A multiscale sparse grid technique for a two-dimensional convection-diffusion problem with exponential layers

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Abstract We investigate the application of a multiscale sparse grid finite element method for computing numerical solutions to a two-dimensional singularly perturbed convection-diffusion problem posed on the unit square. Typically, sparse grid methods are constructed using a hierarchical basis (see, e.g., Bungartz and Griebel [1]). In our approach, the method is presented as a generalisation of the two-scale method described in Liu et al. [3], and is related to the combination technique outlined by Pflaum and Zhou [7]. We show that this method retains the same level of accuracy, in the energy norm, as both the standard Galerkin and two-scale methods. The computational cost associated with the method, however, is $\mathscr{O}(N \log N)$, compared to $\mathscr{O}(N^2)$ and $\mathscr{O}(N^{3/2})$ for the Galerkin and two-scale methods respectively.

1 Introduction

Consider the following two-dimensional convection-diffusion problem:

$$Lu := -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u = f \quad \text{in } \Omega := (0,1)^2, \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

We are interested in the case where the parameter ε may be arbitrarily small, and so the problem is *singularly perturbed*. Special layer-resolving meshes are often used to obtain accurate numerical solutions to such problems, with the piecewise uniform mesh of Shishkin [5] receiving particular attention in the literature. Sparse grid methods for singularly perturbed reaction-diffusion problems, solved on Shishkin meshes, have been analysed [3, 4]. For the convection-diffusion problem, there are computational and theoretical investigations of combination techniques [6, 2]. The

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work of Franz et al. [2] is of particular interest to us: it considers a two-scale combination technique on a Shishkin mesh. It is our goal to develop on that, by applying the technique described in [4]. We use a standard finite element formulation, but with a sparse grid basis for the finite element space. We give a sketch of the analysis that leads to establishing uniform convergence, and present the results of numerical experiments that demonstrate the efficiency of the method.

2 Solution decomposition and Shishkin mesh

We shall assume that the functions $\mathbf{b}$ and f are sufficiently smooth so that (1) has a unique solution in $H_0^1(\Omega)\cap H^2(\Omega)$. For $\mathbf{b}(x,y)=(b_1(x,y),b_2(x,y))$ on $\bar{\Omega}$ we assume that

$$b_1(x,y)>\beta_1>0 \qquad \text{and} \qquad b_2(x,y)>\beta_2>0. \tag{2}$$

As a consequence, the solution to (1) features exponential boundary layers near the boundaries at $x=1$ and $y=1$. To resolve these, we employ a piecewise uniform Shishkin mesh. The transition points between the coarse and fines meshes are determined by the parameters

$$\tau_x=\min\left\{\frac{1}{2},\sigma\frac{\varepsilon}{\beta_1}\ln N\right\} \quad \text{and} \quad \tau_y=\min\left\{\frac{1}{2},\sigma\frac{\varepsilon}{\beta_2}\ln N\right\}. \tag{3}$$

The mesh is then constructed as described in, e.g., [2, §2.2]. It features four distinct subregions (see [2, Fig. 1])

$$\Omega_{II}=[0,1-\tau_x]\times[0,1-\tau_y], \quad \Omega_{BI}=[1-\tau_x,1]\times[0,1-\tau_y],$$
$$\Omega_{IB}=[0,1-\tau_x]\times[1-\tau_y,1], \quad \Omega_{BB}=[1-\tau_x,1]\times[1-\tau_y,1].$$

The behaviour of the solution to (1) is particular to each of these subregions. Following [2, Assumption 2.1], we shall assume that there is a corresponding decomposition of the solution

$$u=v+\omega_{BI}+\omega_{IB}+z, \tag{4}$$

where ω_{BI} is associated with the edge at $x=1$, ω_{IB} is associated with the edge at $y=1$, and z is associated with the corner layer at $(1,1)$. These may be bounded as

$$\left|\frac{\partial^{m+n}v}{\partial x^m\partial^n}(x,y)\right|\le C, \quad \left|\frac{\partial^{m+n}z}{\partial x^m\partial y^n}(x,y)\right|\le C\varepsilon^{-(m+n)}e^{-(\beta_1(1-x)+\beta_2(1-y))/\varepsilon}, \tag{5a}$$

$$\left|\frac{\partial^{m+n}\omega_{BI}}{\partial x^m\partial y^n}(x,y)\right|\le C\varepsilon^{-m}e^{-\beta_1(1-x)/\varepsilon}, \quad \left|\frac{\partial^{m+n}\omega_{IB}}{\partial x^m\partial y^n}(x,y)\right|\le C\varepsilon^{-n}e^{-\beta_2(1-y)/\varepsilon}, \tag{5b}$$

for $0\le m+n\le 3$. For $m+n=4$, we have the following bounds

$$\left\| \frac{\partial^{m+n} v}{\partial x^m \partial y^n}(x,y) \right\|_{0,\Omega} \leq C, \qquad \left\| \frac{\partial^{m+n} z}{\partial x^m \partial y^n}(x,y) \right\|_{0,\Omega} \leq C\varepsilon^{1-m-n}, \tag{6a}$$

$$\left\| \frac{\partial^{m+n} \omega_{BI}}{\partial x^m \partial y^n}(x,y) \right\|_{0,\Omega} \leq C\varepsilon^{-m+1/2}, \qquad \left\| \frac{\partial^{m+n} \omega_{IB}}{\partial x^m \partial y^n}(x,y) \right\|_{0,\Omega} \leq C\varepsilon^{-n+1/2}. \tag{6b}$$

For $p \in [2,\infty)$ and $\phi \in W^{2,p}[0,1]$, the piecewise linear interpolant $I_N \phi$ of ϕ satisfies

$$\begin{aligned} &\|\phi - I_N\phi\|_{0,p,[x_{i-1},x_i]} + h_i \|(\phi - I_N\phi)'\|_{0,p,[x_{i-1},x_i]} \\ &\qquad \leq C \min\left\{ h_i \|\phi'\|_{0,p,[x_{i-1},x_i]}, h_i^2 \|\phi''\|_{0,p,[x_{i-1},x_i]} \right\}. \end{aligned} \tag{7}$$

Define $V_{N_x}([0,1])$ to be the space of piecewise linear functions on the one-dimensional piecewise uniform Shishkin mesh with N_x intervals. The space $V_{N_y}([0,1])$ is defined in the same way. Taking the tensor product of these spaces gives $V_{N_x,N_y}(\bar{\Omega}) = V_{N_x}([0,1]) \times V_{N_y}([0,1])$. Let I_{N_x,N_y} be the piecewise bilinear interpolation operator that projects onto $V_{N_x,N_y}(\bar{\Omega})$. We write $I_{N_x,0}$ and I_{0,N_y} as the interpolation operators that interpolate only in the x- and y-directions respectively. Thus we have

$$I_{N_x,N_y} = I_{N_x,0} \circ I_{0,N_y} = I_{0,N_y} \circ I_{N_x,0}, \tag{8a}$$

$$\frac{\partial}{\partial x} I_{N_x,N_y} = I_{0,N_y} \circ \frac{\partial}{\partial x} I_{N_x,0}, \quad \text{and} \quad \frac{\partial}{\partial y} I_{N_x,N_y} = I_{N_x,0} \circ \frac{\partial}{\partial y} I_{0,N_y}. \tag{8b}$$

From standard inverse inequalities in one dimension one sees that

$$h_x \left\| \frac{\partial \psi}{\partial x} \right\|_{0,K} + k_y \left\| \frac{\partial \psi}{\partial y} \right\|_{0,K} \leq \|\psi\|_{0,K} \quad \forall \psi \in V_{N_x,N_y}(\bar{\Omega}), \tag{9}$$

where K is a mesh rectangle of size $h_x \times k_y$. We also use the following inequalities, which are easily established using standard inductive arguments: for $k \geq 2$

$$\sum_{i=1}^{k-1} i4^{i+1} \leq k4^{k+1}, \quad \sum_{i=1}^{k-1} 4^{i+1} \leq 4^{k+1}, \quad \text{and} \quad \sum_{i=1}^{k-1} i2^{i+1} \leq k2^{k+1}. \tag{10}$$

3 Multiscale Interpolation

The interpretation of multiscale interpolation that we employ is discussed in detail in [4, Section 3.1]. Here we briefly review the main concepts. We define $I_{N,N}$ to be the piecewise bilinear interpolation operator that maps onto $V_{N,N}(\Omega)$. Further to this we take the following definition of the *two-scale interpolation operator* from [2] and [3]:

$$I_{N,N}^{(1)} = I_{N,\mu(N)} + I_{\mu(N),N} - I_{\mu(N),\mu(N)},$$

where $\mu(N)$ is an integer that divides N, and where, for example, $I_{N,\mu(N)}u$ is the piecewise bilinear interpolant of u in $V_{N,\mu(N)}$. Now suppose we choose $\mu(N) = N/2$. We define the Level 1 interpolation operator as:

$$I_{N,N}^{(1)} = I_{N,\frac{N}{2}} + I_{\frac{N}{2},N} - I_{\frac{N}{2},\frac{N}{2}}. \tag{11}$$

By applying this Level 1 operator to the positively signed terms in (11) we arrive at the Level 2 operator:

$$I_{N,N}^{(2)} = I_{N,\frac{N}{2}}^{(1)} + I_{\frac{N}{2},N}^{(1)} - I_{\frac{N}{2},\frac{N}{2}} = I_{N,\frac{N}{4}} + I_{\frac{N}{2},\frac{N}{2}} + I_{\frac{N}{4},N} - I_{\frac{N}{2},\frac{N}{4}} - I_{\frac{N}{4},\frac{N}{2}}. \tag{12}$$

Applying the Level 1 operator to the positively signed terms of (12) gives the Level 3 operator. In general, the Level k operator is constructed by applying the Level 1 operator to the positively signed terms of $I_{N,N}^{(k-1)}$. The multiscale operator constructed in this manner, which we denote $I_{N,N}^{(k)}$, satisfies the following formula (for more detail, see [4, Lemma 3.1]):

$$I_{N,N}^{(k)} = \sum_{i=0}^{k} I_{\frac{N}{2^i},\frac{N}{2^{k-i}}} - \sum_{i=1}^{k} I_{\frac{N}{2^i},\frac{N}{2^{k+1-i}}}, \qquad \text{for } k = 0,1,2,\ldots. \tag{13}$$

For the further analysis of the method we require a bound on the difference between $I_{NN}^{(k)}$ and $I_{N,N}$. We do this by first expressing the difference between interpolants at successive levels in a succinct manner. Lemma 1 shows how the difference between an interpolant at a given Level k and at Level $k-1$ can be written as the product of one-dimensional operators.

Lemma 1. *([4, Lemma 3.2]) Let $I_{N,N}^{(k)}$ be the multiscale interpolation operator defined in (13). Then, for $k = 0,1,2,\ldots$,*

$$I_{N,N}^{(k-1)} - I_{N,N}^{(k)} = \sum_{i=0}^{k-1} \left(I_{\frac{N}{2^i},0} - I_{\frac{N}{2^{i+1}},0} \right) \left(I_{0,\frac{N}{2^{k-1-i}}} - I_{0,\frac{N}{2^{k-i}}} \right). \tag{14}$$

Our main goal in this section is to establish a bound on $\|u - I_{N,N}^{(k)}u\|_{\varepsilon}$, where

$$\|u\|_{\varepsilon} = \{\varepsilon \|\nabla u\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2\}^{1/2}. \tag{15}$$

We do this by first establishing a bound for $\|I_{N,N}^{(k)}u - I_{N,N}^{(k-1)}u\|_{\varepsilon}$.

Lemma 2. *Suppose $\Omega = (0,1)^2$. Let u be a function satisfying the assumptions of Section 2 and $I_{N,N}^{(k)}$ be the multiscale interpolation operator defined in (13). Then there exists a constant C independent of ε, N and k such that, for $k = 1,2,\ldots$,*

$$\|I_{N,N}^{(k)}u - I_{N,N}^{(k-1)}u\|_{\varepsilon} \le C(\varepsilon^{1/2}N^{1-\sigma} + 4^{k+1}N^{-3}\ln N + N^{-\sigma}\ln^{1/2} N + k4^{k+1}N^{-4}).$$

Proof. We wish to show that

$$\|I_{N,N}^{(k)}u - I_{N,N}^{(k-1)}u\|_{0,\Omega} \le C(N^{-\sigma} + k4^{k+1}N^{-4}), \tag{16a}$$

and

$$\varepsilon^{1/2}\|\nabla(I_{N,N}^{(k)}u - I_{N,N}^{(k-1)}u)\|_{0,\Omega} \le C(\varepsilon^{1/2}N^{1-\sigma} + 4^{k+1}N^{-3}\ln N + N^{-\sigma}\ln^{1/2} N). \tag{16b}$$

For brevity, we shall consider only (16b) in detail; the arguments for (16a) are similar (see also, [4, Lemma 3.3]). By (4) and Lemma 1 we have

$$\begin{aligned}
&\varepsilon^{1/2}\|\nabla(I_{N,N}^{(k)}u - I_{N,N}^{(k-1)}u)\|_{0,\Omega}\\
&= \varepsilon^{1/2}\left\|\nabla\left(\sum_{i=0}^{k-1}\left(I_{\frac{N}{2^i},0} - I_{\frac{N}{2^{i+1}},0}\right)\left(I_{0,\frac{N}{2^{k-1-i}}} - I_{0,\frac{N}{2^{k-i}}}\right)(v + \omega_{BI} + \omega_{IB} + z)\right)\right\|_{0,\Omega}.
\end{aligned}$$

We analyse each of the right-hand side components separately. For the smooth component v, we have by (5)–(8) along with (10) that

$$\begin{aligned}
&\varepsilon^{1/2}\left\|\frac{\partial}{\partial x}\sum_{i=0}^{k-1}\left(I_{\frac{N}{2^i},0} - I_{\frac{N}{2^{i+1}},0}\right)\left(I_{0,\frac{N}{2^{k-1-i}}} - I_{0,\frac{N}{2^{k-i}}}\right)v\right\|_{0,\Omega}\\
&\qquad \le C\varepsilon^{1/2}\sum_{i=0}^{k-1}\left(\frac{N}{2^{i+1}}\right)^{-1}\left\|\left(I_{0,\frac{N}{2^{k-1-i}}} - I_{0,\frac{N}{2^{k-i}}}\right)\frac{\partial^2 v}{\partial x^2}\right\|_{0,\Omega}\\
&\qquad \le C\varepsilon^{1/2}\sum_{i=0}^{k-1}\left(\frac{N}{2^{i+1}}\right)^{-1}\left(\frac{N}{2^{k-i}}\right)^{-2}\left\|\frac{\partial^4 v}{\partial x^2\partial y^2}\right\|_{0,\Omega} \le C4^{k+1}\varepsilon^{1/2}N^{-3}.
\end{aligned}$$

For ω_{BI} on $\Omega_{II} \cup \Omega_{IB}$ using an inverse estimate (9) together with (8) and (7) yields

$$\begin{aligned}
&\varepsilon^{1/2}\left\|\frac{\partial}{\partial x}(I_{N,N}^{(k)}\omega_{BI} - I_{N,N}^{(k-1)}\omega_{BI})\right\|_{0,\Omega_{II}\cup\Omega_{IB}}\\
&\qquad \le \varepsilon^{1/2}\left\|\frac{\partial}{\partial x}I_{N,N}^{(k)}\omega_{BI}\right\|_{0,\Omega_{II}\cup\Omega_{IB}} + \varepsilon^{1/2}\left\|\frac{\partial}{\partial x}I_{N,N}^{(k-1)}\omega_{BI}\right\|_{0,\Omega_{II}\cup\Omega_{IB}}\\
&\qquad \le C\varepsilon^{1/2}\left\|\frac{\partial \omega_{BI}}{\partial x}\right\|_{0,\Omega_{II}\cup\Omega_{IB}} \le C\varepsilon^{1/2}N\|\omega_{BI}\|_{0,\Omega_{II}\cup\Omega_{IB}} \le C\varepsilon^{1/2}N^{1-\sigma}.
\end{aligned} \tag{17}$$

On the region $\Omega_{BI} \cup \Omega_{BB}$, by (6–8) and (10) we have

$$
\begin{aligned}
\varepsilon^{1/2} & \left\| \frac{\partial}{\partial x} \sum_{i=0}^{k-1} \left(I_{\frac{N}{2^i},0} - I_{\frac{N}{2^{i+1}},0} \right) \left(I_{0,\frac{N}{2^{k-1-i}}} - I_{0,\frac{N}{2^{k-i}}} \right) \omega_{BI} \right\|_{0,\Omega_{BI}\cup\Omega_{BB}} \\
& \leq C\varepsilon^{1/2} \sum_{i=0}^{k-1} \left(\varepsilon \left(\frac{N}{2^{i+1}} \right)^{-1} \ln N \right) \left(\frac{N}{2^{k-i}} \right)^{-2} \left\| \frac{\partial^4 \omega_{BI}}{\partial x^2 \partial y^2} \right\|_{0,\Omega_{BI}\cup\Omega_{BB}} \\
& \leq C 4^{k+1} N^{-3} \ln N.
\end{aligned}
$$

For the third term ω_{IB} on $\Omega_{II} \cup \Omega_{BI}$ using (5) and an argument similar to (17) yields

$$
\begin{aligned}
\varepsilon^{1/2} & \left\| \frac{\partial}{\partial x} (I_{N,N}^{(k)} \omega_{IB} - I_{N,N}^{(k-1)} \omega_{IB}) \right\|_{0,\Omega_{II}\cup\Omega_{BI}} \\
& \leq C\varepsilon^{1/2} \max_{(x,y)\in\Omega_{II}\cup\Omega_{BI}} e^{-\beta_2(1-y)/\varepsilon} \leq C\varepsilon^{1/2} N^{-\sigma}.
\end{aligned}
$$

On the region $\Omega_{IB} \cup \Omega_{BB}$, (10), together with (6)–(8), leads to

$$
\begin{aligned}
\varepsilon^{1/2} & \left\| \frac{\partial}{\partial x} \sum_{i=0}^{k-1} \left(I_{\frac{N}{2^i},0} - I_{\frac{N}{2^{i+1}},0} \right) \left(I_{0,\frac{N}{2^{k-1-i}}} - I_{0,\frac{N}{2^{k-i}}} \right) \omega_{IB} \right\|_{0,\Omega_{IB}\cup\Omega_{BB}} \\
& \leq C\varepsilon^{1/2} \sum_{i=0}^{k-1} \left(\frac{N}{2^{i+1}} \right)^{-1} \varepsilon^2 \left(\frac{N}{2^{k-i}} \right)^{-2} \ln^2 N \left\| \frac{\partial^4 \omega_{IB}}{\partial x^2 \partial y^2} \right\|_{0,\Omega_{IB}\cup\Omega_{BB}} \\
& \leq C 4^{k+1} \varepsilon N^{-3} \ln^2 N.
\end{aligned}
$$

For the last term, z, by using an inverse estimate (9) and an argument similar to (17) we see that on the region $\Omega_{II} \cup \Omega_{IB}$ we have

$$
\varepsilon^{1/2} \left\| \frac{\partial}{\partial x} (I_{N,N}^{(k)} z - I_{N,N}^{(k-1)} z) \right\|_{0,\Omega_{II}\cup\Omega_{IB}} \leq C\varepsilon^{1/2} N \|z\|_{0,\Omega_{II}\cup\Omega_{IB}} \leq C\varepsilon^{1/2} N^{1-\sigma}.
$$

By (5) we have on the region Ω_{BI} that

$$
\begin{aligned}
\varepsilon^{1/2} \left\| \frac{\partial}{\partial x} (I_{N,N}^{(k)} z - I_{N,N}^{(k-1)} z) \right\|_{0,\Omega_{BI}} & \leq C\varepsilon^{1/2} \sqrt{\int_{\Omega_{BI}} \left\| \frac{\partial z}{\partial x} \right\|_{\infty,\Omega_{BI}}^2 d\Omega_{BI}} \\
& \leq C\varepsilon^{1/2} [\operatorname{meas} \Omega_{BI}]^{1/2} \left\| \frac{\partial z}{\partial x} \right\|_{\infty,\Omega_{BI}} \leq C N^{-\sigma} \ln^{1/2} N.
\end{aligned}
$$

Finally on the region Ω_{BB}, by (6)–(8) one obtains the following bound:

$$
\begin{aligned}
&\varepsilon^{1/2}\left\|\frac{\partial}{\partial x}\sum_{i=0}^{k-1}\left(I_{\frac{N}{2^i},0}-I_{\frac{N}{2^{i+1}},0}\right)\left(I_{0,\frac{N}{2^{k-1-i}}}-I_{0,\frac{N}{2^{k-i}}}\right)z\right\|_{0,\Omega_{BB}}\\
&\quad\le C\varepsilon^{1/2}\sum_{i=0}^{k-1}\left(\varepsilon\left(\frac{N}{2^{i+1}}\right)^{-1}\ln N\right)\left(\varepsilon^2\left(\frac{N}{2^{k-i}}\right)^{-2}\ln^2 N\right)\left\|\frac{\partial^4 z}{\partial x^2\partial y^2}\right\|_{0,\Omega_{BB}}\\
&\qquad\le C\varepsilon^{7/2}N^{-3}\ln^3 N\sum_{i=0}^{k-1}2^{2k-i+1}\varepsilon^{-3}\le C4^{k+1}\varepsilon^{1/2}N^{-3}\ln^3 N.
\end{aligned}
$$

Collecting all these bounds together, observing that $\varepsilon \le N^{-1}$ and then discarding those terms that are bounded by larger terms, we arrive at the following result:

$$
\varepsilon^{1/2}\left\|\frac{\partial}{\partial x}(I_{N,N}^{(k)}u-I_{N,N}^{(k-1)}u)\right\|_{0,\Omega}\le C(\varepsilon^{1/2}N^{1-\sigma}+4^{k+1}N^{-3}\ln N+N^{-\sigma}\ln^{1/2}N).
$$

The corresponding bound for $\varepsilon^{1/2}\|\partial/\partial y(I_{N,N}^{(k)}u-I_{N,N}^{(k-1)}u)\|_{0,\Omega}$ is derived in a similar fashion. Combining these results then completes the proof.

Lemma 3. *Let u and $I_{N,N}^{(k)}$ be defined as in Lemma 2. Then there exists a constant C independent of ε, N and k such that, for $k = 1, 2, \ldots,$*

$$
\|I_{N,N}^{(k-1)}u-I_{N,N}u\|_\varepsilon\le\|I_{N,N}^{(k)}u-I_{N,N}^{(k-1)}u\|_\varepsilon+C(k-1)(\varepsilon^{1/2}N^{1-\sigma}+N^{-\sigma}\ln^{1/2}N).
$$

Proof. The proof of this lemma follows closely that of [4, Lemma 3.6].

Lemma 4. *Let u and $I_{N,N}^{(k)}$ be defined as in Lemma 2. Then there exists a constant C independent of ε, N and k such that for $k = 1, 2, \ldots,$*

$$
\|I_{N,N}^{(k)}u-I_{N,N}u\|_\varepsilon\le C(k(\varepsilon^{1/2}N^{1-\sigma}+N^{-\sigma}\ln^{1/2}N)+4^{k+1}N^{-3}\ln N+k4^{k+1}N^{-4}).
$$

Proof. This result follows from the triangle inequality and Lemmas 2 and 3.

Corollary 1. *Taking $\widetilde{k} = \log_2 N - 1$ and $\sigma \ge 3/2$, there is a constant C independent of N and ε such that*

$$
\|I_{N,N}^{(\widetilde{k})}u-I_{N,N}u\|_\varepsilon\le CN^{-1}\ln N.
$$

Proof. This is a direct consequence of Lemma 4 and observing that $\varepsilon \le N^{-1}$.

Theorem 1. *Let u and $I_{N,N}^{(k)}$ be defined as in Lemma 2. Taking $\widetilde{k} = \log_2 N - 1$ there exists a constant C independent of ε, N and k such that*

$$
\|u-I_{N,N}^{(\widetilde{k})}u\|_\varepsilon\le CN^{-1}\ln N.
$$

Proof. By the triangle inequality, a standard interpolation result, Lemma 4 and Corollary 1 we have

$$
\|u-I_{N,N}^{(\widetilde{k})}u\|_\varepsilon\le\|u-I_{N,N}u\|_\varepsilon+\|I_{N,N}u-I_{N,N}^{(\widetilde{k})}u\|_\varepsilon\le 2CN^{-1}\ln N\le CN^{-1}\ln N.
$$

4 Error analysis

It is known (see, e.g., [8, Theorem 3.109]) that, if $u_{N,N}$ is the finite element solution obtained by the standard Galerkin FEM with bilinear elements, then,

$$\|u - u_{N,N}\|_\varepsilon \le CN^{-1}\ln N. \tag{18}$$

To define the sparse grid finite element solution, first let $\psi_i^N(x)$ be the usual piecewise linear basis function supported on the subinterval $[x_{i-1}, x_{i+1}]$. We define $V_{N,N}^{(k)}(\Omega) \subset H_0^1(\Omega)$ to be the finite dimensional space of piecewise bilinear functions defined on the tensor product Shishkin mesh given by

$$\begin{aligned} V_{N,N}^{(k)}(\Omega) = \operatorname{span}\left\{\psi_i^N(x)\psi_j^{N/2^k}(y)\right\}_{j=1:N/2^k-1}^{i=1:N-1} \\ + \operatorname{span}\left\{\psi_i^{N/2}(x)\psi_j^{N/2^{k-1}}(y)\right\}_{j=1:N/2^{k-1}-1}^{i=1:N/2-1} \\ + \cdots + \operatorname{span}\left\{\psi_i^{N/2^{k-1}}(x)\psi_j^{N/2}(y)\right\}_{j=1:N/2-1}^{i=1:N/2^{k-1}-1} \\ + \operatorname{span}\left\{\psi_i^{N/2^k}(x)\psi_j^{N}(y)\right\}_{j=1:N-1}^{i=1:N/2^k-1}. \end{aligned}$$

In general the choice of basis we make for this space is dependent on whether k is odd or even. When k is odd, the basis is chosen as follows:

$$\begin{aligned} \bigcup_{l=0}^{(k-1)/2} \left\{\psi_i^{N/2^l}\psi_j^{N/2^{k-l}}\right\}_{j=1:N/2^{k-l}-1}^{i=1:2:N/2^l-1} \\ \bigcup \left\{\psi_i^{N/2^{(k+1)/2}}\psi_j^{N/2^{(k-1)/2}}\right\}_{j=1:N/2^{(k-1)/2}-1}^{i=1:N/2^{(k+1)/2}-1} \\ \bigcup_{l=(k+3)/2}^{k} \left\{\psi_i^{N/2^l}\psi_j^{N/2^{k-l}}\right\}_{j=1:2:N/2^{k-l}-1}^{i=1:N/2^l-1}. \end{aligned}$$

When k is even, the basis is chosen as follows:

$$\begin{aligned} \bigcup_{l=0}^{k/2-1} \left\{\psi_i^{N/2^l}\psi_j^{N/2^{k-l}}\right\}_{j=1:N/2^{k-l}-1}^{i=1:2:N/2^l-1} \\ \bigcup \left\{\psi_i^{N/2^{k/2}}\psi_j^{N/2^{k/2}}\right\}_{j=1:N/2^{k/2}-1}^{i=1:N/2^{k/2}-1} \\ \bigcup_{l=k/2+1}^{k} \left\{\psi_i^{N/2^l}\psi_j^{N/2^{k-l}}\right\}_{j=1:2:N/2^{k-l}-1}^{i=1:N/2^l-1}. \end{aligned}$$

Equipped with this choice of basis and sparse grid finite element space the corresponding multiscale sparse grid finite element method is: *find* $u^{(k)} \in V_{N,N}^{(k)}$ *such that*

$$\int_\Omega \varepsilon \nabla u_{N,N}^{(k)} \nabla v_{N,N} + \int_\Omega \mathbf{b} \cdot \nabla u_{N,N}^{(k)} v_{N,N} = \int_\Omega f v_{N,N} \quad \text{for all } v_{N,N} \in V_{N,N}^{(k)}. \tag{19}$$

Theorem 2. *Let u be the solution to (1), subject to the assumptions of Section 2, and let* $u_{N,N}^{(\widetilde{k})}$ *be the solution to (19), where* $\widetilde{k} = \log_2 N - 1$. *Then there exists a constant C independent of* ε *and N such that*

$$\|u - u_{N,N}^{(\widetilde{k})}\|_\varepsilon \le CN^{-1} \ln N.$$

Proof. Noting the result from Theorem 1 and following an argument similar to [8, Theorem 3.109] gives the desired result.

5 Numerical results

We verify the bounds of 2 with numerical results, based on a test problem taken from [2, §4]:

$$-\varepsilon \Delta u - (2+x)u_x - (3+y^3)u_y + u = f \quad \text{in} \quad \Omega = (0,1)^2, \tag{20a}$$

and $u = 0$ on $\partial\Omega$, with f such that

$$u(x,y) = \cos(x\pi/2)[1 - e^{-2x/\varepsilon}](1-y)^3[1 - e^{-3y/\varepsilon}], \tag{20b}$$

which exhibits exponential boundary layers at $x = 0$ and $y = 0$. In Table 1 we show results computed when this is solved using both the standard Galerkin method with bilinear elements, and the multiscale sparse grid method (19). In the top table we take $N = 2^8$ and show that the errors for both methods are robust for small ε, as proved in Theorem 2 (also, compare with [2, Table 3]). In the lower table, we take $\varepsilon = 10^{-8}$, and present results for $N = 2^6, 2^7, \ldots, 2^{10}$ for both methods, and for $N = 2^{11}$ for the sparse grid method. This verifies the almost first-order convergence proved in Theorem 2, and shows that the error associated with the multiscale method is only slightly larger than for the Galerkin method. However, as shown, the sparse grid method is far more efficient (times are solve-times for the linear systems, measured in seconds, using a direct solver in MATLAB 8.1 (R2013a) on a single core of an AMD Opteron 2427, 2200 MHz processor with 32Gb of RAM, averaged over three runs).

$N=256$	$\varepsilon=1$	$\varepsilon=10^{-2}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-6}$	$\varepsilon=10^{-8}$	$\varepsilon=10^{-10}$
Galerkin	2.771e-03	3.580e-02	3.560e-02	3.560e-02	3.560e-02	3.560e-02
Multiscale	3.000e-01	4.157e-02	3.730e-02	3.726e-02	3.726e-02	3.726e-02

$\varepsilon=10^{-8}$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$	$N=2^{11}$
Galerkin	1.064e-01	6.223e-02	3.560e-02	2.003e-02	1.113e-02	-
Time (s)	*0.04*	*0.20*	*1.13*	*6.99*	*131.06*	-
Multiscale	1.111e-01	6.509e-02	3.726e-02	2.097e-02	1.165e-02	6.410e-03
Time (s)	*0.01*	*0.03*	*0.16*	*0.94*	*6.26*	*53.47*

Table 1: The Galerkin, $\|u-u_{N,N}\|_\varepsilon$, and multiscale, $\|u-u_{N,N}^{(k)}\|_\varepsilon$, methods applied to solving (20a)

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