w_{0,1}(z;k)$. The behaviour (3) and the local surface pressure agree with

$$[w_0, p_0] \sim [z - 4\mathrm{i}k/3\, z^{3/2} + O(z^2),\, -2k\,\Im(z^{1/2}) + O(z)] \quad (z \to 0)\,, \tag{5a,b}$$

$$w_1 \sim az + bz^{1/2} + O(z) \quad (a, b \in \mathbb{R}\,,\ z \to 0) \tag{5c}$$

as $\delta = O(x)$ $(x \to 0_+)$ for the free shear layer. We fix the coefficients a, b in Sect. 2.1.

Attached boundary layer flow

The continuity equation (1a) is satisfied identically by $[u,v] = [\psi_y, -\psi_x]$ where ψ is a streamfunction. Now $[\sigma_x, \sigma_y] := -[\langle u_f^2\rangle, \langle u_f v_f\rangle]$ are the Reynolds stresses, δ is the local thickness of the BL. Following [2, 8], this is initially two-tiered and governed by a single small turbulent velocity scale, $u_t(x;Re) := \sqrt{Re^{-1}\,\partial_y u|_{y=0}}$, i.e. the local skin friction velocity. Thus, the (first unknown) intensity gauge factor T here relaxes the classical asymptotic structure of a firmly attached, fully developed turbulent BL.

This structure holds even for compressible flow as long as the Mach number formed with u_t and the speed of sound evaluated at the surface is small [1].

The outer, largely inviscid main region of the BL is characterised by the coordinate $\eta := y/\delta$ and a small streamwise velocity deficit of $O(u_t)$:

$$\left\{\Big[\frac{u_s y - \psi}{u_t \delta}, \frac{\sigma_y}{T u_t^2}\Big], \frac{\delta}{T\gamma}\right\} \sim \big\{[F, \Sigma](x, \eta; k),\, \Delta(x;k)\big\} + O(\gamma) \quad \Big(\gamma := \frac{u_t}{u_s} \to 0\Big)\,. \tag{6}$$

The scaled streamfunction F, shear stress Σ, and BL thickness Δ are $O(1)$-quantities and satisfy the accordingly expanded form of the x-component of (1b) $(i = 1)$.

In the so-called viscous wall layer molecular and Reynolds shear stresses are both of $O(u_t^2)$. From this its scaling in the common "+"-representation

$$[\psi/(u_t\delta_v),\, \sigma_{x,y}/(T u_t^2)] = [\psi^+,\, \sigma^+_{x,y}](s, y^+; Re)\,, \quad y^+ := y/\delta_v\,, \quad \delta_v := 1/(Re T u_t) \tag{7}$$

ensues. The streamwise component of (1b) $(i = 1)$ is then rewritten as

$$\sigma_y^+ + \frac{\partial^2\psi^+}{\partial^2 y^+} \sim 1 + p^+ y^+ + \frac{\delta_v}{T}\left[\frac{1}{u_t}\int_0^{y^+} \frac{\partial^2(u_t\psi^+)}{\partial y^+ \partial s}\,\mathrm{d}t - \frac{\partial\psi^+}{\partial s}\,\frac{\partial^2\psi^+}{\partial y^+\partial s}\right] + O(\delta_v) \tag{8}$$

after integration with respect to y. Here p^+ represents the imposed pressure gradient,

$$p^+ = -u_s u_s'/(ReTu_t^3)\ , \tag{9}$$

and we have anticipated that $T \ll 1$ since the integrated form of $(\delta_v/u_t^2)\partial(u_t^2\sigma_x^+)/\partial x$ gives the dominant neglected remainder term in (8). Matching σ_y in both layers subject to the for $y^+ \to \infty$ vanishing viscous term on the left-hand side of (8) confirms the above identification of u_t. Hence, the right-hand side of (8) starts with the rescaled skin friction, and p^+ and δ_v/T, measuring the strength of the inertia terms in (8), appear to be small so that the near-wall flow is termed a developed one:

$$[\psi^+, \sigma_y^+] \sim [\psi_0^+, \sigma_{y0}^+](y^+) + O(p^+)\ ,\quad [\psi_0^+, \sigma_{y0}^+] \sim [y^+ \ln y^+/\kappa,\, 1]\ \ (y^+ \to \infty)\ , \tag{10a,b}$$

matching $\partial u/\partial y$ gives the celebrated logarithmic law of the wall in (10b), involving the von Kármán constant κ; matching u provides the closure-free skin friction law

$$\gamma \sim \varepsilon - (2/\kappa)\varepsilon^2 \ln \varepsilon + O(\varepsilon^2) \quad [\mathrm{d}\gamma/\mathrm{d}x = O(\varepsilon^2)]\ ,\quad \varepsilon := \kappa/\ln(T^2 Re)\ . \tag{11}$$

We stress that $u_s = O(1)$ initially. A distinct deviation of the BL from a laminar one, having a lateral extent of $O(Re^{-1/2})$, means a predominance of the Reynolds over the viscous stresses in its main portion, simultaneously implying the two-layer splitting and a small velocity deficit as $\varepsilon \ll 1$ or $T \gg Re^{-1/2}$. Thus (9), (11) predict $p^+ = O\big[(\ln Re)^3/(TRe)\big]$, which completes the analysis of the BL at this stage.

Having recapitulated the structure of the wall layer in the spirit of [8], we are able to study gross separation, commencing as x becomes sufficiently small, in a most generic manner. As a remarkable fact of the asymptotic concept, it is this local mechanism,that fixes T and, according to (6), the thickness δ of the oncoming BL.

2 Moderate delay: generic triple deck applied to wall flow

Since self-induced separation of strictly laminar flow requires $k = O(Re^{-1/16})$ [10], one deals with the least-degenerate case by assuming $T \gg Re^{-1/2}$ and $k = O(1)$, which shall coin the notion of *moderately* retarded separation. A central question is whether regularising the potential-flow singularity (5) singles out a (unique) value of k, i.e. fixes the distance x_d and, finally, the picture of separation at the body scale.

Three key observations made at incipient separation [8] deserve a critical review:

1. the velocity defect remains of $O(\varepsilon)$ in the predominantly inviscid main layer;
2. at its base, a Reynolds-stress blending layer might form;
3. the viscous wall-layer flow remains fully developed to leading order.

The first and second issue are envisaged next; their careful investigation alleviates the insight why and how the third inevitably invokes the formation of the triple-deck (TD) structure addressed, embedded in but largely unaffected by the core flow.

2.1 Small-defect Euler stage

The streamwise scale shortens under the action of (3). Then the corresponding growth of the surface-normal pressure gradient is the only new effect that becomes relevant as in the bulk of the BL. Furthermore, F, Σ, Δ, and hence δ attain finite limits as $x \to 0_-$ we indicate by a subscript d. Matching the oncoming BL described by (6) for $x \to 0_-$ and the potential flow as given by (5), one finds $a = -F_{d1}$, $b = -4F_{d1}$, $F_{d1} := F_d(1;k)$ [8]. This furthermore implies $\delta_d = O(T\varepsilon)$, $\sigma_{x,y} = O(\delta_d)$ in the so evoked square region described by $(X,Y) := (x,y)/\delta_d = O(1)$ where the BL limit ceases to be valid as $\partial_y p/\partial_x p = O(1)$. Inspection of (1,*b*) shows that the Reynolds stress gradients do not enter the problems for the coefficient functions $\psi_{0,1,2,3}(X,Y;k)$, $P_{2,3}(X,Y;k)$ in the arising double expansion

$$[\psi/(u_d\delta_d) - \Psi_0 - \varepsilon\Psi_1,\, p] \sim \sqrt{\delta_d}\,[\Psi_2, P_2] + \varepsilon\sqrt{\delta_d}\,[\Psi_3, P_3] + O(\varepsilon^2, \delta_d) \tag{12}$$

(and a corresponding one of δ). Then (12) states a level of approximation governing an Euler stage: once the first integral $(\partial_{xx} + \partial_{yy})\psi \sim -\omega(\psi)$, $\omega \sim \varepsilon u_d F_d''(Y;k)$ is the vorticity due to the incident BL, of the vorticity transport equation derived from (1) is solved in accordingly expanded form, $P_{2,3}$ follow by virtue of Bernoulli's law.

The expansion process reveals the hierarchy of Dirichlet problems:

$$(\partial_{XX} + \partial_{YY})[\Psi_{0,1,2,3}] = [0, -F_d'', 0, -\Psi_2 F_d'''(Y;k)]\,, \quad \Psi_{0,1,2,3}|_{X\le 0, Y=0} = 0\,, \tag{13a,b}$$

and $\Psi_{0,1,2,3}$ subject to conditions of matching with the external flow ($Y \to \infty$) in view of (5) and the BL flow upstream ($X \to -\infty$); those with the separated shear layer emerging for large values of X and the missing near-wall conditions for $X > 0$ are considered in the course of the analysis. First, one detects a "frozen" small velocity defect as $[\Psi_0, \Psi_1] = [Y, -F_d(Y;k)]$: any further harmonic contributions to $\Psi_{0,1}$ vanish in the above limits and exhibit zero Y-derivatives for $X = 0$ due to the vanishing associated pressure variations; so they vanish at all by (13*b*).

Homogeneous second-order problem

The behaviour (5*a*) suggests $\Psi_2 = -(4k/3)\Re[Z^{3/2}W(Z;k)]$, $Z := X + \mathrm{i}Y$ $(Y \ge 0)$ with some holomorphic function W satisfying $W \to 1$ $(Z \to \infty)$ and $\Im W = 0$ $(X \le 0)$, see (13*b*). Because (5*c*) expresses zero potential-flow pressure acting on the separated BL and the pressure in the adjacent recirculation region can only originate in the weak viscous backflow, linearised Bernoulli's law gives $\partial_Y \Psi_2|_{Y=0} = \Im W = 0$ $(X > 0)$. We then arrive at Wiener–Hopf-type boundary conditions on the real axis, typically describing separation of high-*Re* flows at an inviscid scale, cf. [10]. Now any non-trivial $\Im W$ vanishes at $Z = \infty$ $(Y > 0)$ and on the X-axis apart from the origin where $\Im W$ and thus $\Re W$ must be singular. But such a strengthening of the potential-flow singularity (5*a*,*b*) by the Euler stage has to be discarded. We then have $\Im W \equiv 0$, $W \equiv 1$; (5*a*,*b*) remains unaltered first: no viscous effects, no regularisation process.

Inhomogeneous third-order problem

The chance that the Euler limit is at least capable to settle the value of k is attributed to the arising Poisson problem governing Ψ_3, cf. (13). This attracted attention first in the context of turbulent trailing-edge flows in terms of a semi-analytical/numerical treatment [3]; note that F_d'' is given by the numerical solution of the leading-order BL problem governing the flow upstream with $F_d(0;k)=0$ and $F_d' \equiv 0$ ($Y \geq 0$, i.e. outside the BL). However, the formal change of variables $(X,Y)=(Z+\mathrm{i}Y,Y)$, $\Psi_3=(4k/3)\Re W_3(Z,Y;k)$ gives Ψ_3 in closed form as elucidated tersely next.

By taking into account the form of Ψ_2, we arrive at the problem for W_3:

$$\partial_Y(\partial_Y+2\mathrm{i}\partial_Z)W_3=-Z^{3/2}F_d''(Y;k)\,, \quad W_3 \sim 3\mathrm{i}F_{d1}Z^{1/2} \quad (Z\to\infty\,,\ Y\geq 1) \quad (14a,b)$$

with (14b) representing the match with the induced potential flow according to (5b). Once (14a) is integrated with respect to Y (Z fixed), the further change of variables $(Z,Y)=(Z^*+2\mathrm{i}Y,Y)$ allows for the second integration, which finally yields

$$\Psi_3=\frac{4k}{3}\Re\left[Z^{3/2}F_d'(Y)-3\mathrm{i}\int_0^Y\sqrt{Z-2\mathrm{i}(Y-t)}\,F_d'(t)\,\mathrm{d}t\right]. \quad (15)$$

Herein the bracketed term satisfies (14) and is of $O(Z^{3/2}\ln Z)$ as $Z\to 0$ where the match between F and ψ_0^+ according to (10b) accounts for the logarithmic variation. Hence, Ψ_3 given by (15) not only meets (13b) but also two further requirements:

a) $(\partial_Y\Psi_3+F_d''\Psi_2)_{Y=0,X>0}=0$, arising by Bernoulli's law and complementing (13b) when we assume that even $P_3(X,0;k)$ vanishes downstream of separation,
b) $W\sim BZ^{1/2}$ ($Z\to 0$) for some constant $B\in\mathbb{R}$.

Let us for the moment accept b). Then any contribution $\bar{\Psi}$, say, to Ψ_3 adding to (15) is harmonic and satisfies Wiener–Hopf boundary conditions as does Ψ_2. These and the fact that $\bar{\Psi}$ must be of $o(Z^{1/2})$ as $Z\to\infty$ gives $\bar{\Psi}\sim c|Z|^{-1/2}\cos(\arg(Z)/2)$ in this limit with $\pi\geq\arg(Z)\geq 0$ and some constant C. In analogy to the above analysis of Ψ_2, writing $\bar{\Psi}=\Re[Z^{-1/2}\bar{W}(Z;k)]$ means $\Im\bar{W}=0$ for $Y=0$ and $Y=\infty$ and thus $\Im\bar{W}\equiv 0, \bar{W}\equiv C$ for the newly introduced analytic function $\bar{W}$. Avoiding the singularity at $Z=0$ means $C=0$, so (15) is the only acceptable representation for Ψ_3.

As an important finding, the absence of a singularity stronger than that indicated by item b) renders (15) valid for all admissible values of k: the originally desired selection does not take place. In fact, one can readily demonstrate that a slight shift of the origin $x=0$ along the smooth body surface by weakly disturbing a prescribed value of k only shifts the local asymptotic structure of the flow correspondingly. This ambiguity contrasts with separation from a fixed corner, yielding a unique HK flow.

In the aforementioned near-wall behaviour of F', only at separation a half-power term follows the conventional leading terms: $F_d'\sim -\kappa^{-1}\ln Y+C+O(Y^{1/2})$ ($Y\to 0$, with some $C(k)>0$). It reflects the emergence of an intermediate layer. This is provoked by the singular behaviour (3) and accounts for the blending of the then, according to (11), rapidly varying value of σ_{y0}^+ on top of the wall layer and the "frozen" value 1 of Σ as $Y\to 0$. However, that new sublayer is meaningless in the current setting because it only applies for $\delta \gg Re^{-1/3}$ [8] but the subsequent TD analysis gives the opposite. Even more important, the flow there is still predominantly inviscid as the relative velocity defect is still of $O(\varepsilon)$, but the regularisation of the potential-flow singularity in (5) is accomplished entirely by the BL-internal TD. This is different from the situation for large values of k in Sect. 3 where the regularisation due to a self-induced pressure takes place at the scale of the inviscid-flow.

2.2 Triple-deck structure

The predominance of inertia terms and stress gradients in the main and the wall layer, respectively, have these only interacting loosely, even around separation. In turn, the viscous wall-layer flow is subject to a distinctly different shortening of streamwise scales. Specifically, inspection of (7), (9), (11) shows that in the linearised version of (8) according to (10a) the dominant perturbations of $O(p^+)$ govern a balance of the stress and pressure gradients with the convective terms as $-x$ has decreased to $O[1/(ReT^2\varepsilon u_d)]$. In (8) inertia terms then dominate for smaller values of $-x$, but, according to (3), even those disturbances attain finite limits [8].

This scenario for developed viscous flow and a strong adverse pressure gradient is tied in with the one that occurs when a non-developed viscous BL flow is subjected to a driving surface pressure $p_s(x;k) := p_0(x,0;k)$ that stays finite as $|p_s'| \to \infty$ $(x \to 0_-)$. We distinguish if variations of p_s are small, case A), or of $O(1)$, case B):

A) laminar BL, p_s' mostly adverse where $p_s' \to 0$ as $Re \to \infty$ avoids the well-known non-removable Goldstein singularity further upstream;
B) (i) laminar BL (then p_s' favourable, prominent example: aforementioned separation from concave corner [10, chap. 2.1]), (ii) (developed) turbulent wall layer.

The no-slip condition requires the BL equations remaining fully intact in a so-called lower deck (LD) around the origin $x = y = 0$ and where $(x,y) := O(\delta_{TD}, \delta_{LD})$, say. In order to regularise the otherwise singular behaviour of p and the skin friction when changing sign, however, p no longer equals p_s to leading order but is induced in a square so-called upper deck (UD) of extent δ_{TD} on top of the so formed main deck (MD). This just continues the layer upstream addressed in cases A) and B). In both we now assign the thickness δ_v and the velocity scale u_t to it. Evaluating ψ for $x = 0_-$ then describes the predominantly inviscid flow free of pressure variations in the MD, cf. the arguments underlying the forms of Ψ_0, Ψ_1 subsequent to (13). The displacement of the streamlines by the LD is shifted unaltered towards the UD, there in turn causing inviscid-flow disturbances. This mechanism completes the feedback cycle of the ensuing TD, for laminar flow studied exhaustively e.g. in [10].

The remainder of this section is devoted to the general scaling of the TD, see Fig. 2a, deduced by inspection analysis, specifically, the dependences of δ_{TD}, δ_{LD} on Re in case B). For the sake of conciseness, "$\sim$" now replaces the Landau symbol.

Denotes u_{LD} a velocity scale for the LD, evaluation of (1b) for $i = 1$ there yields

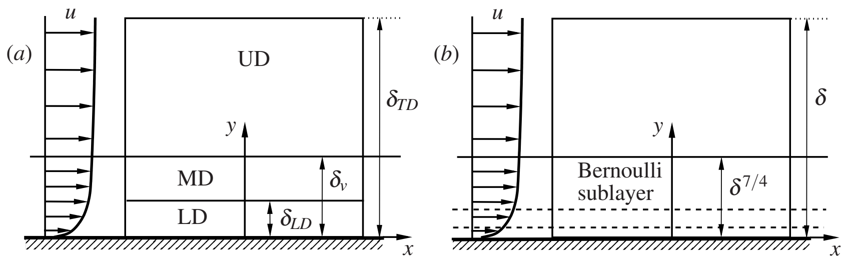

Fig. 2 Structure of triple deck (*a*) vs. of Rayleigh stage, here oncoming BL three-tiered (*b*).

$$u\,\partial_x u + \cdots \sim -\partial_x p + Re^{-1}\partial_{yy}u\,, \quad u_{LD}/\delta_{TD} = 1/(Re\,\delta_{LD}^2)\,. \qquad (16a,b)$$

We derive two further relations involving the first unknown scales u_{LD}, δ_{LD}, δ_{TD}, δ_v:

$$u_{LD} = u_t(\delta_{LD}/\delta_v)^r \quad (r>0)\,, \quad \delta_{TD}^3/\delta_{LD}^5 = Re^2 u_d\,. \qquad (17a,b)$$

The first reflects the wall-normal rise of u in the LD into the overlap with the MD. The predominance of the inertia terms there has u varying proportional to $u_t(y+A)^r$ in leading order, with r depending on the behaviour of the oncoming flow as $x \to 0_-$ and the x-dependence of the displacement function A being part of the solution of the TD problem. In turn, $u_t\delta_{LD}/\delta_v$ gives the magnitude of the u-, $u_t\delta_{LD}$ that of the ψ-perturbation exerted by the LD displacement in the MD. The latter produces a v-disturbance $-\partial_x\psi$, provoking u- and p-disturbances of the same order of magnitude $u_d\delta_{LD}/\delta_{TD}$ by linearisation of the flow about the locally frozen oncoming one; we either identify u_t with u_d (laminar BL) or use (11) (turbulent wall layer) and consider (10*b*) when matching the MD and the UD with a passive buffer layer. Those p-variations react on the LD, giving $u_{LD}^2 \sim u_d\delta_{LD}/\delta_{TD}$ by (16*a*) and (17*b*) by (16*b*).

In case A), we set $\delta_v = Re^{-1/2}$ and $r=1$ as u in the LD matches $u \sim y$ at the base of the unperturbed BL upstream with positive skin friction. Then (16*b*), (17) imply the conventional TD scaling. We recall that $u^2 \sim p \sim Re^{-1/4}$ in the LD: for massive separation, (3) gives $p \sim u_d^2 k\delta_{TD}^{1/2}$ and the classical results $k \sim Re^{-1/16}$, $u_d \sim 1$.

In case B), we are concerned with an external HK flow parametrised by k typically independent of Re; the relationship for p in the LD holds as in case A). Here (16) gives $u_d^2 k\delta_{TD}^{1/2} \sim \delta_{TD}^2/(Re^2\delta_{LD}^4)$. From this and (17*b*) we extract

$$\delta_{TD} = u_d^{-4/3}k^{-10/9}Re^{-4/9}\,, \quad \delta_{LD} = u_d^{-1}(kRe)^{-6/9}\,. \qquad (18)$$

In the most interesting situation (ii), we have again $r=1$ in (17*a*). Eliminating u_{LD} with the aid of (16*b*) yields the estimate for δ_v, using (11), (7), (6) completes the scaling even of the oncoming turbulent BL and shows that the submerged TD of case B) here implies $\delta_{TD}/\delta \sim 1/\ln Re$ (k fixed) only:

$$\delta_v = u_d^{-2/3}k^{4/3}\varepsilon Re^{-5/9}\,, \quad T = u_d^{-1/3}k^{-4/3}\varepsilon^{-2}Re^{-4/9}\,, \quad \varepsilon \sim \kappa/(9\ln Re)\,. \qquad (19a,b,c)$$

These considerations apply also to nominally unsteady flows (i.e. in the LD where δ_{TD}/u_{LD} forms the time scale), in case A) also to supersonic external flows.

In the present case B) (ii), the TD problem is cast into standard form independent of k, i.e. just depending on $(\hat{x},\hat{y}) := (x,y)/\delta_{TD}$, $\bar{y} := y/\delta_{LD}$, and first devised and solved numerically for laminar gross separation [8]. The translational invariance of its solution against a $\hat{x}$-shift agrees with the indeterminacy of k. In the UD, we expand $\psi \sim \delta_{TD} u_d \hat{y} + \cdots + (4\delta_{TD}^{3/2} k/3)\,\hat{\psi}(\hat{x},\hat{y}) + \cdots$ and $\hat{\psi}$ being part of the solution as $\hat{\psi} \sim -\Re(\hat{z}-\hat{b})^{3/2}$ $(\hat{z} \to \infty)$ with a constant $\hat{b} \in \mathbb{R}$ condensing the origin shift. That form is due to the match with (12) and the singularity (5a) recovered by Ψ_2 and justifies a) and b) in Sect. 2.1: $\hat{b} = 0$ by (15). This fixes the location of separation with an accuracy of $O(\delta_{TD})$ — once a self-consistent global flow picture has fixed k.

3 Strong delay: Rayleigh stage and beyond

The downstream-moving TD first shrinks as k increases to $O(1)$ to admit the "immersed" structure of case B) (ii) and grow again for $k \gg 1$ by (18), (19) and (4) so that its substructure breaks down for d varying algebraically with Re. However, its invalidity is already encountered in a new limit $d \sim 1/k^2 \sim \varepsilon^{1/2}/(-\ln\varepsilon)^{1/4}$ fixing d [6]: the BL has grown as the linear decay of u_s signals the advent of the rear stagnation point such that the relative velocity defect is locally of $O(1)$ and the scenario of Sect. 2.1 no longer applies. This finally gives rise to a Rayleigh stage around separation or the much weaker constraint $T \ll 1/\ln Re$ rather than (19b) [6]: Fig. 2b. Here the fully nonlinear inertia terms in (1b) ($i = 1$) are retained in the sublayer where Bernoulli's law prevails. Solving the (homogeneous) Rayleigh problem subject to Wiener–Hopf-type boundary conditions is a topic of the current activities.

Increasing k (and T) further raises a novel stage of ultimately delayed separation [7] with the vorticity again convected by the stagnating external flow as in (14a,b).

Regarding moderately retarded separation, unsettled questions concern the merge of the free shear layers closing the stagnation zone for larger values of k (cf. Fig. 1) and and the additional delay by distributed wall roughness markedly modifying (10).

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