Second order uniformly convergent numerical method for a coupled system of singularly perturbed reaction-diffusion problems with discontinuous source term

S. Chandra Sekhara Rao and Sheetal Chawla

Abstract In this work we consider a coupled system of $m(\geq 2)$ linear singularly perturbed equations of reaction-diffusion type coupled in the reaction terms with discontinuous source term. The leading term of each equation is multiplied by a small positive parameter. These singular perturbation parameters are assumed to be distinct in magnitude. Overlapping boundary and interior layers can appear in the solution. A numerical method is constructed that involve an appropriate piecewise-uniform Shishkin mesh, which is fitted to both the boundary and interior layers. The parameter-uniform convergence of the numerical approximations is examined.

1 Introduction

We consider a coupled system of linear singularly perturbed boundary value problems with discontinuous source term in the interval $\Omega = (0,1)$. A single discontinuity in the source term is assumed to occur at a point $d \in \Omega$. Let $\Omega_1 = (0,d)$ and $\Omega_2 = (d,1)$ and the jump at d in any function Ψ is given as $[\Psi](d) = \Psi(d+) - \Psi(d-)$. The corresponding boundary value problem is:

$$\boldsymbol{L}\boldsymbol{u} := -\boldsymbol{E}\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}, \qquad x \in \Omega_1 \cup \Omega_2, \tag{1}$$

$$\boldsymbol{u}(0) = \boldsymbol{p}, \;\; \boldsymbol{u}(1) = \boldsymbol{q}, \tag{2}$$

where $\boldsymbol{E}$=diag$(\varepsilon_1,\ldots,\varepsilon_m)$ with small parameters $\varepsilon_1,\ldots,\varepsilon_m$, are such that $0 < \varepsilon_1 \leq \ldots \leq \varepsilon_m \leq 1$,

$$\boldsymbol{A}(x) = (a_{ij}(x))_{m\times m} \quad \text{and} \quad \boldsymbol{f}(x) = (f_i(x))_{m\times 1} \tag{3}$$

S. Chandra Sekhara Rao and Sheetal Chawla
Department of Mathematics, Indian Institute of Technology Delhi, Hauz khas, New Delhi-110016, India, e-mail: scsr@maths.iitd.ac.in (S. Chandra Sekhara Rao), chawlaasheetal@gmail.com (Sheetal Chawla)

are given. We assume that the coupling matrix satisfies the following conditions:

$$a_{ii}(x) > \sum_{j\neq i, j=1}^{m} |a_{ij}(x)|, \text{ for } 1 \leq i \leq m, \text{ and } a_{ij}(x) \leq 0 \text{ for } i \neq j, \tag{4}$$

and for some constant α, we have

$$0 < \alpha < \min_{x\in\overline{\Omega}, 1\leq i\leq m} \sum_{j=1}^{m} (a_{ij}(x)). \tag{5}$$

Since $\boldsymbol{f}$ is discontinuous at d, the solution $\boldsymbol{u}$ does not necessarily have a continuous second order derivative at d, but the first derivative of $\boldsymbol{u}$ exists and is contiunous.

A coupled system of $m \geq 2$ singularly perturbed reaction-diffusion equations were considered in [3]-[5]. Parameter-uniform numerical method of second order was considered in [4] with continuous source term. Rao and Chawla [7] considered a coupled system of two singularly perturbed linear reaction-diffusion equations with discontinuous source term. The scheme was proved to be almost first order uniformly convergent in which the diffusion parameter associated with each equation of the system had a different order of magnitude. Paramasivam et. al. [5] proved the same result for an arbitrary number of equations. We have considered $m \geq 2$ different diffusion parameters and source term in the system has a discontinuity at a point in the interior of the domain, and improved order of convergence upto order almost two.

This paper is arranged as follows. Section 2 presents the properties of the exact solution of the problem. In Section 3 piecewise-uniform variant of Shishkin mesh, is introduced, then the finite difference scheme that approximates the singularly perturbed problem is constructed. In Section 4 the statement and proof of the parameter-uniform error estimate is given. Results of numerical experiments are presented in Section 5.

Notations We use C to denote a generic positive constant and $\mathbf{C} = (C, C, \ldots, C)^T$ to denote a generic positive constant vector which are independent of the perturbation parameters and the discretization parameter N, but may not be same at each occurrence. Define $\boldsymbol{v} \leq \boldsymbol{w}$ if $v_i \leq w_i$, for $1 \leq i \leq m$. We consider the maximum norm and denote it by $\| \,.\, \|_S$, where S is a closed subset of $\overline{\Omega}$. We define $\| v \|_S = \max_{x\in S} |v(x)|$ and $\| \boldsymbol{v} \|_S = \max\{\| v_1 \|_S, \| v_2 \|_S, \ldots, \| v_m \|_S\}$.

2 Properties of the exact solution

Theorem 1. *The problem (1)-(2) has a solution* $\boldsymbol{u} = (u_1, \ldots, u_m)^T$ *with* $u_1, \ldots, u_m \in C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$.

Proof. The proof is similar to the proof of Theorem 2.1. in [5].

Theorem 2. *Suppose $u_1,\dots,u_m \in C^0(\overline{\Omega}) \cap C^2(\Omega_1 \cup \Omega_2)$. Further suppose that $\boldsymbol{u} = (u_1,\dots,u_m)^T$ satisfies $\boldsymbol{u}(0) \geq \boldsymbol{0}$, $\boldsymbol{u}(1) \geq \boldsymbol{0}$, $\boldsymbol{L}\boldsymbol{u}(x) \geq \boldsymbol{0}$ in $\Omega_1 \cup \Omega_2$ and $[\boldsymbol{u}'](d) \leq \boldsymbol{0}$. Then $\boldsymbol{u}(x) \geq \boldsymbol{0}$, for all $x \in \overline{\Omega}$.*

Proof. The result can be proved by following similar arguments considered in [7].

Lemma 1. *Let* A*(x) satisfy (4)-(5). If $\boldsymbol{u} = (u_1,\dots,u_m)^T$ be the solution of (1)-(2), then,*

$$\|\boldsymbol{u}\|_{\overline{\Omega}} \leq \max\{\|\boldsymbol{u}(0)\|, \|\boldsymbol{u}(1)\|, \frac{1}{\alpha}\|\boldsymbol{f}\|_{\Omega_1\cup\Omega_2}\}.$$

Lemma 2. *Let* A*(x) satisfy (4)-(5) and let* **u** *be the exact solution of (1)-(2). Then for each $i = 1,\dots,m$, $x \in \Omega_1 \cup \Omega_2$.*

$$|u_i^{(k)}(x)| \leq C\varepsilon_i^{-\frac{k}{2}}(\|\boldsymbol{u}(0)\| + \|\boldsymbol{u}(1)\| + \|\boldsymbol{f}\|_{\Omega_1\cup\Omega_2}) \text{ for } k = 0,1,2.$$
$$|u_i^{(3)}(x)| \leq C\varepsilon_1^{-\frac{1}{2}}\varepsilon_i^{-1}(\|\boldsymbol{u}(0)\| + \|\boldsymbol{u}(1)\| + \|\boldsymbol{f}\|_{\Omega_1\cup\Omega_2} + \sqrt{\varepsilon_1}\|\boldsymbol{f}'\|_{\Omega_1\cup\Omega_2}), \text{ and}$$
$$|u_i^{(4)}(x)| \leq C\varepsilon_1^{-1}\varepsilon_i^{-1}(\|\boldsymbol{u}(0)\| + \|\boldsymbol{u}(1)\| + \|\boldsymbol{f}\|_{\Omega_1\cup\Omega_2} + \varepsilon_1\|\boldsymbol{f}''\|_{\Omega_1\cup\Omega_2}).$$

To derive sharper bounds on the derivatives of the solution, the solution is decomposed into a sum, composed of a regular component $\boldsymbol{v}$ and a singular component $\boldsymbol{w}$. That is, $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$. The regular component $\boldsymbol{v}$, is defined as the solution of the following problem:
$\boldsymbol{L}\boldsymbol{v}(x) = \boldsymbol{f}(x), \quad x \in \Omega_1 \cup \Omega_2, \quad \boldsymbol{v}(x) = A^{-1}(x)\boldsymbol{f}(x), \; x \in \{0, d-, d+, 1\}.$
and the singular component $\boldsymbol{w}$ is defined as the solution of the following problem:

$$\boldsymbol{L}\boldsymbol{w}(x) = \boldsymbol{0}, \qquad x \in \Omega_1 \cup \Omega_2;$$
$$\boldsymbol{w}(x) = \boldsymbol{u}(x) - \boldsymbol{v}(x), \;\; x \in \{0,1\}, \;\; [\boldsymbol{w}](d) = -[\boldsymbol{v}](d), \;\; [\boldsymbol{w}'](\mathrm{d}) = -[\boldsymbol{v}'](d).$$

The following layer functions are used in defining the bounds on derivatives, for $1 \leq i \leq m$:

$$B_{\varepsilon_{l_i}}(x) := \exp(-x\sqrt{\alpha/\varepsilon_i}) + \exp(-(d-x)\sqrt{\alpha/\varepsilon_i}), \tag{6}$$

$$B_{\varepsilon_{r_i}}(x) := \exp((d-x)\sqrt{\alpha/\varepsilon_i}) + \exp(-(1-x)\sqrt{\alpha/\varepsilon_i}). \tag{7}$$

Theorem 3. *Let $\boldsymbol{A}(x)$ satisfy (4)-(5). Then the regular component* **v** *and its derivatives satisfy the bounds for all $x \in \Omega_1 \cup \Omega_2$, $i = 1,\dots m$, and $k = 0,1,2,3,4$,*

$$|v_i^{(k)}(x)|_{\Omega_1\cup\Omega_2} \leq \begin{cases} C(1+\varepsilon_i^{(1-\frac{k}{2})}B_{\varepsilon_{l_i}}(x)), & x \in \Omega_1, \\ C(1+\varepsilon_i^{(1-\frac{k}{2})}B_{\varepsilon_{r_i}}(x)), & x \in \Omega_2. \end{cases}$$

Lemma 3. *For $1 \leq i \leq j \leq m$ and $0 < s \leq 3/2$, there exists a unique point $x_{i,j}^{(s)} \in (0, \frac{d}{2})$ such that $\varepsilon_i^{-s}\mathrm{B}_{\varepsilon_{l_i}}(x_{i,j}^{(s)}) = \varepsilon_j^{-s}\mathrm{B}_{\varepsilon_{l_j}}(x_{i,j}^{(s)})$. Also, $\varepsilon_i^{-s}\mathrm{B}_{\varepsilon_{l_i}}(d - x_{i,j}^{(s)}) = \varepsilon_j^{-s}\mathrm{B}_{\varepsilon_{l_j}}(d - x_{i,j}^{(s)})$. On $[0, x_{i,j}^{(s)}) \cup (d - x_{i,j}^{(s)}, d)$ we have $\varepsilon_i^{-s}\mathrm{B}_{\varepsilon_{l_i}}(x) > \varepsilon_j^{-s}\mathrm{B}_{\varepsilon_{l_j}}(x)$ and on $(x_{i,j}^{(s)}, d - x_{i,j}^{(s)})$ we have $\varepsilon_i^{-s}\mathrm{B}_{\varepsilon_{l_i}}(x) < \varepsilon_j^{-s}\mathrm{B}_{\varepsilon_{l_j}}(x)$. Similarly we can prove the similar result for $x_{i,j}^{(s)} \in \Omega_2$.*

Theorem 4. *Let $\boldsymbol{A}(x)$ satisfy (4)-(5). Then the regular component $\boldsymbol{v}$ and its derivatives satisfy the bounds for all $x \in \Omega_1 \cup \Omega_2,\ i = 1, \ldots, m$, and $k = 0, 1, 2, 3, 4$,*

$$|v_i^{(k)}(x)|_{\Omega_1 \cup \Omega_2} \leq \begin{cases} C\left(1 + \sum_{q=i}^{m} \frac{B_{\varepsilon_{l_q}}(x)}{\varepsilon_q^{\frac{k}{2}-1}}\right), & x \in \Omega_1, \\ C\left(1 + \sum_{q=i}^{m} \frac{B_{\varepsilon_{r_q}}(x)}{\varepsilon_q^{\frac{k}{2}-1}}\right), & x \in \Omega_2. \end{cases}$$“‘

Lemma 4. *Let $\boldsymbol{A}(x)$ satisfy (4)-(5). Then the singular component $\boldsymbol{w}$ of the solution $\boldsymbol{u}$ of (1)-(2) satisfies for $i = 1, \ldots, m$,*

$$|w_i(x)| \leq C \begin{cases} B_{\varepsilon_{l_m}}(x), & x \in \Omega_1, \\ B_{\varepsilon_{r_m}}(x), & x \in \Omega_2, \end{cases} \quad |w_i^{(k)}(x)| \leq C \begin{cases} \sum_{q=i}^{m} \frac{B_{\varepsilon_{l_q}}(x)}{\varepsilon_q^{\frac{k}{2}}}, & x \in \Omega_1, \\ \sum_{q=i}^{m} \frac{B_{\varepsilon_{r_q}}(x)}{\varepsilon_q^{\frac{k}{2}}}, & x \in \Omega_2, \text{ for } k = 1, 2 \end{cases}$$

$$|w_i^{(3)}(x)| \leq C \begin{cases} \sum_{q=1}^{m} \frac{B_{\varepsilon_{l_q}}(x)}{\varepsilon_q^{\frac{3}{2}}}, & x \in \Omega_1, \\ \sum_{q=1}^{m} \frac{B_{\varepsilon_{r_q}}(x)}{\varepsilon_q^{\frac{3}{2}}}, & x \in \Omega_2, \end{cases} \quad |\varepsilon_i w_i^{(4)}(x)| \leq C \begin{cases} \sum_{q=1}^{m} \frac{B_{\varepsilon_{l_q}}(x)}{\varepsilon_q}, & x \in \Omega_1, \\ \sum_{q=1}^{m} \frac{B_{\varepsilon_{r_q}}(x)}{\varepsilon_q}, & x \in \Omega_2. \end{cases}$$

Theorem 5. *Let $\boldsymbol{A}(x)$ satisfy (4)-(5). Then the singular component $\boldsymbol{w}$ can be decomposed in this way as follows, for $1 \leq i \leq m$:*

$$w_i(x) = \sum_{q=1}^{m} w_{i,\varepsilon_q}(x) = \sum_{q=1}^{m} y_{i,\varepsilon_q}(x)$$

$$|w''_{i,\varepsilon_q}(x)| \leq C \begin{cases} \frac{B_{\varepsilon_{l_q}}(x)}{\varepsilon_q}, & x \in \Omega_1, \\ \frac{B_{\varepsilon_{r_q}}(x)}{\varepsilon_q}, & x \in \Omega_2, \end{cases} \quad |w'''_{i,\varepsilon_q}(x)| \leq C \begin{cases} \frac{B_{\varepsilon_{l_q}}(x)}{\varepsilon_q^{3/2}}, & x \in \Omega_1, \\ \frac{B_{\varepsilon_{r_q}}(x)}{\varepsilon_q^{3/2}}, & x \in \Omega_2, \end{cases}$$

$$|\varepsilon_i\, y''_{i,\varepsilon_q}(x)| \leq C \begin{cases} B_{\varepsilon_{l_q}}(x), & x \in \Omega_1, \\ B_{\varepsilon_{r_q}}(x), & x \in \Omega_2, \end{cases} \quad |\varepsilon_i\, y'''_{i,\varepsilon_q}(x)| \leq C \begin{cases} \frac{B_{\varepsilon_{l_q}}(x)}{\sqrt{\varepsilon_q}}, & x \in \Omega_1, \\ \frac{B_{\varepsilon_{r_q}}(x)}{\sqrt{\varepsilon_q}}, & x \in \Omega_2, \end{cases}$$

$$|\varepsilon_i\, y^{(4)}_{i,\varepsilon_q}(x)| \leq C \begin{cases} \frac{B_{\varepsilon_{l_q}}(x)}{\varepsilon_q}, & x \in \Omega_1, \\ \frac{B_{\varepsilon_{r_q}}(x)}{\varepsilon_q}, & x \in \Omega_2. \end{cases}$$

Proof. Define a function $w_{1,\varepsilon_m}(x)$ as follows:

$$w_{i,\varepsilon_m}(x) = \begin{cases} \Sigma_{k=0}^{3} \frac{[(x - x_{m-1,m}^{(3/2)})^k]}{k!} w_i^{(k)}(x_{m-1,m}^{(3/2)}) & \text{for } x \in [0, x_{m-1,m}^{(3/2)}), \\ w_i(x) & \text{for } x \in [x_{m-1,m}^{(3/2)}, d - x_{m-1,m}^{(3/2)}], \\ \Sigma_{k=0}^{3} \frac{[(x - d + x_{m-1,m}^{(3/2)})^k]}{k!} w_i^{(k)}(d - x_{m-1,m}^{(3/2)}) & \text{for } x \in (d - x_{m-1,m}^{(3/2)}, d). \end{cases}$$

For $m-1 \le q \le 2$, define the functions $w_{1,\varepsilon_q}(x)$ as follows:

$$w_{i,\varepsilon_q}(x) = \begin{cases} \Sigma_{k=0}^{3} \frac{[(x-x^{(3/2)}_{q-1,q})^k]}{k!} w_i^{(k)}(x^{(3/2)}_{q-1,q}) & \text{for } x \in [0, x^{(3/2)}_{q-1,q}), \\ w_i(x) - \sum_{t=q+1}^{m} w_{i,\varepsilon_t}(x) & \text{for } x \in [x^{(3/2)}_{q-1,q}, d - x^{(3/2)}_{q-1,q}], \\ \Sigma_{k=0}^{3} \frac{[(x-d+x^{(3/2)}_{q-1,q})^k]}{k!} w_i^{(k)}(d - x^{(3/2)}_{q-1,q}) & \text{for } x \in (d - x^{(3/2)}_{q-1,q}, d), \end{cases}$$

and

$$w_{i,\varepsilon_1}(x) = w_i(x) - \sum_{q=2}^{m} w_{i,\varepsilon_q}(x).$$

Now proceed similar to the proof of Lemma 14 in [4] on the intervals Ω_1 and Ω_2 separately. Similarly to derive the higher order derivative bounds on the components of $w_i(x)$, the functions $y_{1,\varepsilon_q}(x)$, for $1 \le q \le m$ are defined as below.

$$y_{i,\varepsilon_m}(x) = \begin{cases} \Sigma_{k=0}^{4} \frac{[(x-x^{(1)}_{m-1,m})^k]}{k!} w_i^{(k)}(x^{(1)}_{m-1,m}) & \text{for } x \in [0, x^{(1)}_{m-1,m}), \\ w_i(x) & \text{for } x \in [x^{(1)}_{m-1,m}, d - x^{(1)}_{m-1,m}], \\ \Sigma_{k=0}^{4} \frac{[(x-d+x^{(1)}_{m-1,m})^k]}{k!} w_i^{(k)}(d - x^{(1)}_{m-1,m}) & \text{for } x \in (d - x^{(1)}_{m-1,m}, d). \end{cases}$$

For $m-1 \le q \le 2$, define the functions $y_{1,\varepsilon_q}(x)$ as follows:

$$y_{i,\varepsilon_q}(x) = \begin{cases} \Sigma_{k=0}^{4} \frac{[(x-x^{(1)}_{q-1,q})^k]}{k!} w_i^{(k)}(x^{(1)}_{q-1,q}) & \text{for } x \in [0, x^{(1)}_{q-1,q}), \\ w_i(x) - \sum_{t=q+1}^{m} y_{i,\varepsilon_t}(x) & \text{for } x \in [x^{(1)}_{q-1,q}, d - x^{(1)}_{q-1,q}], \\ \Sigma_{k=0}^{4} \frac{[(x-d+x^{(1)}_{q-1,q})^k]}{k!} w_i^{(k)}(d - x^{(1)}_{q-1,q}) & \text{for } x \in (d - x^{(1)}_{q-1,q}, d), \end{cases}$$

and

$$y_{i,\varepsilon_1}(x) = w_i(x) - \sum_{q=2}^{m} y_{i,\varepsilon_q}(x).$$

3 Discretization of the problem

A piecewise uniform variant of Shishkin mesh with N mesh-intervals is now constructed on $\Omega_1^N \cup \Omega_2^N$ which uses the following transition parameters:

$$\sigma_{\varepsilon_{l_m}} := \min\left\{\frac{d}{4}, 2\sqrt{\frac{\varepsilon_m}{\alpha}} \ln N\right\}, \quad \sigma_{\varepsilon_{r_m}} := \min\left\{\frac{(1-d)}{4}, 2\sqrt{\frac{\varepsilon_m}{\alpha}} \ln N\right\},$$

$$\sigma_{\varepsilon_{l_k}} := \min\left\{\frac{\sigma_{\varepsilon_{l_{k+1}}}}{2}, 2\sqrt{\frac{\varepsilon_k}{\alpha}} \ln N\right\}, \; \sigma_{\varepsilon_{r_k}} := \min\left\{\frac{\sigma_{\varepsilon_{r_{k+1}}}}{2}, 2\sqrt{\frac{\varepsilon_k}{\alpha}} \ln N\right\},$$

for $k = m-1, \ldots, 1$. The interior points of the mesh are denoted by
$\Omega^N = \{x_i : 1 \le i \le \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \le i \le N-1\} = \Omega_1^N \cup \Omega_2^N$.

Let $h_i = x_i - x_{i-1}$ be the i^{th} mesh step and $\hbar_i = \frac{h_i + h_{i+1}}{2}$, clearly $x_{\frac{N}{2}} = d$ and $\overline{\Omega}^N = \{x_i : i = 0, 1, \ldots, N\}$. Let $N = 2^l$, $l \geq 8$ be any positive integer.

The interval $[0,d]$ is subdivided into $2m+1$ subintervals $[0, \sigma_{\varepsilon_{l_1}}], \ldots, (\sigma_{\varepsilon_{l_{m-1}}}, \sigma_{\varepsilon_{l_m}}], (\sigma_{\varepsilon_{l_m}}, d - \sigma_{\varepsilon_{l_m}}], (d - \sigma_{\varepsilon_{l_m}}, d - \sigma_{\varepsilon_{l_{m-1}}}], \ldots, (d - \sigma_{\varepsilon_{l_1}}, d]$. On each subinterval $[0, \sigma_{\varepsilon_{l_1}}]$ and $(d - \sigma_{\varepsilon_{l_1}}, d]$, a uniform mesh of $N/2^{m+2}$ mesh intervals are placed. On each subinterval $(\sigma_{\varepsilon_{l_k}}, \sigma_{\varepsilon_{l_{k+1}}}]$ and $(d - \sigma_{\varepsilon_{l_{k+1}}}, \sigma_{\varepsilon_{l_k}}]$, for$1 \leq k \leq m-1$, a uniform mesh of $N/2^{m-k+3}$ mesh intervals, and on $(\sigma_{\varepsilon_{l_m}}, d - \sigma_{\varepsilon_{l_m}}]$ a uniform mesh of $N/4$ mesh intervals are placed. Similarly, we divide the interval $[d, 1]$ into subintervals $[d, d + \sigma_{\varepsilon_{r_1}}], \ldots,\ (d + \sigma_{\varepsilon_{r_{m-1}}}, d + \sigma_{\varepsilon_{rm}}], (d + \sigma_{\varepsilon_{rm}}, 1 - \sigma_{\varepsilon_{rm}}], (1 - \sigma_{\varepsilon_{rm}}, 1 - \sigma_{\varepsilon_{r_{m-1}}}], \ldots, (1 - \sigma_{\varepsilon_{r_1}}, 1]$. On each subinterval $(d, d + \sigma_{\varepsilon_{r_1}}]$ and $(1 - \sigma_{\varepsilon_{r_1}}, 1]$, a uniform mesh of $N/2^{m+2}$ mesh intervals, on each subinterval $(d + \sigma_{\varepsilon_{r_k}}, d + \sigma_{\varepsilon_{r_{k+1}}}]$ and $(1 - \sigma_{\varepsilon_{r_{k+1}}}, 1 - \sigma_{\varepsilon_{r_k}}]$, for $1 \leq k \leq m-1$, a uniform mesh of $N/2^{m-k+3}$ mesh intervals, and on $(d + \sigma_{\varepsilon_{rm}}, 1 - \sigma_{\varepsilon_{rm}}]$ a uniform mesh of $N/4$ mesh intervals are placed.

Define the discrete finite difference operator $\boldsymbol{L}^N$ as follows:

$$\boldsymbol{L}^N \boldsymbol{U}(x_i) = -\boldsymbol{E}\delta^2 \boldsymbol{U}(x_i) + \boldsymbol{A}(x_i)\boldsymbol{U}(x_i) = \boldsymbol{f}(x_i), \qquad \text{for all } \ x_i \in \Omega^N, \tag{8}$$

$$\boldsymbol{U}(x_0) = \boldsymbol{p}, \quad \boldsymbol{U}(x_N) = \boldsymbol{q}, \tag{9}$$

and at $x_{N/2} = d$ the scheme is given by:

$$\boldsymbol{L}^N \boldsymbol{U}(d) = -\boldsymbol{E}\ \delta^2 \boldsymbol{U}(d) + \boldsymbol{A}(d)\boldsymbol{U}(d) = \overline{\boldsymbol{f}}(d), \tag{10}$$

where

$$\delta^2 Z(x_i) = \frac{(D^+ Z(x_i) - D^- Z(x_i))}{\hbar_i}, \qquad D^+ Z(x_i) = \frac{Z(x_{i+1}) - Z(x_i)}{h_{i+1}},$$

$$D^- Z(x_i) = \frac{Z(x_i) - Z(x_{i-1})}{h_i}, \quad \text{and} \quad \overline{f}(d) = \frac{h_{\frac{N}{2}} f(d - h_{\frac{N}{2}}) + h_{\frac{N}{2}+1} f(d + h_{\frac{N}{2}+1})}{h_{\frac{N}{2}} + h_{\frac{N}{2}+1}}.$$

Lemma 5. *Suppose the mesh function* $\mathbf{W}$ *satisfy* $\mathbf{W}_0 \geq \boldsymbol{0}, \mathbf{W}_N \geq \boldsymbol{0}$ *and* $\boldsymbol{L}^N \boldsymbol{W} \geq \boldsymbol{0}$ *for all* $x_i \in \overline{\Omega}^N$, *then* $\boldsymbol{W} \geq \boldsymbol{0}$ *for all* $x_i \in \overline{\Omega}^N$.

Lemma 6. *If* $\boldsymbol{U}$ *be the numerical solution of (8)-(9), then,*

$$\|\boldsymbol{U}\|_{\overline{\Omega}^N} \leq \max\left\{ \|\boldsymbol{U}(0)\|, \|\boldsymbol{U}(1)\|, \frac{1}{\alpha}\|\boldsymbol{f}\|_{\Omega_1^N \cup\ \Omega_2^N} \right\}.$$

4 Error analysis

The truncation error will be bounded using the following result which can be verified by a Taylor expansion. Let $\phi_j \in C^4(\Omega_1 \cup \Omega_2)$. Then for $i = 1, \ldots, N-1$ we have

$$|\varepsilon_j(\frac{d^2}{dx^2}-\delta^2)\phi_j(x_i)| \leq \begin{cases} C\varepsilon_j(x_{i+1}-x_{i-1})|\phi_j|_3 & (11)\\ C\varepsilon_j h^2|\phi_j|_4,\ x_{i+1}-x_i = x_i - x_{i-1} = h & (12)\\ C\varepsilon_j \max\limits_{x\in[x_{i-1},x_{i+1}]}|\phi_j''(x_i)|, & (13)\end{cases}$$

where $j=1,\dots,m$, $i\neq\frac{N}{2}$, $|z_j|_t := \max|\frac{d^t z}{dx^t}|,\ \ \forall t\in \mathrm{N}$.

Using the bounds on the derivatives on the regular components derived in Section 2 on (11) and (12), we have

$$|\,((\boldsymbol{L}^N-\boldsymbol{L})\boldsymbol{v})_j(x_i)\,| \leq \begin{cases} CN^{-2} & for\ x_i \notin \{\sigma_{\varepsilon_{l_k}}, d-\sigma_{\varepsilon_{l_k}}, d+\sigma_{\varepsilon_{r_k}}, 1-\sigma_{\varepsilon_{r_k}}\},\\ C\sqrt{\varepsilon_j}(h_{\varepsilon_k}+h_{\varepsilon_{k+1}}) & for\ x_i \in \{\sigma_{\varepsilon_{l_k}}, d-\sigma_{\varepsilon_{l_k}}, d+\sigma_{\varepsilon_{r_k}}, 1-\sigma_{\varepsilon_{r_k}}\}, j>k,\\ C\frac{\varepsilon_j}{\sqrt{\varepsilon_k}}(h_{\varepsilon_k}+h_{\varepsilon_{k+1}}) & for\ x_i \in \{\sigma_{\varepsilon_{l_k}}, d-\sigma_{\varepsilon_{l_k}}, d+\sigma_{\varepsilon_{r_k}}, 1-\sigma_{\varepsilon_{r_k}}\}, j\leq k.\end{cases}$$

To evaluate the error estimates for the singular components on different subintervals proceed as follows:

Case (i) For $x_i \in [\sigma_{\varepsilon_{l_m}}, d-\sigma_{\varepsilon_{l_m}}]\cup[d+\sigma_{\varepsilon_{r_m}}, 1-\sigma_{\varepsilon_{r_m}}]$.

Consider first that $x_i\in[\sigma_{\varepsilon_{l_m}},\frac{d}{2}]$. Using (13) and bounds on singular components, we have, for $k=1,\dots,m$,

$$|\,((\boldsymbol{L}^N-\boldsymbol{L})\boldsymbol{w})_j(x_i)\,| \leq C\varepsilon_j\sum_{q=j}^{m}\frac{B_{\varepsilon_{l_q}}(x)}{\varepsilon_q} \leq C\parallel B_{\varepsilon_{l_m}}\parallel_{[x_{i-1},x_{i+1}]} = B_{\varepsilon_{l_m}}(x_{i-1}) \leq CN^{-2}.$$

We can prove a similar result when $x_i\in[\frac{d}{2},d-\sigma_{\varepsilon_{l_m}}]$. Similar arguments prove a similar result for the subinterval $[d+\sigma_{\varepsilon_{r_m}},1-\sigma_{\varepsilon_{r_m}}]$. Hence, for $x_i\in[\sigma_{\varepsilon_{l_m}},d-\sigma_{\varepsilon_{l_m}}]\cup[d+\sigma_{\varepsilon_{r_m}},1-\sigma_{\varepsilon_{r_m}}]$ we have,

$$|\,((\boldsymbol{L}^N-\boldsymbol{L})\boldsymbol{w})_k(x_i)\,| \leq CN^{-2}.$$

Case (ii) For $x_i\in(0,\sigma_{\varepsilon_{l_1}})\cup(d-\sigma_{\varepsilon_{l_1}},d)\cup(d,d+\sigma_{\varepsilon_{r_1}})\cup(1-\sigma_{\varepsilon_{r_1}},1)$.

Since the mesh is uniform in these intervals, using (12) and the bounds on the singular components yields

$$|\,((\boldsymbol{L}^N-\boldsymbol{L})\boldsymbol{w})_j(x_i)\,| \leq h_i^2\|\varepsilon_j w_j^{(4)}\| \leq h_i^2\sum_{q=1}^{m}\frac{B_{\varepsilon_{l_q}}(x)}{\varepsilon_q} \leq C(N^{-1}\ln N)^2.$$

Case (iii) For $x_i\in(\sigma_{\varepsilon_{l_k}},\sigma_{\varepsilon_{l_{k+1}}})\cup(d-\sigma_{\varepsilon_{l_{k+1}}},d-\sigma_{\varepsilon_{l_k}})\cup(d+\sigma_{\varepsilon_{r_k}},d+\sigma_{\varepsilon_{r_{k+1}}})\cup(d-\sigma_{\varepsilon_{r_{k+1}}},d-\sigma_{\varepsilon_{r_k}})$, where $1\leq k\leq m-1$.

Using the decomposition in Theorem 5 of singular components and bounds on singular components gives,

$$|\,((\boldsymbol{L}^N-\boldsymbol{L})\boldsymbol{w})_j(x_i)\,| = |\sum_{q=1}^{m-1}\varepsilon_j(\frac{d^2}{dx^2}-\delta^2)y_{j,\varepsilon_q}(x_i)+\varepsilon_j(\frac{d^2}{dx^2}-\delta^2)y_{j,\varepsilon_m}(x_i)|. \quad (14)$$

Consider the first part of (14) and using the bounds on singular components, we obtain

$$|\sum_{q=1}^{m-1}\varepsilon_j(\frac{d^2}{dx^2}-\delta^2)y_{j,\varepsilon_q}(x_i)| \leq \|\sum_{q=1}^{m-1}\varepsilon_j y''_{j,\varepsilon_q}\|_{[x_{i-1},x_{i+1}]} \leq CB_{\varepsilon_{l_{m-1}}}(x_{i-1}) \leq CN^{-2}.$$

Using the bounds on singular components for the second part of (14), we have

$$|\varepsilon_j(\frac{d^2}{dx^2}-\delta^2)y_{j,\varepsilon_m}(x_i)| \leq \frac{\varepsilon_j h_i^2}{12}\parallel y_{j,\varepsilon_m}^{(4)}\parallel \leq C(N^{-1}\ln N)^2.$$

Case (iv) For $x_i\in\{\sigma_{\varepsilon_{l_k}},d-\sigma_{\varepsilon_{l_k}},d+\sigma_{\varepsilon_{r_k}},1-\sigma_{\varepsilon_{r_k}}\}$, where $1\leq k\leq m-1$.

Using the decomposition of the singular components and bounds on singular components defined in Theorem 5 gives

$$|\ ((\boldsymbol{L}^N-\boldsymbol{L})\boldsymbol{w})_j(x_i)\ |\le |\sum_{q=1}^{m-1}\varepsilon_j(\frac{d^2}{dx^2}-\delta^2)w_{j,\varepsilon_q}(x_i)+\varepsilon_j(\frac{d^2}{dx^2}-\delta^2)w_{j,\varepsilon_m}(x_i)|. \quad (15)$$

Consider the first part of (15) for the case $j\le k$, and using the definition of point $x_{i,j}^{(s)}$ we have

$$|\sum_{q=1}^{m-1}\varepsilon_j(\frac{d^2}{dx^2}-\delta^2)w_{j,\varepsilon_q}(x_i)|\le \|\sum_{q=1}^{m-1}\varepsilon_j w''_{j,\varepsilon_q}\|_{[x_{i-1},x_{i+1}]}\le CN^{-2}.$$

and if, $j>k$, using the bounds on singular component and the analysis in Case (i)

$$|\sum_{q=1}^{m-1}\varepsilon_j(\frac{d^2}{dx^2}-\delta^2)w_{j,\varepsilon_q}(x_i)|\le \|\sum_{q=1}^{m-1}\varepsilon_j w''_{j,\varepsilon_q}\|_{[x_{i-1},x_{i+1}]}\le CN^{-2}.$$

For the second part of (15), use bounds on singular components defined in Theorem 5, to obtain

$$|\varepsilon_j(\tfrac{d^2}{dx^2}-\delta^2)y_{j,\varepsilon_m}(x_i)|\le C\varepsilon_j(h_{\varepsilon_k}+h_{\varepsilon_{k+1}})\parallel w'''_{j,\varepsilon_m}\parallel\le C\frac{\varepsilon_j}{\sqrt{\varepsilon_{k+1}\varepsilon_k}}N^{-1}.$$

Case (v) At the point $x_{N/2}=d$,$h_{N/2}=h_{N/2+1}=h$ and $\sigma_{\varepsilon_{l_1}}=\sigma_{\varepsilon_{r_1}}=2\sqrt{\frac{\varepsilon_1}{\alpha}}\ln N$,

$$|(\boldsymbol{L}^N(\boldsymbol{U}-\boldsymbol{u}))_j(d)|\ \le\ C(N^{-1}\ln N), \qquad 2\le j\le m.$$

The result can be proved by following the similar calculations considered in [7].

Theorem 6. *Let* $\boldsymbol{u}$ *be the solution of the problem (1)-(2) and* $\boldsymbol{U}$ *be the solution of discrete problem on the variant of Shishkin mesh defined in Section 3, then*

$$\|\boldsymbol{U}-\boldsymbol{u}\|_{\overline{\Omega}^N}\le C(N^{-1}\ln N)^2.$$

Proof. Define the mesh functions $\eta_{1k},\eta_{2k},\eta_3,\eta_4$ for $k=1,\dots,m$ to be

$$\eta_{1k}(x_i):=\Pi_{j=1}^{i}\left(1+\sqrt{\frac{\alpha}{2\varepsilon_k}}h_j\right), \qquad \eta_{2k}(x_i):=\Pi_{j=1}^{i}\left(1+\sqrt{\frac{\alpha}{2\varepsilon_k}}h_j\right)^{-1},$$

$$\eta_3(x_i)=\Pi_{j=1}^{i}\left(1+\sqrt{\frac{\alpha}{2\varepsilon_1}}h_j\right), \qquad \eta_4(x_i)=\Pi_{j=1}^{i}\left(1+\sqrt{\frac{\alpha}{2\varepsilon_1}}h_j\right)^{-1}.$$

These mesh functions satisfies the following properties:

$$D^-\eta_{1k}(x_i)=\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_k}(1+\sqrt{\alpha}h_i/\sqrt{2\varepsilon_k})}\eta_{1k}(x_i), \qquad D^+\eta_{1k}(x_i)=\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_k}}\eta_{1k}(x_i),$$

$$D^+\eta_{2k}(x_i)=-\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_k}(1+\sqrt{\alpha}h_{i+1}/\sqrt{2\varepsilon_k})}\eta_{2k}(x_i), \qquad D^-\eta_{2k}(x_i)=-\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_k}}\eta_{2k}(x_i),$$

$$D^-\eta_3(x_i)=\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_1}(1+\sqrt{\alpha}h_i/\sqrt{2\varepsilon_1})}\eta_3(x_i), \qquad D^+\eta_3(x_i)=\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_1}}\eta_3(x_i),$$

$$D^+\eta_4(x_i)=-\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_1}(1+\sqrt{\alpha}h_{i+1}/\sqrt{2\varepsilon_1})}\eta_4(x_i), \qquad D^-\eta_4(x_i)=-\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_1}}\eta_4(x_i).$$

Define the barrier functions $\theta_k, \theta_{kd}, \theta_d$ for $k = 1,\ldots,m$ as follows:

$$\theta_k(x_i) = \begin{cases} \frac{x_i}{\sigma_{\varepsilon_{l_k}}}, \ 0 \le x_i \le \sigma_{\varepsilon_{l_k}}, \\ 1, \quad \sigma_{\varepsilon_{l_k}} \le x_i \le 1-\sigma_{\varepsilon_{r_k}}, \\ \frac{1-x_i}{\sigma_{\varepsilon_{r_k}}}, \ 1-\sigma_{\varepsilon_{r_k}} \le x_i \le 1, \end{cases}$$

$$\theta_{kd}(x_i) = \begin{cases} \frac{\eta_{1k}(x_i)}{\eta_{1k}(d-\sigma_{\varepsilon_{l_k}})}, & 0 \le x_i \le d-\sigma_{\varepsilon_{l_k}}, \\ 1, & d-\sigma_{\varepsilon_{l_k}} \le x_i \le d+\sigma_{\varepsilon_{r_k}}, \\ \frac{\eta_{2k}(x_i)}{\eta_{2k}(d+\sigma_{\varepsilon_{r_k}})}, & d+\sigma_{\varepsilon_{r_k}} \le x_i \le 1, \end{cases}$$

and

$$\theta_d(x_i) = \begin{cases} \frac{\eta_3(x_i)}{\eta_3(d)}, & 0 \le x_i \le d, \\ \frac{\eta_4(x_i)}{\eta_4(d)}, & d \le x_i \le 1. \end{cases}$$

Now define the mesh function for $i \neq \frac{N}{2}$,

$$\Theta^{\pm}(x_i) = C(N^{-1}\ln N)^2(1+\sum_{k=1}^{m}\theta_k(x_i)+\theta_{kd}(x_i))(1,\ldots,1)^T \pm (\boldsymbol{U}-\boldsymbol{u})(x_i),$$

and for $i = \frac{N}{2}$,

$$\Theta^{\pm}(x_i) = C(N^{-1}\ln N)^2(1+\theta_d(x_i))(1,\ldots,1)^T \pm (\boldsymbol{U}-\boldsymbol{u})(x_i).$$

Using the discrete maximum principle defined in Lemma 5, we conclude that

$$\|\boldsymbol{U}-\boldsymbol{u}\|_{\overline{\Omega}^N} \le C(N^{-1}\ln N)^2.$$

5 Numerical results

Example 5.1 Consider the singularly perturbed reaction-diffusion equations with discontinuous source term

$$\begin{aligned} &-\varepsilon_1 u_1''(x)+3u_1(x)-(1-x)u_2(x)-(1-x)u_3(x) = f_1(x), && x \in \Omega_1 \cup \Omega_2, \\ &-\varepsilon_2 u_2''(x)-2u_1(x)+(4+x)u_2(x)-u_3(x) = f_2(x), && x \in \Omega_1 \cup \Omega_2, \\ &-\varepsilon_3 u_3''(x)-2u_1(x)-3u_2(x)+(6+x)u_3(x) = f_3(x), && x \in \Omega_1 \cup \Omega_2, \\ &\quad \boldsymbol{u}(0) = \boldsymbol{0}, \qquad \boldsymbol{u}(1) = \boldsymbol{0}, \end{aligned}$$

where

$$f_1(x) = \begin{cases} e^x & \text{for } 0 \le x \le 0.5 \\ 2 & \text{for } 0.5 < x \le 1, \end{cases} \qquad f_2(x) = \begin{cases} \cos x & \text{for } 0 \le x \le 0.5 \\ 4 & \text{for } 0.5 < x \le 1, \end{cases}$$

and

$$f_3(x) = \begin{cases} (1+x^2) & \text{for } 0 \le x \le 0.5 \\ 3 & \text{for } 0.5 < x \le 1. \end{cases}$$

For the construction of piecewise-uniform Shishkin mesh $\overline{\Omega}_N$, we take $\alpha = 0.95$ for Example 5.1. Since the exact solution of the Example 5.1 is not known, we estimate the error for $\boldsymbol{U}$ by comparing it to the numerical solution $\widetilde{\boldsymbol{U}}$ obtained on the mesh $\widetilde{x}_j$ that contains the mesh points of the original mesh and their mid-points. That is, $\widetilde{x}_{2j} = x_j, j = 0,...,N, \widetilde{x}_{2j+1} = (x_j + x_{j+1})/2, j = 0,...,N-1$. For different values of N and $\varepsilon_1, \varepsilon_2, \varepsilon_3$, we compute the maximum point-wise errors $D^N_{\varepsilon_1,\varepsilon_2,\varepsilon_3} := \|(\boldsymbol{U} - \widetilde{\boldsymbol{U}})(x_j)\|_{\overline{\Omega}^N}$, where the singular perturbation parameters take values from the set $S_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = \{(\varepsilon_1, \varepsilon_2, \varepsilon_3)|, \varepsilon_1 = 10^{-j}, 0 \le j \le 12, \varepsilon_2 = 10^{-l}, 0 \le l \le j, \varepsilon_3 = 10^{-k}, 0 \le k \le l\}$. For each fixed j, the choices for l are: $l = 0, 1, \ldots\ldots, j$ and for each fixed l, the choices for k are: $k = 0, 1, \ldots\ldots, l$. That is, for a fixed ε_1, there are $0, 1, \ldots\ldots, j$ choices of ε_2, and for fixed ε_1 and ε_2 there are $0, 1, \ldots\ldots, l$ choices of ε_3. Then the parameter-uniform error is computed as $D^N := \max\limits_{S_{\varepsilon_1,\varepsilon_2,\varepsilon_3}} \{D^N_{\varepsilon_1,\varepsilon_2,\varepsilon_3}\}$. The maximum point-wise errors for all possible values of $\varepsilon_1, \varepsilon_2, \varepsilon_3$, the parameter uniform error for each N and the order of convergence which is calculated using the formula $p^N = \dfrac{\ln(D^N) - \ln(D^{2N})}{\ln(2\ln N) - \ln(\ln(2N))}$, are shown in the Table 1. It can be seen from this table that the present method gives almost second order parameter uniformly convergent results as proved in the Theorem 6.

Table 1 Maximum point-wise errors $D^N_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$, D^N and $\varepsilon_1, \varepsilon_2, \varepsilon_3-$uniform rate of convergence p^N for Example 5.1.

$\varepsilon_1 = 10^{-j}$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$	$N = 4096$
0	1.60E-06	3.99E-07	9.99E-08	2.49E-08	6.13E-09
1	1.07E-05	2.43E-06	6.67E-07	1.51E-07	3.81E-08
2	7.27E-05	1.78E-05	4.45E-06	1.11E-06	2.84E-07
3	6.95E-04	1.75E-04	4.37E-05	1.09E-05	2.75E-06
4	6.31E-03	1.70E-03	4.34E-04	1.09E-04	2.73E-05
5	1.81E-02	7.78E-03	2.19E-03	6.77E-04	2.02E-04
6	1.90E-02	1.16E-02	4.11E-03	1.33E-03	4.05E-04
7	3.51E-02	1.16E-02	4.11E-03	1.33E-03	4.05E-04
8	7.40E-02	3.57E-02	1.29E-02	3.96E-03	1.13E-03
9	7.90E-02	3.86E-02	1.49E-02	5.02E-03	1.57E-03
10	7.90E-02	3.86E-02	1.49E-02	5.02E-03	1.57E-03
11	7.90E-02	3.86E-02	1.49E-02	5.02E-03	1.57E-03
12	7.90E-02	3.86E-02	1.49E-02	5.02E-03	1.57E-03
D^N	7.90E-02	3.86E-02	1.49E-02	5.02E-03	1.57E-03
p^N	1.24	1.61	1.82	1.92	

Acknowledgements The authors gratefully acknowledge the valuable comments and suggestions from the anonymous referee and Prof. Petr Knobloch. The research work of the second author is supported by Council of Scientific and Industrial Research, India.

References

1. Falco, C. de, O'Riordan, E.: Interior Layers in a reaction-diffusion equation with a discontinuous diffusion coefficient. Int. J. Numer. Anal. Model. **7**, 444-461 (2010)
2. Farrell, P.A., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Singularly perturbed differential equations with discontinuous source terms, Proceedings of Analytical and Numerical Methods for Convection-Dominated and Singularly Perturbed Problems, Nova Science Publishers, NewYork, USA, 23-32 (2000)
3. Linss, T., Madden N.: Layer-adapted meshes for a linear system of coupled singularly perturbed reaction-diffusion problems, IMA J. Num. Anal., **29**, 109-125 (2009)
4. Paramasivam, M., Valarmathi, S., Miller, J.J.H.: Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear reaction-diffusion system, Math. Commun. **15**, 587-612 (2010)
5. Paramasivam, M., Miller, J.J.H., Valarmathi, S.: Parameter-uniform convergence for a finite difference method for a singulalry perturbed linear reaction-diffusion system with discontinuous source terms. Int. J. Numer. Anal. Model. **11**, 385-399 (2014)
6. Protter, M.H., Weinberger, H.F.: Maximum principles in differential equations, Prentice-Hall, Englewood Cliffs (1967)
7. Rao, S.C.S., Chawla, S.: Interior layers in coupled system of two singularly perturbed reaction-diffusion equations with discontinuous source term. NAA 2012, LNCS 8236, 445-453 (2012)