Use of standard difference scheme on uniform grids for solving singularly perturbed problems under computer perturbations

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Abstract A Dirichlet problem is considered for a singularly perturbed ordinary differential convection-diffusion equation with a perturbation parameter ε ($\varepsilon \in (0,1]$) multiplying the highest-order derivative in the equation. This problem is approximated by the standard monotone finite difference scheme on a uniform grid. Such a scheme does not converge ε-uniformly in the maximum norm when the number of grid nodes grows. Moreover, under its convergence, the scheme is not ε-uniformly well conditioned and stable to data perturbations of the discrete problem and/or computer perturbations. For small values of ε, perturbations of the grid solution can significantly exceed (and even in order of magnitude) the error in the unperturbed solution. For a computer difference scheme (the standard scheme in the presence of computer perturbations), technique is developed for theoretical and experimental study of convergence of perturbed grid solutions. For computer perturbations, conditions are obtained (depending on the parameter ε and the number of grid intervals N), for which the solution of the computer scheme converges in the maximum norm with the same order as the solution of the standard scheme in the absence of perturbations.

1 Introduction

Numerical methods using classical finite difference schemes on uniform grids (standard difference schemes) (see., e.g., [3, 6] and the bibliography therein) are widely used for solving complicated theoretical and applied problems. The problems with boundary layers constitute a large class among such problems. Only for some (rather narrow) class of singularly perturbed problems, special numerical methods are developed in which accuracy of solutions in the maximum norm does not depend on

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the parameter ε defining the width of the boundary layer (see., e.g., [2, 4, 5, 7, 8] and the bibliography therein). For this reason, for solving large classes of problems with boundary layers, as a rule, standard numerical methods are applied.

It is known that in the case of singularly perturbed problem, the solution of the standard scheme in the absence of perturbations converges with the growing number of grid nodes, provided that the step-size h in the mesh across the layer is much less than the value of the perturbing parameter ε ($\varepsilon \in (0,1]$, $h = 1/N$, $h \ll \varepsilon$, N is the number of mesh intervals). For the modern "advanced" supercomputers, in general, this theoretical condition seems not much restrictive. However, here a new problem appears. Standard schemes do not converge ε-uniformly, moreover, under their convergence, they are not ε-uniformly stable to data perturbations; in the case of the convection-diffusion problem, see, e.g., [8, 9, 10, 11, 12] and the bibliography therein.

In the process of solving the grid problem on a computer, perturbations of the grid solution arise caused by computer perturbations. For small values of ε, such perturbations of the grid solution can be comparable with the error in the unperturbed grid solution and even significantly exceed this error. Thus, the applicability of standard schemes for numerical solving singularly perturbed problems requires a detailed study.

In the present paper, in the case of the Dirichlet problem for a singularly perturbed ordinary convection-diffusion differential equation, an approach is proposed to the development of technique for theoretical and numerical study of grid solutions in the presence of computer perturbations. Results of numerical experiments, illustrating the theoretical results, are presented and discussed. A number of results related to the research subject is published in the papers [8, 9, 10, 11, 12]. Detailed results of the research will be presented in the journal publication.

2 Problem formulation; standard difference scheme

On the set $\overline{D} = D \cup \Gamma$, $D = (0,1)$, we consider the Dirichlet problem for the singularly perturbed ordinary differential convection-diffusion equation [1]

$$L_{(1)}u(x) \equiv \left\{ \varepsilon a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} - c(x) \right\} u(x) = f(x), \quad x \in D, \tag{1}$$

$$u(x) = \varphi(x), \quad x \in \Gamma.$$

Here $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are the left and right parts of the boundary Γ; the functions $a(x)$, $b(x)$, $c(x)$, $f(x)$ are assumed to be sufficiently smooth on $\overline{D}$,

[1] The notation $L_{(j)}$ ($M_{(j)}$, $G_{h(j)}$) means that these operators (constants, grids) are introduced in formula (j).

moreover[2]

$$m \le a(x), b(x), c(x) \le M, \quad |f(x)| \le M, \quad x \in \overline{D}, \quad |\varphi(x)| \le M, \quad x \in \Gamma,$$

the parameter ε takes arbitrary values in $(0,1]$. For small values of the parameter ε, a boundary layer appears in a neighborhood of the set Γ_1.

We consider a standard difference scheme on the uniform grid $\overline{D}_h$ with the step-size $h = 1/N$, where $N+1$ is the number of nodes $x = x^i$ in the grid $\overline{D}_h$, $i = 0, 1, \ldots, N$. Problem (1) is approximated by the difference scheme [6]

$$\Lambda z(x) \equiv \{\varepsilon\, a(x)\, \delta_{\overline{x}\widehat{x}} + b(x)\, \delta_x - c(x)\} z(x) = f(x), \;\; x \in D_h, \tag{2}$$

$$z(x) = \varphi(x), \;\; x \in \Gamma_h;$$

here $D_h = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$, and $\delta_{\overline{x}\widehat{x}} z(x)$ is the central second-order difference derivative, and $\delta_x z(x)$ is the forward first-order difference derivative.

For the error in the grid solution $z(x) - u(x)$, we have the following estimate under the conditions of Theorem 1 below (similar to estimate (3) in [10]):

$$\|u - z\|_{\overline{D}_h} \le M \left(\varepsilon + N^{-1}\right)^{-1} N^{-1}, \tag{3a}$$

from which, under the condition

$$N^{-1} = o(\varepsilon), \tag{3b}$$

the next estimate follows:

$$\|u - z\|_{\overline{D}_h} \le M \nu; \quad \nu = \nu(\varepsilon, N) = \varepsilon^{-1} N^{-1}; \tag{3c}$$

here ν is the parameter of accuracy for the unperturbed difference scheme.

The following theorem on convergence of the standard difference scheme (2) holds (similar to Theorem 1 from [12]):

Theorem 1. *Let the solution $u(x)$ of the problem (1) satisfy the estimate*

$$|d^k/dx^k\, u(x)| \le M\, (1 + \varepsilon^{1-k} + \varepsilon^{-k} \exp^{-m\,\varepsilon^{-1} x}), \quad x \in \overline{D}, \;\; k \le K, \;\; K = 3.$$

Then the solution of the standard finite difference scheme (2) converges to the solution $u(x)$ of the boundary value problem provided (3b) with the estimate (3c); for the solution of the scheme, estimate (3a) holds.

[2] By M (or m), we denote sufficiently large (small) positive constants independent of the parameter ε and of the discretization parameters.

3 Standard difference scheme in the presence of computer perturbations

3.1 Matrix forms of difference schemes

Standard difference scheme (2) has the following matrix form:

$$AY = F. \tag{4}$$

Here A is a three-diagonal $(N+1)\times(N+1)$-matrix (a_{ij}); Y and F are vectors from the space $\mathbb{R}^{N+1}$ with the uniform vector norm $\|\cdot\|$. The components of the matrix A and vectors Y and F are determined by the relations

$$a_{i,i-1} = -\varepsilon\, h^{-2}\, a(x_i), \quad a_{i,i} = 2\,\varepsilon\, h^{-2}\, a(x_i) + h^{-1}\, b(x_i) + c(x_i),$$

$$a_{i,i+1} = -\varepsilon\, h^{-2}\, a(x_i) - h^{-1}\, b(x_i), \quad 2 \le i \le N; \quad Y_i = z(x_i), \quad 1 \le i \le N+1;$$

$$F_1 = \varphi(x_1), \quad F_i = -f(x_i), \quad 2 \le i \le N, \quad F_{N+1} = \varphi(x_{N+1});$$

here $x_i = x^{i+1}$, $x^i \in \overline{D}_h$, and $a_{1,1} = a_{N+1,N+1} = 1$.

For difference scheme (2) in the presence of data perturbations, we have the following matrix form:

$$A^* Y^* = F^*. \tag{5}$$

Here A^* is the perturbed matrix (a^*_{ij}), Y^* and F^* are perturbed vectors, $A^* = A + \delta A$, $Y^* = Y + \delta Y$, $F^* = F + \delta F$. In the componentwise notation of the matrix δA and the vectors δF and δY, we have

$$\delta a_{i,i-1} = -\varepsilon\, h^{-2}\, \delta a_i^{i-1}, \quad \delta a_{ii} = 2\,\varepsilon\, h^{-2}\, \delta a_i^i + h^{-1}\, \delta b_i^i + \delta c_i^i, \tag{6}$$

$$\delta a_{i,i+1} = -\varepsilon\, h^{-2}\, \delta a_i^{i+1} - h^{-1}\, \delta b_i^{i+1}, \quad 2 \le i \le N;$$

$$\delta F_1 = \delta\varphi(x_1), \;\; \delta F_i = -\delta f(x_i), \;\; 2 \le i \le N, \quad \delta F_{N+1} = \delta\varphi(x_{N+1}); \;\; \delta Y_i = \delta z(x_i).$$

3.2 Estimates for computer perturbation and solution

We denote by $\triangle$ the parameter characterizing “allowable” perturbations caused by computer calculations.

Let $z^*_{\triangle}(x), x \in \overline{D}_h$ be the solution of the difference scheme in the presence of data perturbations, i.e., the solution of the difference scheme in the matrix form (5), (6) under the condition that the data perturbations satisfy the following condition (see [10, 12]):

$$|\delta a_i^j|, \; |\delta b_i^j|, \; |\delta c_i^i|, \; |\delta f(x_i)| \le \triangle, \;\; 2 \le i \le N; \quad |\delta\varphi(x_i)| \le \triangle, \;\; i = 1, N+1. \tag{7}$$

For the grid function $z^*_\triangle(x) - z(x)$, i.e., the perturbation of the grid solution $z(x)$ caused by computer perturbations (or, in short, the computer perturbation), using a technique in [1] (§13), we obtain the following estimate in the variables $\varepsilon, \delta, \triangle$:

$$\| z^*_\triangle - z \|_{\overline{D}_h} \le M \varepsilon^{-1} \delta^{-2} \triangle. \tag{8a}$$

This estimate is equivalent to the following estimate in the variables $\varepsilon, N, \triangle$:

$$\| z^*_\triangle - z \|_{\overline{D}_h} \le M \varepsilon N^2 \triangle. \tag{8b}$$

For the error in the computer solution $z^*_\triangle(x) - u(x)$, by virtue of the estimate

$$\| u - z^*_\triangle \|_{\overline{D}_h} \le \| u - z \|_{\overline{D}_h} + \| z^*_\triangle - z \|_{\overline{D}_h} \tag{9a}$$

taking into account estimates (3), (8), we obtain the estimate in the variables ε, δ, $\triangle$:

$$\| u - z^*_\triangle \|_{\overline{D}_h} \le M_1 \delta + M_2 \varepsilon^{-1} \delta^{-2} \triangle \ \le M \left[\ \delta + \varepsilon^{-1} \delta^{-2} \triangle \ \right], \tag{9b}$$

where $M_1 = M_{(3)}$, $M_2 = M_{(8a)}$. In the variables ε, N, $\triangle$, we have the estimate

$$\| u - z^*_\triangle \|_{\overline{D}_h} \le M_1 (\varepsilon + N^{-1})^{-1} N^{-1} + M_2 \varepsilon N^2 \triangle. \tag{9c}$$

The following theorem is valid (similar to Theorem 5 from [12]):

Theorem 2. *Let the conditions of Theorem* 1 *be satisfied. Then for the computer perturbation* $z^*_\triangle(x) - z(x)$ *and the error in the computer solution* $z^*_\triangle(x) - u(x)$*, the estimates* (8) *and* (9) *hold, respectively.*

4 Numerical investigation of model boundary value problem

For a model boundary value problem, using results of numerical experiments, we study errors in the grid solution $z(x) - u(x)$ and computer perturbations; results of numerical experiments are compared with theoretical results.

4.1 Difference schemes for model boundary value problem

Consider the boundary value problem

$$L_{(10)} u(x) \equiv \left\{ \varepsilon a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} \right\} u(x) = f(x), \quad x \in D, \tag{10}$$
$$u(x) = \varphi(x), \quad x \in \Gamma.$$

Here $\overline{D}=[0,1]$, $a(x)=1$, $b(x)=2$, $f(x)=-2$, $\varphi(x)=0$. The solution of problem (10) is written out explicitly:

$$u(x)=(1-e^{-2\varepsilon^{-1}})^{-1}\,(1-e^{-2\varepsilon^{-1}x})-x,\quad x\in\overline{D}.$$

We approximate problem (10) by the standard difference scheme

$$\Lambda z(x)\equiv\{\varepsilon\,\delta_{\overline{x}\widehat{x}}+2\,\delta_x\}z(x)=-2,\ \ x\in D_h,\quad z(x)=0,\ \ x\in\Gamma_h. \tag{11}$$

In the case of perturbations in the data, the following perturbed standard difference scheme corresponds to difference scheme (11):

$$\begin{aligned}\Lambda^* z^*(x)\equiv\{\varepsilon\,a^*(x)\,\delta_{\overline{x}\widehat{x}}+b^*(x)\,\delta_x\}\,z^*(x)\,&=\,f^*(x),\quad x\in D_h,\\ z^*(x)\,&=\,\varphi^*(x),\quad x\in\Gamma_h.\end{aligned} \tag{12a}$$

The perturbed data in the scheme (12) are determined by the relations

$$\begin{aligned}a^*(x)&=a_{(10)}(x)+\delta a^i_{i+1},\quad b^*(x)=b_{(10)}(x)=2,\\ f^*(x)&=f_{(10)}(x)=-2,\quad x=x^i,\quad x^i\in\overline{D}_h;\\ \varphi^*(x)&=\varphi_{(10)}(x)=0,\quad x\in\Gamma_h,\end{aligned} \tag{12b}$$

i.e., the coefficient of the second derivative is only perturbed, and only in the left node of the three-point pattern. In numerical experiments, we set

$$\delta a^i_{i+1}=-\delta a,\quad \delta a=10^{-8};\quad i=1,2,\ldots,N-1. \tag{12c}$$

This corresponds to the difference scheme in the presence of computer perturbations in the case of condition (7), where $\delta a=\triangle$, and $\triangle=10^{-8}$. Thus, we have $z^*_\triangle(x)=z^*_{(12)}(x)$.

In the case of boundary value problem (10), we are interested in the behavior of the error in the solution of standard difference scheme (11)

$$\delta_u=\delta_u(\varepsilon,N)=\|\,u-z\,\|_{\overline{D}_h} \tag{13a}$$

and the perturbations in the solution of computer difference scheme (12)

$$\delta_z=\delta_z(\varepsilon,N;\triangle)=\|\,z^*_\triangle-z\,\|_{\overline{D}_h} \tag{14a}$$

depending on the parameter ε, the number of grid intervals N and the value $\triangle$, as well as the comparison of the experimental with the theoretical results.

Technique to study $\delta_u=\delta_u(\varepsilon,N)$ is well known (see., e.g., [2], Ch. 2 for the convection-diffusion problem). In a similar way, studying $\delta_z=\delta_z(\varepsilon,N;\triangle)$ is performed.

Tables of errors in the solution of the difference scheme and perturbations in the solution of the computer scheme in the variables ε and N have sufficiently complicated character making it difficult to analyze the computer difference scheme. The results obtained are qualitatively consistent with estimate (3c) from Theorem 1 and with estimate (8b) from Theorem 2, but they are not shown here. Instead the results in the variables ε and N, in the next subsection, results of numerical experiments are presented in other (*automodel*) variables, which makes the obtained numerical results informative.

4.2 *Study of errors in standard and computer schemes in the automodel variables*

For model boundary value problem (10), we discuss the behavior of the error in the solution of the standard scheme and the computer perturbation of the grid solution with regard to their theoretical estimates (3c) and (8b).

Consider the error in the solution of the standard scheme (11), using the variables ε and β, where $\beta = \varepsilon N$ is the *automodel variable* in the case of the *standard* difference scheme (11).

Table 1 Errors in the grid solution $\overline{\delta}_u = \overline{\delta}_u(\varepsilon, \beta)$ of the standard scheme (11) for various values ε and β, and also the value $\{\beta \max_{\varepsilon} \overline{\delta}_u(\varepsilon, \beta)\}$ for various values β

$\varepsilon \setminus \beta$	2^0	2^2	2^4	2^6	2^8	2^{10}
1		$3.96e^{-2}$	$1.27e^{-2}$	$3.37e^{-3}$	$8.54e^{-4}$	$2.14e^{-4}$
2^{-2}	$1.90e^{-1}$	$7.59e^{-2}$	$2.17e^{-2}$	$5.64e^{-3}$	$1.42e^{-3}$	$3.57e^{-4}$
2^{-4}	$1.98e^{-1}$	$7.65e^{-2}$	$2.18e^{-2}$	$5.67e^{-3}$	$1.43e^{-3}$	$3.59e^{-4}$
2^{-6}	$1.98e^{-1}$	$7.65e^{-2}$	$2.18e^{-2}$	$5.67e^{-3}$	$1.43e^{-3}$	$3.59e^{-4}$
2^{-8}	$1.98e^{-1}$	$7.65e^{-2}$	$2.18e^{-2}$	$5.67e^{-3}$	$1.43e^{-3}$	$3.59e^{-4}$
2^{-10}	$1.98e^{-1}$	$7.65e^{-2}$	$2.18e^{-2}$	$5.67e^{-3}$	$1.43e^{-3}$	$3.59e^{-4}$
$\{\beta \max_{\varepsilon} \overline{\delta}_u(\varepsilon, \beta)\}$	**0.198**	**0.306**	**0.358**	**0.366**	**0.368**	**0.369**

In Table 1, errors in the grid solution $\overline{\delta}_u = \overline{\delta}_u(\varepsilon, \beta)$ of the standard scheme (11) are given for various values ε and β, where $\beta = \beta(\varepsilon, N) = \varepsilon N$. Here also the values $\{\beta \max_{\varepsilon} \overline{\delta}_u(\varepsilon, \beta)\}$ are given for various values β. Note that

$$\overline{\delta}_u = \overline{\delta}_u(\varepsilon, \beta) = \overline{\delta}_u(\varepsilon, \beta(\varepsilon, N)) = \delta_{u(13)}(\varepsilon, N).$$

From the numerical experiments it follows that for a fixed β, the values $\overline{\delta}_u(\varepsilon, \beta)$ rather weakly depend on the values of the parameter ε and they stabilize quickly with decreasing ε. The values $\{\beta \max_{\varepsilon} \overline{\delta}_u(\varepsilon, \beta)\}$ are weakly dependent on β, and they stabilize quickly with increasing of β; the maximum of these values does not exceed 0.369.

Thus, in the case of the model problem, for the error in the grid solution $\delta_{u\,(13)}(\varepsilon, N)$, using the results in Table 1, we obtain the experimental estimate

$$\delta_u(\varepsilon, N) \le M_1\, \varepsilon^{-1} N^{-1}, \tag{13b}$$

where (according to Table 1) we have

$$M_1 = \max_{\beta}\{\beta \max_{\varepsilon} \overline{\delta}_u(\varepsilon, \beta)\} \approx 0.369.$$

The estimate (13) for the error in the solution of the standard difference scheme (11) is fully consistent with the estimate (3c) from Theorem 1.

Consider the computer perturbations, using the variables ε and γ, where $\gamma = \varepsilon N^2$ is the **automodel variable** in the case of the **computer** difference scheme (12).

In Table 2, computer perturbations of the grid solutions $\widetilde{\delta}_z = \widetilde{\delta}_z(\varepsilon, \gamma; \triangle)$ are given for various values ε and γ, where $\gamma = \gamma(\varepsilon, N) = \varepsilon N^2$. Here also the values $\{(\gamma\triangle)^{-1} \max_{\varepsilon} \widetilde{\delta}_z(\varepsilon, \gamma;\ \triangle)\}$ are given for various values γ. Note that

$$\widetilde{\delta}_z = \widetilde{\delta}_z(\varepsilon, \gamma; \triangle),\ \ \widetilde{\delta}_z(\varepsilon,\ \gamma(\varepsilon, N);\ \triangle) = \delta_{z\,(14)}(\varepsilon, N;\ \triangle).$$

Table 2 Perturbations of the grid solution $\widetilde{\delta}_z = \widetilde{\delta}_z(\varepsilon, \gamma; \triangle)$ of scheme (12) for various values ε and γ, and also the values $\{(\gamma\triangle)^{-1} \max_{\varepsilon}(\widetilde{\delta}_z(\varepsilon, \gamma; \triangle))\}$ for various values γ

$\varepsilon \backslash \gamma$	2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}
1	$5.35e^{-8}$	$2.24e^{-7}$	$9.18e^{-7}$	$3.71e^{-6}$	$1.49e^{-5}$	$5.98e^{-5}$
2^{-2}	$3.15e^{-7}$	$1.29e^{-6}$	$5.22e^{-6}$	$2.10e^{-5}$	$8.42e^{-5}$	$3.37e^{-4}$
2^{-4}	$5.16e^{-7}$	$2.07e^{-6}$	$8.32e^{-6}$	$3.33e^{-5}$	$1.33e^{-4}$	$5.34e^{-4}$
2^{-6}	$5.93e^{-7}$	$2.38e^{-6}$	$9.54e^{-6}$	$3.82e^{-5}$	$1.53e^{-4}$	$6.12e^{-4}$
2^{-8}	$6.22e^{-7}$	$2.49e^{-6}$	$9.99e^{-6}$	$4.00e^{-5}$	$1.60e^{-4}$	$6.41e^{-4}$
2^{-10}	$6.33e^{-7}$	$2.53e^{-6}$	$1.01e^{-5}$	$4.06e^{-5}$	$1.62e^{-4}$	$6.51e^{-4}$
$\{(\gamma\triangle)^{-1} \max_{\varepsilon}(\widetilde{\delta}_z(\varepsilon, \gamma; \triangle))\}$	**0.247**	**0.249**	**0.249**	**0.249**	**0.249**	**0.250**

From Table 2, it follows that, for fixed values γ, the computer perturbations $\delta_z = \widetilde{\delta}_z(\varepsilon, \gamma;\ \triangle)$ sufficiently weakly depend on the parameter ε, moreover, they are stabilized quickly with decreasing ε. For fixed values ε, the perturbations $\widetilde{\delta}_z(\varepsilon, \gamma;\ \triangle)$ change significantly with increasing γ; these perturbations grow with increasing γ at the rate close to linear one for all values ε. Here also the values (ratio)

$$\{(\gamma\triangle)^{-1} \max_{\varepsilon}(\widetilde{\delta}_z(\varepsilon, \gamma;\ \triangle))\}$$

are given for various values γ. These values weakly depend on the γ, and they stabilize quickly with increasing γ; the maximum of that ratio does not exceed a value of 0.250.

Thus, in the case of the model problem for the computer perturbations $\delta_z(\varepsilon, N; \triangle)$, using the results in Table 2, we obtain the experimental estimate in the variables $\{\varepsilon, N,\ \triangle\}$

$$\delta_z(\varepsilon, N; \triangle) \le M_2\, \varepsilon N^2 \triangle, \tag{14b}$$

where (according to Table 2) we have

$$M_2 = \max_{\gamma} \{(\gamma\triangle)^{-1} \max_{\varepsilon}(\widetilde{\delta}_z(\varepsilon, \gamma; \triangle))\} \approx 0.250.$$

The estimate (14) is fully consistent with the estimate (8b) from Theorem 2.

4.3 On convergence of the computer solution

Taking into account the estimates (13), (14), for the error of the perturbed computer solution $\delta_u^* = \delta_{u/\triangle}^* = \|u - z_\triangle^*\|$, we obtain the experimental estimate in the variables $\{\varepsilon, N,\ \triangle\}$:

$$\delta_u^* \le M_1\, (\varepsilon + N^{-1})^{-1}\, N^{-1} + M_2\, \varepsilon N^2\, \triangle, \tag{15}$$

$M_1 \approx 0.369$, $M_2 \approx 0.250$. The estimate (15) is fully consistent with the estimate (9c) from Theorem 2.

The estimate (15) which is unimprovable with respect to order of incoming values, allows us to specify conditions imposed on the computer perturbations under which the computer solution converges to the solution of the boundary value problem with the same convergence order as the solution of the standard scheme in the absence of perturbations. From the estimate (15), it follows that provided

$$\triangle \le M\, \varepsilon^{-2}\, N^{-3} \tag{16a}$$

for the computer solution $z_\triangle^*(x)$, the following estimate holds

$$\|u - z_\triangle^*\| \le M\, \varepsilon^{-1}\, N^{-1}, \tag{16b}$$

which is the same in order as the estimate (3) for the solution $z(x)$ of the unperturbed difference scheme (2).

5 Conclusions

For the Dirichlet problem for a singularly perturbed ordinary differential convection-diffusion equation, a technique is presented for theoretical and experimental studies of influence of the computer perturbations on the perturbations of the grid solutions. The results of numerical experiments are also showed and analyzed. For the computer perturbations, conditions are obtained under which the solution of the computer scheme converges in the maximum norm with the same accuracy order as the solution of the standard scheme in the absence of perturbations. The fulfillment of these conditions allows one to use standard schemes for solving singularly perturbed problems. The obtained experimental results are consistent with the theoretical results.

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