On the application of algebraic flux correction schemes to problems with non-vanishing right-hand side

Petr Knobloch

Abstract It is demonstrated that the application of an algebraic flux correction (AFC) scheme to a singularly perturbed steady convection–diffusion equation with a non-vanishing right-hand side does not lead to satisfactory results in the boundary layer region. It is proved that it is not possible to construct an AFC scheme of the type considered for which the solution is accurate in the whole computational domain for any convection–diffusion problem with non-vanishing right-hand side.

1 Introduction

It is well known that Galerkin finite element discretizations of convection-dominated problems are not appropriate since the approximate solutions are usually polluted by spurious oscillations. A common remedy is to modify the variational formulation of the Galerkin FEM. An alternative approach modifies the algebraic form of the Galerkin FEM with the aim to satisfy the discrete maximum principle. In the present paper we consider an approach of this type, called algebraic flux correction (AFC). AFC schemes have been constructed, e.g., in [8, 6, 7] and recently analyzed in [2, 1]. The aim of this paper is to demonstrate that for problems with non-vanishing right-hand side the quality of AFC solutions may be not satisfactory along boundary layers and to investigate this phenomenon theoretically.

The plan of the paper is as follows. In Section 2, we introduce a general AFC scheme. Then, in Section 3, we present a particular example of limiters used in the AFC scheme. For this choice, numerical results of the AFC scheme applied to a convection–diffusion problem are presented in Section 4. Finally, in Section 5, the accuracy of a general AFC scheme is investigated theoretically.

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2 An algebraic flux correction scheme

Consider a linear boundary value problem for which the maximum principle holds. Let us discretize this problem by the finite element method. Then, the discrete solution can be represented by a vector $\mathrm{U} \in \mathbb{R}^N$ of its coefficients with respect to a basis of the respective finite element space. Let us assume that the last $N-M$ components of U ($0 < M < N$) correspond to nodes where Dirichlet boundary conditions are prescribed whereas the first M components of U are computed using the finite element discretization of the underlying partial differential equation. Then $\mathrm{U} \equiv (u_1, \dots, u_N)$ satisfies a system of linear equations of the form

$$\sum_{j=1}^{N} a_{ij}\, u_j = g_i\,, \qquad i = 1, \dots, M\,, \tag{1}$$

$$u_i = u_i^b\,, \qquad i = M+1, \dots, N\,. \tag{2}$$

The starting point of the AFC algorithm is the finite element matrix $\mathbb{A} = (a_{ij})_{i,j=1}^N$ corresponding to the above-mentioned finite element discretization in the case where homogeneous natural boundary conditions are used instead of the Dirichlet ones. We introduce a symmetric artificial diffusion matrix $\mathbb{D} = (d_{ij})_{i,j=1}^N$ possessing the entries

$$d_{ij} = d_{ji} = -\max\{a_{ij}, 0, a_{ji}\} \qquad \forall\, i \neq j\,, \qquad\qquad d_{ii} = -\sum_{j \neq i} d_{ij}\,. \tag{3}$$

Then the algebraic flux correction scheme is the following system of nonlinear equations:

$$\sum_{j=1}^{N} a_{ij}\, u_j + \sum_{j=1}^{N} (1 - \alpha_{ij})\, d_{ij}\, (u_j - u_i) = g_i\,, \qquad i = 1, \dots, M\,, \tag{4}$$

$$u_i = u_i^b\,, \qquad i = M+1, \dots, N\,, \tag{5}$$

where the limiters $\alpha_{ij} = \alpha_{ij}(u_1, \dots, u_N) \in [0,1]$ satisfy

$$\alpha_{ij} = \alpha_{ji}\,, \qquad i, j = 1, \dots, N\,. \tag{6}$$

We refer to [1] for a derivation of the equation (4).

The nonlinear problem (4), (5) is solvable under a continuity assumption on α_{ij}:

Theorem 1. *Let the matrix $(a_{ij})_{i,j=1}^M$ be positive definite. For any $i, j \in \{1, \dots, N\}$, let $\alpha_{ij} : \mathbb{R}^N \to [0,1]$ be such that $\alpha_{ij}(u_1, \dots, u_N)(u_j - u_i)$ is a continuous function of $u_1, \dots, u_N$. Finally, let the functions α_{ij} satisfy* (6)*. Then there exists a solution of the nonlinear problem* (4)*,* (5).

Proof. See [1], Theorem 3.3.

3 An example of the choice of α_{ij}

In this section we present a concrete choice of the limiters α_{ij} proposed in [6]. This choice is often used in computations and it was shown in [1] that it satisfies the assumptions of Theorem 1 and hence leads to a solvable nonlinear problem (4), (5).

The definition of the coefficients α_{ij} considered in this section relies on the values

$$P_i^+ = \sum_{\substack{j=1 \\ a_{ji} \le a_{ij}}}^{N} f_{ij}^+ , \quad P_i^- = \sum_{\substack{j=1 \\ a_{ji} \le a_{ij}}}^{N} f_{ij}^- , \quad Q_i^+ = -\sum_{j=1}^{N} f_{ij}^- , \quad Q_i^- = -\sum_{j=1}^{N} f_{ij}^+$$

computed for $i = 1, \dots, N$. Here $f_{ij} = d_{ij}(u_j - u_i)$ and we use the notation $f_{ij}^+ = \max\{0, f_{ij}\}$ and $f_{ij}^- = \min\{0, f_{ij}\}$. Using these quantities, one defines

$$R_i^+ := \min\left\{1, \frac{Q_i^+}{P_i^+}\right\}, \quad R_i^- := \min\left\{1, \frac{Q_i^-}{P_i^-}\right\}, \qquad i = 1, \dots, N .$$

If P_i^+ or P_i^- vanishes, we set $R_i^+ := 1$ or $R_i^- := 1$, respectively. Finally, for any $i, j \in \{1, \dots, N\}$ such that $a_{ji} \le a_{ij}$, one sets

$$\alpha_{ij} := \begin{cases} R_i^+ & \text{if } f_{ij} > 0 , \\ 1 & \text{if } f_{ij} = 0 , \\ R_i^- & \text{if } f_{ij} < 0 , \end{cases} \qquad \alpha_{ji} := \alpha_{ij} .$$

4 Application to a convection–diffusion equation

Let us apply the algebraic flux correction scheme (4), (5) to the numerical solution of the steady-state convection–diffusion equation

$$-\varepsilon\,\Delta u + \mathbf{b} \cdot \nabla u = g \quad \text{in } \Omega , \qquad u = u_b \quad \text{on } \partial\Omega , \tag{7}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a polygonal boundary $\partial\Omega$, $\varepsilon > 0$ is a constant, and $\mathbf{b} \in W^{1,\infty}(\Omega)^d$ with $\nabla \cdot \mathbf{b} = 0$, $g \in L^2(\Omega)$, and $u_b \in H^{\frac{1}{2}}(\partial\Omega) \cap C(\partial\Omega)$ are given functions. It is well known that the problem (7) has a unique weak solution that satisfies the maximum principle.

Let $\mathscr{T}_h$ be a triangulation of Ω consisting of triangles possessing the usual compatibility properties. First, we introduce the standard Galerkin finite element discretization of (7) based on a conforming piecewise linear finite element space. This discretization can be written in the form of the linear system (1), (2) and it is known to be inappropriate for solving (7) in the convection-dominated regime. The corresponding algebraic flux correction scheme is obtained as described in Section 2 and we use the limiters α_{ij} defined in Section 3. Then the discrete maximum principle

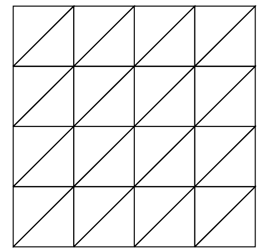

Fig. 1 Type of the triangulation used for computations

is satisfied provided that $\mathscr{T}_h$ is a Delaunay triangulation (i.e., the sum of any pair of angles opposite a common edge is smaller than, or equal to, π), see [1].

In [1], numerical studies are presented for the algebraic flux correction scheme applied to a convection–diffusion–reaction equation. They show the dependence of the errors of the discrete solutions measured in various norms on the discretization parameter and the type of the triangulation. Here we qualitatively compare the approximate solution obtained using the AFC scheme with the exact solution and approximate solutions obtained by two different discretization techniques. We consider the following example.

Example 1. Problem (7) is considered with $\Omega = (0,1)^2$, $\varepsilon = 10^{-8}$, and $\mathbf{b} = (2,3)^T$. The right-hand side g and the boundary condition u_b are chosen in such a way that

$$u(x,y) = x y^2 - y^2 \exp\left(\frac{2\,(x-1)}{\varepsilon}\right) - x \exp\left(\frac{3\,(y-1)}{\varepsilon}\right) + \exp\left(\frac{2\,(x-1)+3\,(y-1)}{\varepsilon}\right)$$

is the solution of (7).

The triangulation $\mathscr{T}_h$ of the above domain Ω used in all computations was of the type shown in Fig. 1 and consisted of 800 triangles. The solution u is depicted in Fig. 2 (top left). In the first row of Fig. 2, one can also see the solution of the streamline upwind/ Petrov–Galerkin (SUPG) method [3], which is one of the most efficient procedures for solving convection-dominated problems. It provides accurate solutions away from layers but does not not preclude small nonphysical oscillations localized in layer regions. This clearly shows that the SUPG method does not satisfy the discrete maximum principle. In the left of the second row of Fig. 2, one can see the solution of the AFC scheme with the limiters α_{ij} from Section 3. Again, the solution is accurate away from layers but it is not correct in layer regions. Nevertheless, the solution does not contain any spurious oscillations since the method satisfies the discrete maximum principle, as we mentioned above. Finally, Fig. 2 also shows the solution of the Mizukami–Hughes method [9, 5]. Like the AFC scheme, the Mizukami–Hughes method is a nonlinear method satisfying the discrete maximum principle. We observe that the Mizukami–Hughes solution is

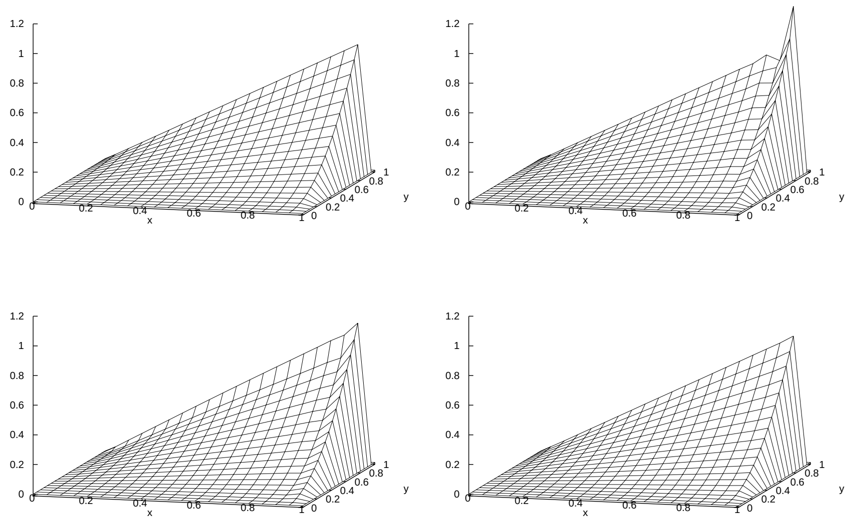

Fig. 2 Example 1: exact solution (top left), SUPG method (top right), algebraic flux correction scheme (bottom left), Mizukami–Hughes method (bottom right)

qualitatively correct and a detailed comparison with the exact solution reveals that it is accurate in the whole computational domain.

Many other nonlinear stabilized methods are based on adding additional terms to the SUPG discretization, aimed to suppress spurious oscillations without smearing the layers, see [4]. By adjusting parameters of the methods, it is sometimes possible to obtain approximate solutions comparable to the solution of the Mizukami–Hughes method. Now a natural question is whether the algebraic flux correction scheme can also provide solutions of such a high quality if the limiters α_{ij} are chosen in an appropriate way. We shall show in the next section that it is not possible.

5 Accuracy of general algebraic flux correction schemes

The aim of this section is to investigate whether algebraic flux correction schemes applied to convection–diffusion problems in two dimensions can provide accurate approximate solutions in the whole computational domain like, e.g., the Mizukami–Hughes method or some other nonlinear methods mentioned at the end of the preceding section. Solutions of many further stabilized methods still possess spurious oscillations in layer regions but are very accurate away from layers. If an algebraic flux correction scheme should be competitive at least with these methods, it should have this property. Therefore, in what follows, we consider only algebraic flux correction schemes whose solutions are accurate away from layers.

We shall investigate a general algebraic flux correction scheme that, apart from the above accuracy assumption, satisfies only the assumptions made in Section 2. Thus, we do not assume any particular definition of the limiters α_{ij}. We can even relax the definition of the artificial diffusion matrix $\mathbb{D}$ and instead of (3) assume

$$\begin{aligned} d_{ij} = d_{ji} &\le -\max\{a_{ij},0,a_{ji}\}\,, \quad i,j=1,\dots,M\,, \quad i\neq j\,,\\ d_{ij} = d_{ji} &\le -\max\{a_{ij},0\}\,, \quad i=1,\dots,M,\ j=M+1,\dots,N\,. \end{aligned}$$

Let us consider the following boundary value problem in $\Omega=(0,1)^2$:

$$-\varepsilon\,\Delta u + \mathbf{b}\cdot\nabla u = g \quad \text{in } \Omega\,, \tag{8}$$

$$u(0,y) = u(1,y) = 0 \quad \text{for } y\in(0,1)\,, \tag{9}$$

$$u(x,0) = u(x,1)\,,\ u_y(x,0) = u_y(x,1) \quad \text{for } x\in(0,1)\,, \tag{10}$$

where $\varepsilon>0$, $\mathbf{b}=(b_1,b_2)$ with $b_1>0$, and $g>0$ are constants. In principle, this is a one-dimensional problem, since $u(x,y)=\bar u(x)$ for any $x,y\in\overline{\Omega}$, where $\bar u(x)$ solves the problem

$$-\varepsilon\,\bar u'' + b_1\,\bar u' = g \quad \text{in } (0,1)\,, \qquad \bar u(0)=\bar u(1)=0\,.$$

Note that

$$\bar u(x) = \omega\,x - \omega\,\frac{e^{-\delta(1-x)}-e^{-\delta}}{1-e^{-\delta}}\,, \quad x\in[0,1]\,, \tag{11}$$

where $\omega=g/b_1$ and $\delta=b_1/\varepsilon$. Thus, if $\varepsilon\ll b_1$, one sees that $u(x,y)\approx\omega\,x$ on most of Ω and a boundary layer occurs along the line $x=1$.

Let $\mathscr{T}_h$ be again a triangulation of Ω of the type shown in Fig. 1 and let the number of vertices in each direction be $N+1$. Then $\mathscr{T}_h$ contains $2N^2$ triangles and its vertices have the coordinates (x_i,y_j) with $x_i=i\,h$, $y_j=j\,h$, and $h=1/N$, where $i,j=0,\dots,N$. We denote

$$\mathscr{P} = \{(x_i,y_j)\,;\ i=0,\dots,N,\ j=1,\dots,N\}\,, \qquad \mathscr{P}^D = \{(x,y)\in\mathscr{P}\,;\ x\in\{0,1\}\}\,.$$

For any $P\in\mathscr{P}$, let φ_P be the standard basis function of the piecewise linear space

$$W_h = \{v\in C(\overline{\Omega})\,;\ v|_T\in P_1(T)\ \forall\,T\in\mathscr{T}_h,\ v(x,0)=v(x,1)\ \forall\,x\in(0,1)\}$$

assigned to the vertex P, i.e., $\varphi_P(P)=1$ and $\varphi_P(Q)=0$ for all $Q\in\mathscr{P}\setminus\{P\}$. Then the algebraic flux correction scheme (4), (5) applied to (8)–(10) discretized using the space W_h can be written in the form

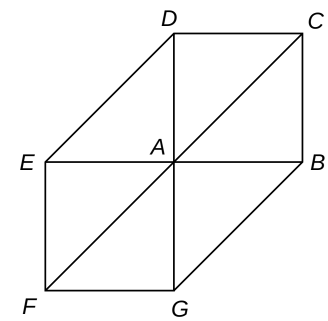

Fig. 3 Elements of $\mathscr{T}_h$ sharing an interior vertex A

$$\sum_{Q\in\mathscr{P}} a_{PQ}\,u_Q + \sum_{Q\in\mathscr{P}} (1-\alpha_{PQ})\,d_{PQ}\,(u_Q-u_P) = g_P \qquad \forall\, P\in\mathscr{P}\setminus\mathscr{P}^D\,, \tag{12}$$

$$u_P = 0 \qquad \forall\, P\in\mathscr{P}^D\,, \tag{13}$$

where, for any $P,Q\in\mathscr{P}$,

$$a_{PQ} = \varepsilon\,(\nabla\varphi_Q,\nabla\varphi_P) + (\mathbf{b}\cdot\nabla\varphi_Q,\varphi_P)\,, \qquad g_P = (g,\varphi_P)\,, \tag{14}$$

$$\alpha_{PQ} = \alpha_{PQ}(\{u_R\}_{R\in\mathscr{P}}) \in [0,1]\,, \qquad \alpha_{PQ} = \alpha_{QP}\,, \tag{15}$$

$$d_{PQ} = d_{QP} \le -\max\{a_{PQ},0,a_{QP}\} \qquad \text{if } P,Q\in\mathscr{P}\setminus\mathscr{P}^D\,,\ P\neq Q\,, \tag{16}$$

$$d_{PQ} = d_{QP} \le -\max\{a_{PQ},0\} \qquad \text{if } \quad P\in\mathscr{P}\setminus\mathscr{P}^D\,,\ Q\in\mathscr{P}^D\,. \tag{17}$$

The notation $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $L^2(\Omega)^d$.

Let A be an interior vertex of the triangulation $\mathscr{T}_h$. Then the elements of $\mathscr{T}_h$ containing the vertex A are arranged as in Fig. 3. We denote by $B,\dots,G$ the remaining vertices of these triangles, see again Fig. 3. Then

$$\begin{aligned}
a_{AA} &= 4\,\varepsilon\,, & g_A &= g\,h^2\,,\\
a_{AB} = a_{EA} &= -\varepsilon + \frac{h}{6}\,(2\,b_1-b_2)\,, & a_{AE} = a_{BA} &= -\varepsilon + \frac{h}{6}\,(-2\,b_1+b_2)\,,\\
a_{AC} = a_{FA} &= \frac{h}{6}\,(b_1+b_2)\,, & a_{AF} = a_{CA} &= \frac{h}{6}\,(-b_1-b_2)\,,\\
a_{AD} = a_{GA} &= -\varepsilon + \frac{h}{6}\,(-b_1+2\,b_2)\,, & a_{AG} = a_{DA} &= -\varepsilon + \frac{h}{6}\,(b_1-2\,b_2)\,.
\end{aligned}$$

Consequently,

$$\max\{a_{AB},a_{BA}\} = \max\{a_{AE},a_{EA}\} = -\varepsilon + \frac{h}{6}\,|2\,b_1-b_2|\,, \tag{18}$$

$$\max\{a_{AC},a_{CA}\} = \max\{a_{AF},a_{FA}\} = \frac{h}{6}\,|b_1+b_2|\,, \tag{19}$$

$$\max\{a_{AD},a_{DA}\} = \max\{a_{AG},a_{GA}\} = -\varepsilon + \frac{h}{6}\,|b_1-2\,b_2|\,. \tag{20}$$

To simplify our considerations, let us assume that

$$h\,b_1 \geq 6\,\varepsilon \qquad \text{and} \qquad b_2 \in [-b_1, b_1]\,.$$

Then the maxima in (18) are nonnegative. Since the diffusion matrix satisfies the conditions for the discrete maximum principle, it is not necessary to define the artificial diffusion matrix $\mathbb{D}$ using the maxima (18)–(20), but one can define $\mathbb{D}$ using the convection matrix only (i.e., (18)–(20) with $\varepsilon = 0$). Another possibility is to use an intermediate variant based on the sum of the convection matrix and the diffusion matrix with ε replaced by $\bar{\varepsilon} \in (0, \varepsilon)$, see the discussion in [2] on the optimal choice of the artificial diffusion in a related method of AFC type. In what follows, we shall consider this more general choice of the matrix $\mathbb{D}$. In particular, we have

$$d_{AB} = d_{BA} = d_{AE} = d_{EA} = \bar{\varepsilon} - \frac{h}{6}\,(2\,b_1 - b_2)\,, \tag{21}$$

$$d_{AC} = d_{CA} = d_{AF} = d_{FA} = -\frac{h}{6}\,(b_1 + b_2)\,, \tag{22}$$

$$d_{AD} = d_{DA} = d_{AG} = d_{GA} = \min\left\{0, \bar{\varepsilon} - \frac{h}{6}\,|b_1 - 2\,b_2|\right\}, \tag{23}$$

where

$$\bar{\varepsilon} \in [0, \varepsilon]\,.$$

Note that $\mathbb{D}$ then satisfies the assumptions (16), (17).

Remark 1. Since $a_{AE} \leq 0$, $a_{AF} \leq 0$, the assumption (17) enables to set $d_{AE} = d_{EA} = d_{AF} = d_{FA} = 0$ if $E, F \in \mathscr{P}^D$. However, the values of these entries of $\mathbb{D}$ have no influence on our further proceeding and hence we shall use the values from (21), (22). If $B, C \in \mathscr{P}^D$, the assumption (17) does not enable to use other values of d_{AB}, d_{BA}, d_{AC}, d_{CA} than the inequality in (16) since $a_{BA} \leq 0$, $a_{CA} \leq 0$.

It is reasonable to require that the solution of (12), (13) is constant in the y direction as it is for the exact solution and also for the Galerkin solution. Then $u_B = u_C$, $u_A = u_D = u_G$, and $u_E = u_F$, so that (12) with $P = A$ reduces to

$$\begin{aligned}
-\,\varepsilon\,(u_E - 2\,u_A + u_B) + \frac{h\,b_1}{2}\,(u_B - u_E)& \\
+ [(1-\alpha_{AB})\,d_{AB} + (1-\alpha_{AC})\,d_{AC}]\,(u_B - u_A)& \\
+ [(1-\alpha_{AE})\,d_{AE} + (1-\alpha_{AF})\,d_{AF}]\,(u_E - u_A) &= g\,h^2\,.
\end{aligned} \tag{24}$$

We set

$$\gamma_A = (1-\alpha_{AB})\,d_{AB} + (1-\alpha_{AC})\,d_{AC}$$

for any vertices $A, B, C \in \mathscr{P}$ arranged as in Fig. 3. Note that

$$\bar{\varepsilon} - \frac{h\,b_1}{2} \leq \gamma_A \leq 0\,. \tag{25}$$

Obviously, $d_{AF} = d_{DE}$. Moreover, since the solution of (12), (13) does not depend on the y coordinate, one has $\alpha_{AF} = \alpha_{DE}$. Then the symmetry of d_{PQ} and α_{PQ} implies

$$\begin{aligned}(1-\alpha_{AE})\,d_{AE} + (1-\alpha_{AF})\,d_{AF} &= (1-\alpha_{AE})\,d_{AE} + (1-\alpha_{DE})\,d_{DE}\\ &= (1-\alpha_{EA})\,d_{EA} + (1-\alpha_{ED})\,d_{ED} = \gamma_E\,.\end{aligned}$$

Since the approximate solution does not depend on the y coordinate, one can denote by u_i the value of the approximate solution at any vertex A having the x coordinate equal to x_i, $i \in \{0,\dots,N\}$. Similarly, one can denote γ_A by γ_i for any such vertex A. Then (24) can be written in the form

$$\begin{aligned}&-\varepsilon\,(u_{i-1} - 2\,u_i + u_{i+1}) + \frac{h\,b_1}{2}\,(u_{i+1} - u_{i-1})\\ &+\gamma_i\,(u_{i+1} - u_i) - \gamma_{i-1}\,(u_i - u_{i-1}) = g\,h^2\,, \qquad i = 1,\dots,N-1\,,\end{aligned} \tag{26}$$

with $u_0 = u_N = 0$. Summing up the equations (26) over $i = l,\dots,N-1$ with any $l \in \{1,\dots,N-2\}$, one obtains

$$\begin{aligned}&\varepsilon\,(u_{N-1} + u_l - u_{l-1}) + \frac{h\,b_1}{2}\,(u_{N-1} - u_l - u_{l-1})\\ &-\gamma_{N-1}\,u_{N-1} - \gamma_{l-1}\,(u_l - u_{l-1}) = (N-l)\,g\,h^2\,,\end{aligned}$$

which implies that

$$u_{N-1} = \frac{(N-l)\,g\,h^2 - \varepsilon\,(u_l - u_{l-1}) + \frac{h b_1}{2}\,(u_l + u_{l-1}) + \gamma_{l-1}\,(u_l - u_{l-1})}{\varepsilon + \frac{h b_1}{2} - \gamma_{N-1}}\,. \tag{27}$$

Assuming

$$0 < \varepsilon \ll h\,b_1\,,$$

the exact solution on $(0,1-h)\times(0,1)$ is indistinguishable from the linear function ωx in a finite precision arithmetic, cf. (11). Since we assume that the solution of the algebraic flux correction scheme is accurate away from layers, one has

$$u_i \approx \omega\,h\,i\,, \qquad i = 0,\dots,k\,, \tag{28}$$

with some $k < N$. The philosophy of algebraic flux correction schemes implies that the limiters α_{ij} should equal 1 for a linear function, at least sufficiently far from the boundary. Thus, for some $l \le k$, one has

$$\gamma_{l-1} \approx 0\,. \tag{29}$$

The relations (28) and (29) imply that

$$\text{the numerator of (27)} \quad \approx \quad \omega\,h\left[b_1\left(1 - \frac{h}{2}\right) - \varepsilon\right] > 0\,.$$

Thus, it follows from (27) and (25) that

$$
\begin{aligned}
u_{N-1} &\geq \frac{(N-l)\,g\,h^2 - \varepsilon\,(u_l - u_{l-1}) + \frac{h\,b_1}{2}\,(u_l + u_{l-1}) + \gamma_{l-1}\,(u_l - u_{l-1})}{h\,b_1 + \varepsilon} \\
&\approx \omega\,h\,\frac{b_1\left(1 - \frac{h}{2}\right) - \varepsilon}{h\,b_1 + \varepsilon} = \omega\,x_{N-1}\,\frac{h\,b_1}{h\,b_1 + \varepsilon} + \frac{\omega\,h}{2}\,\frac{h\,b_1 - 2\,\varepsilon}{h\,b_1 + \varepsilon} \\
&\approx \bar{u}(x_{N-1}) + \frac{g\,h}{2\,b_1}\,.
\end{aligned}
$$

This result shows that u_{N-1} is larger than the value of the exact solution at x_{N-1}, at least by $g\,h/(2\,b_1)$, which corresponds to the observation in the preceding section and also in many other computations. Since we considered general limiters α_{ij}, we conclude that it is not possible to construct an algebraic flux correction scheme of the type considered in this paper for which the approximate solution would be accurate in the whole computational domain for any convection–diffusion problem with non-vanishing right-hand side.

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