Outflow conditions for the Navier-Stokes equations with skew-symmetric formulation of the convective term

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Abstract The classical do-nothing condition is very often prescribed at outflow boundaries for fluid dynamical problems. However, it has a severe drawback in the context of the Navier-Stokes equations, because not even existence of weak solutions can be shown. The reason is that this boundary condition does not exhibit any control about inflow across such boundaries. This has also severe impact onto the stability of numerical algorithms for flows at higher Reynolds number. A modification of this boundary condition is one possibility to circumvent these drawbacks. This paper addresses such modifications in the context of the skew-symmetric formulation of the convective term. Moreover, we introduce a parameter which gives the possibility to downsize possible inflow even more and to enhance the stability further. Numerical examples illustrate the effectiveness of the approach.

1 Introduction

The classical do-nothing condition (CDN) is very often prescribed at outflow boundaries for fluid dynamical problems. However, not even existence of weak solutions can be shown, if this condition is used for the Navier-Stokes equation, see [6]. The reason is that this boundary condition does not exhibit any control about inflow across such boundaries, see [4]. This has also severe impact onto the stability of numerical algorithms for flows at higher Reynolds numbers. The directional do-nothing (DDN) boundary condition is one possibility to circumvent this disadvantage. In particular, existence of weak solutions are proved in [4], and in several applications the stability is enhanced compared to the classical do-nothing condition, see e.g. [1, 7]. The issue of appropriate boundary conditions for the Navier-Stokes and Euler system is also recently addressed by Becker et al. in [2]. Therein, a Nitsche

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method with a proper scaling is proposed in order to obtain control over the kinetic energy.

This paper addresses a variant of the DDN condition in the context of the skew-symmetric formulation of the convective term. We show existence of weak solutions and, in the case of small data, also uniqueness. Moreover, we introduce a parameter which gives the possibility to downsize possible inflow even more and to enhance the stability further.

We consider the stationary incompressible Navier-Stokes equation in the bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$,

$$\begin{aligned} (u\cdot\nabla)u - \nu\Delta u + \nabla p = f \quad &\text{in } \Omega\,, \\ \operatorname{div} u = 0 \quad &\text{in } \Omega\,. \end{aligned}$$

Here, $p : \Omega \to \mathbb{R}$ denotes the pressure and $u : \Omega \to \mathbb{R}^d$ the velocity field. The usual variational spaces for these functions are $L^2(\Omega)$ for p and the Sobolev space $H^1(\Omega)^d$ for u. The constant $\nu > 0$ is the viscosity. The right hand side f is assumed to be in $L^2(\Omega)^d$. The norm in the Sobolev space $W^{m,p}(\Omega)$ is denoted by $\|\cdot\|_{W^{m,p}(\Omega)}$. For the $L^2(\Omega)$-norm we suppress the index and simply write $\|\cdot\|$. The corresponding L^2-scalar product is denoted by $(\cdot,\cdot)$.

The boundary $\partial\Omega = S_0 \cup S_1$ is split into a Dirichlet part S_0 with homogeneous Dirichlet conditions

$$u = 0 \quad \text{on } S_0$$

and an outflow part S_1. Although S_1 is called "outflow" boundary, the flow may also be directed (locally) into Ω due to possible presence of vortices. The classical do-nothing condition (CDN) reads

$$\nu\frac{\partial u}{\partial n} - pn = 0 \quad \text{on } S_1. \tag{1}$$

This condition is easy to implement for discretization methods based on variational formulations, because it arises naturally by integration by parts of the viscous term and the pressure gradient. The corresponding semi-linear form for $\phi \in V := \{\phi \in H^1(\Omega)^d \mid \phi = 0 \text{ a.e. on } S_0\}$ and $\chi \in Q := L^2(\Omega)$ reads

$$B(u,p;\phi,\chi) = ((u\cdot\nabla)u,\phi) + (\nu\nabla u,\nabla\phi) - (p,\operatorname{div}\phi) + (\operatorname{div} u,\chi).$$

The severe drawback of the corresponding variational formulation is that not even existence of weak solutions can be shown due to the absence of a uniform stability property of possible solutions, see e.g. [4].

2 Directional do-nothing condition

In order to distinguish between the outflow and the inflow portion of S_1 we use for a scalar quantity v the notation

$$v_+ := \max(v,0) \quad \text{and} \quad v_- := \min(v,0),$$

such that $v = v_+ + v_-$. In contrast to (1) we consider in this work the directional do-nothing condition (DDN):

$$\nu \frac{\partial u}{\partial n} - pn - \frac{1+\beta}{2}(u\cdot n)_- u = 0 \quad \text{on } S_1, \tag{2}$$

with free parameter $\beta \geq 0$. This boundary condition was analyzed in [4] for the particular case $\beta = 0$. Moreover, we consider here the case of the skew-symmetric formulation of the convective term in combination with such DDN condition.

The skew-symmetric formulation of the convective term is obtained for $u, \phi \in V$ and $\operatorname{div} u = 0$ by partial integration:

$$\begin{aligned}
((u\cdot\nabla)u,\phi) &= \frac{1}{2}\left(((u\cdot\nabla)u,\phi) - (u, \operatorname{div}(\phi\otimes u)) + \int_{\partial\Omega}(u\cdot n)u\cdot\phi\,ds\right)\\
&= \frac{1}{2}(((u\cdot\nabla)u,\phi) - (u,(u\cdot\nabla)\phi)) + \\
&\quad \frac{1}{2}\int_{S_1}[(u\cdot n)_+ + (u\cdot n)_-]\,u\cdot\phi\,ds.
\end{aligned}$$

The arising boundary integral does not vanish because we do not have Dirichlet conditions on S_1. We will treat the DDN condition (2) in the weak sense. This leads us to the following associated bilinear form (linear in (u,p) and (ϕ,χ)):

$$\begin{aligned}
A(w)(u,p;\phi,\chi) &= \frac{1}{2}(((w\cdot\nabla)u,\phi) - (u,(w\cdot\nabla)\phi)) + (\nu\nabla u,\nabla\phi)\\
&\quad -(p,\operatorname{div}\phi) + (\operatorname{div} u,\chi)\\
&\quad +\frac{1}{2}\int_{S_1}[(w\cdot n)_+ - \beta(w\cdot n)_-]\,u\cdot\phi\,ds.
\end{aligned}$$

The variational equation becomes for $X := V\times Q$:

$$(u,p)\in X: \quad A(u)(u,p;\phi,\chi) = (f,\phi) \quad \forall(\phi,\chi)\in X. \tag{3}$$

The variational formulation with skew-symmetric convection terms and CDN condition is also given by (3) with parameter $\beta = -1$. The advantage of this bilinear form A for $\beta \geq 0$ consists in the following two stability properties:

- The skew-symmetric form of the convective term is very convenient, because for the convective term it holds:

$$\frac{1}{2}\left(((w\cdot\nabla)u,u)-(u,(w\cdot\nabla)u)\right)=0 \qquad \forall w\in V.$$

In particular, this sum vanishes even though w is not divergence free.

- Due to the signs of the boundary terms of DDN-type we get non-negativity for the semi-linear form

$$A(w)(u,p;u,p)\geq 0.$$

We will see in the next section that these properties are important to ensure existence (and uniqueness) of solutions.

The formulation (3) has certain similarities to the one proposed in [5] but it is not identical. They include the Stokes solution in the variational formulation and use the symmetric stress tensor σ instead of $\partial_n u$. In the particular case that the Stokes solution vanishes and for a certain choice of parameters, their formulation has similarities to (3) with $\beta=0$.

3 Existence of weak solutions

For showing stability of the semi-linear form A we make use of the following non-negative quantity:

$$|||u|||_\beta := \left(\nu\|\nabla u\|^2+\frac{1}{2}\int_{S_1}[(u\cdot n)_+-\beta(u\cdot n)_-]\,u^2\,ds\right)^{1/2}.$$

Obviously holds $|||u|||_\beta \geq \nu\|\nabla u\|\geq 0$ for arbitrary non-negative β.

Proposition 1. *Let $\Omega\subset\mathbb{R}^d$, $d\in\{2,3\}$, a bounded Lipschitz domain, $f\in L^2(\Omega)^d$, $S_1\subseteq\partial\Omega$ a C^1 boundary, and $\beta\geq 0$. Then a weak solution (u,p) of (3) exists. Solutions of (3) are bounded:*

$$|||u|||_\beta \leq c_\Omega \nu^{-1/2}\|f\|. \tag{4}$$

Proof. Testing the semi-linear form A diagonally with $\phi=u$ and $\chi=p$ with a possible solution (u,p) of (3) leads to the equality:

$$\begin{aligned}(f,u) &= A(u)(u,p;u,p)\\ &= \nu\|\nabla u\|^2+\frac{1}{2}\int_{S_1}[(u\cdot n)_+-\beta(u\cdot n)_-]\,u^2\,ds\\ &= |||u|||_\beta^2.\end{aligned}$$

Applying Cauchy-Schwarz and the Friedrichs inequality (with constant c_Ω) yields the upper bound:

$$|||u|||_\beta^2 \leq \|f\|\|u\| \;\leq\; c_\Omega\|f\|\nu^{-1/2}|||u|||_\beta.$$

This implies (4). Hence, possible solutions are uniformly bounded. Now, we apply standard arguments with finite-dimensional Galerkin solutions $u_n \in V_n$, $\dim u_n = n \in N$. The uniform bound (4) is sufficient to deduce the existence of a limit solution $u \in V$ for $n \to \infty$. The pressure p is obtained by help of the inf-sup condition. The C^1 property of S_1 is needed for the application of a trace theorem. For details we refer to [4] where the proof is presented in more detail for the case $\beta = 0$. □

4 Uniqueness of weak solutions for small data

Proposition 2. *Under the same conditions as Proposition 1 and the additional assumption of small data*

$$\|f\| \le c\nu^2,$$

the solution (u, p) of (3) is unique. The constant c only depends on Ω and S_1.

Proof. Let (u_1, p_1) and (u_2, p_2) be two solutions of (3). Reverse integration by parts of the convective terms yields for both solutions ($i = 1, 2$):

$$\begin{aligned}(f,\phi) &= A(u_i)(u_i,p_i;\phi,\chi)\\ &= ((u_i\cdot\nabla)u_i,\phi) + \nu(\nabla u_i,\nabla\phi) - (p_i,\operatorname{div}\phi) + (\operatorname{div} u_i,\chi)\\ &\quad - \frac{1+\beta}{2}\int_{S_1}(u_i\cdot n)_- u_i\cdot\phi\,ds.\end{aligned}$$

With the particular choice $\phi = e := u_1 - u_2$ and $\chi = q := p_1 - p_2$ we obtain the equality

$$\begin{aligned}0 &= ((u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2, e) + \nu\|\nabla e\|^2\\ &\quad - \frac{1+\beta}{2}\int_{S_1}((u_1\cdot n)_- u_1 - (u_2\cdot n)_- u_2)\cdot e\,ds.\end{aligned}$$

The convective and the boundary terms can be written as

$$\begin{aligned}(u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2 &= (e\cdot\nabla)u_1 + (u_2\cdot\nabla)e,\\ (u_1\cdot n)_- u_1 - (u_2\cdot n)_- u_2 &= (e\cdot n)_- u_1 + (u_2\cdot n)_- e.\end{aligned}$$

This leads to the equality

$$\begin{aligned}\nu\|\nabla e\|^2 &= -((e\cdot\nabla)u_1 + (u_2\cdot\nabla)e, e)\\ &\quad + \frac{1+\beta}{2}\int_{S_1}((e\cdot n)_- u_1 + (u_2\cdot n)_- e)\cdot e\,ds.\end{aligned}$$

The second term on the right hand side can be reformulated by partial integration:

$$
\begin{aligned}
-((u_2\cdot\nabla)e,e) &= -\frac{1}{2}\int_{S_1}(u_2\cdot n)|e|^2\,ds\\
&\le -\frac{1}{2}\int_{S_1}(u_2\cdot n)_-|e|^2\,ds.
\end{aligned}
$$

Using the non-positivity of the following boundary integral

$$
\frac{\beta}{2}\int_{S_1}(u_2\cdot n)_-|e|^2\,ds \le 0,
$$

yields the inequality:

$$
\nu\|\nabla e\|^2 \le -((e\cdot\nabla)u_1,e)+\frac{1+\beta}{2}\int_{S_1}(e\cdot n)_-u_1\cdot e\,ds.
$$

The convective term can be bounded in the classical way by Hölder's inequality and the embedding of Sobolev, $H^1(\Omega)\subset L^q(\Omega)$, with $1\le q\le 6$ for $d\in\{2,3\}$:

$$
\begin{aligned}
((e\cdot\nabla)u_1,e) &\le \|(e\cdot\nabla)u_1\|_{L^{3/2}}\|e\|_{L^3}\\
&\le \|e\|_{L^6}\|\nabla u_1\|\|e\|_{L^3}\\
&\le C\|\nabla u_1\|\|\nabla e\|^2.
\end{aligned}
$$

Taking into account that S_1 is C^1-regular, the remaining boundary integral can be bounded by the trace theorem, $W^{1,1}(\Omega)\subset L^1(S_1)$:

$$
\begin{aligned}
\int_{S_1}(e\cdot n)_-u_1\cdot e\,ds &\le \|e^2u_1\|_{L^1(S_1)}\\
&\le C\|e^2u_1\|_{W^{1,1}(\Omega)}\\
&= C\left(\|e^2u_1\|^2_{L^1(\Omega)}+\|\nabla(e^2u_1)\|^2_{L^1(\Omega)}\right)^{1/2}.
\end{aligned}
$$

Application of the Sobolev embedding (in similar fashion as above) and Friedrichs inequality yields:

$$
\int_{S_1}(e\cdot n)_-u_1\cdot e\,ds \le C\|\nabla u_1\|\|\nabla e\|^2.
$$

Hence, we arrive at the upper bound

$$
\nu\|\nabla e\|^2 \le C\|\nabla u_1\|\|\nabla e\|^2.
$$

In the case of multiple solutions, $e\neq 0$, it follows

$$
\|\nabla u_1\| \ge C^{-1}\nu.
$$

Further, we know due to the stability result (4) that the upper bound holds

$$\|\nabla u_1\| \leq c_\Omega \nu^{-1} \|f\|.$$

The combination of these two inequalities implies in the case $e \neq 0$:

$$\|f\| \geq \frac{1}{c_\Omega C} \nu^2.$$

This states the assertion that the data must be large enough to admit multiple solutions. □

Remark 1. We cannot expect uniqueness without the smallness assumption in Proposition 2, since we encounter similar difficulties as for Navier-Stokes in the standard case of homogeneous Dirichlet conditions on entire $\partial\Omega$.

5 Numerical results

In this section, we illustrate the stability properties of the DDN condition and its classical counterpart (CDN) in two examples. The simulations are done with piecewise bilinear elements, so called Q_1-elements for pressure and velocities. The absence of a discrete inf-sup condition for this equal-order pair is cured by local projection stabilization (LPS). The convective term is stabilized by a LPS technique as well. For details, we refer to [3]. We consider a stationary problem of a rotating vortex driven by a right-hand side, and a standard time-dependent backward facing step problem.

5.1 Standing vortex

In this example, we consider two computational domains, the larger reference domain $\Omega_{ref} = (-2,1) \times (-1,1)$ and the cut-domain $\Omega = (-1,1) \times (-1,1)$. At the boundaries $x = -2$, $x = 1$, $y = -1$, and $y = 1$ we consider homogeneous Dirichlet conditions for u. The outflow conditions (CDN and DDN) are implemented at the left boundaries, i.e. for Ω_{ref} at $x = -2$ and for Ω at $x = -1$. The right-hand side is given by

$$f(x) = s(|x - x_0|)x^t,$$

where x^t is the vector obtained by rotating x counter-clockwise by the angle $\pi/2$, $x_0 = (1/2, 0)$ and s is the scaling function

$$s(r) = \begin{cases} r^2(r-1/2)^2 & \text{if } r \leq 1/2 \\ 0 & \text{else.} \end{cases}$$

This right-hand side enforces a rotation around the origin.

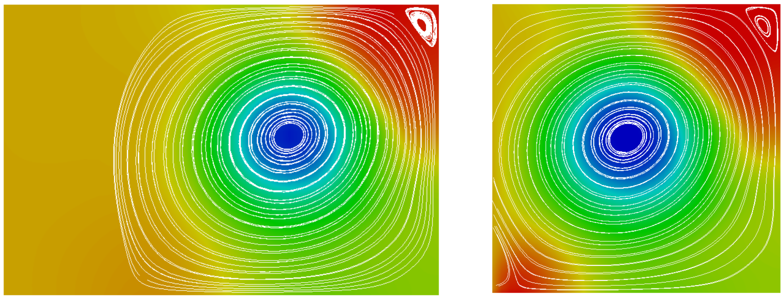

Fig. 1 Pressure field (colors) and streamlines for $\mu = 7 \cdot 10^{-4}$ in the large domain Ω_{ref} (left figures), and in the cut domain Ω (right figures) obtained with the classical do-nothing condition (CDN). A spurious pressure peak (red color) appears in the lower left corner.

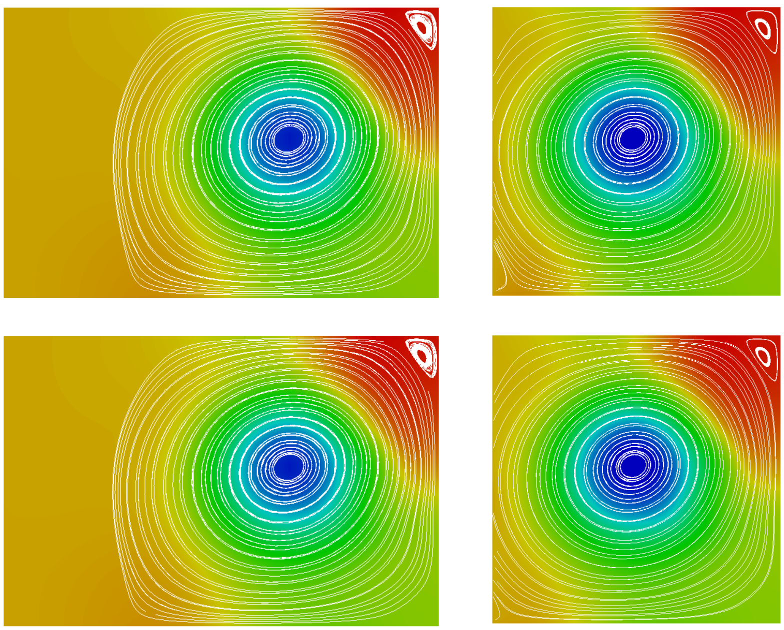

Fig. 2 Pressure field (colors) and streamlines for $\mu = 7 \cdot 10^{-4}$ in the large domain Ω_{ref} (left figures), and in the cut domain Ω (right figures). The upper row shows the use of the directional do-nothing condition (DDN) with $\beta = 0$, the lower row corresponds to $\beta = 2$.

In Figure 1 the pressure field and streamlines for the Navier-Stokes system with the classical do-nothing condition (CDN) are shown. Comparison of the solutions for Ω and Ω_{ref} shows that the CDN does a pretty good job. However, some differences can be observed in the flow field close to the lower left corner in Ω. Moreover, a small recirculation zone appears which is not the case in the larger domain Ω_{ref}. An even more evident difference is a broad pressure peak in the same area.

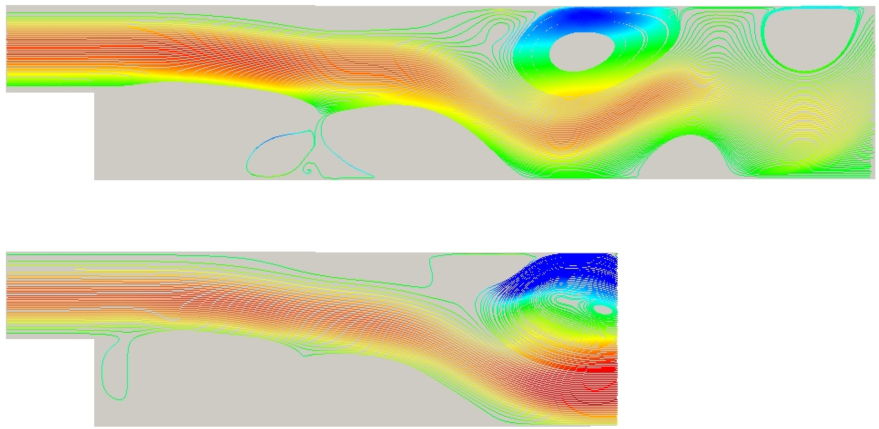

Fig. 3 Solution of the backward-facing step problem with the CDN condition and two different lengths of the domain. The shorter configuration (lower figure) shows a secondary recirculation zone, which is too large.

The solutions with the directional do-nothing (DDN) conditions are shown in the upper and lower row of Figure 2 for $\beta = 0$ and $\beta = 2$, respectively. The (artificial) recirculation zone at the lower left corner of Ω reduces with higher values of β. The spurious pressure peak disappears as well. In this example, the use of the DDN condition is obviously better than the CDN condition.

5.2 Backward-facing step

The second example consists of the backward-facing step configuration at Reynolds number $Re = 8,000$. The flow becomes time-dependent, because it is strongly convection-dominated. For time integration we use the trapezoidal rule (Crank-Nicolson) with constant time step $\Delta t = 0.125$. As initial condition we choose a parabolic flow in the upper part of the domain, consistent with the parabolic inflow at the left boundary.

In Figure 3 we depict streamlines at time $t = 25$ by use of the CDN condition for two different lengths of the domain, $L = 7$ and $L = 10$. The streamlines are colored according to the horizontal velocity component. In the shorter domain, the location of the outflow boundary S_1 is located inside the secondary recirculation zone. Therefore, the use of CDN leads to a too dominant recirculation. The larger domain leads to a smaller amount of recirculation. Using the DDN condition with $\beta = 0$ results in solutions displayed in Figure 4. The solution for the longer domain coincides perfectly with the one with CDN in the long configuration. Due to the smaller amount of recirculation at the outflow boundary at $x = 10$, the choice of boundary condition

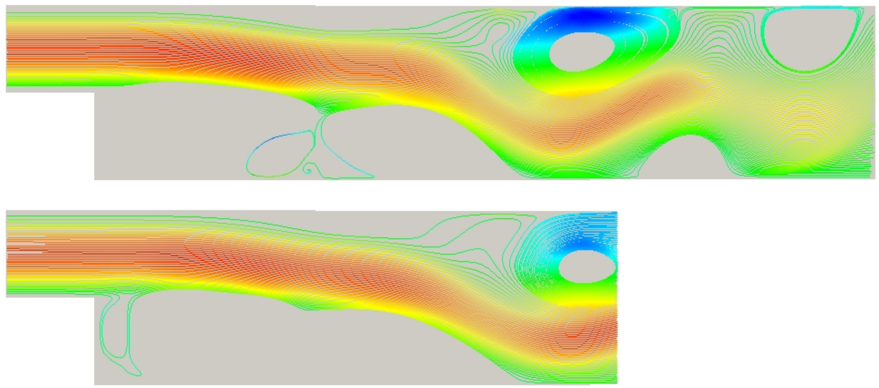

Fig. 4 Solution of the backward-facing step problem with the DDN condition ($\beta = 0$) and two different lengths of the domain. Although the fit of the two solutions is not perfect, the solution of the shorter configuration is much better than the corresponding one with CDN.

is less critical in this case. However, the solution with DDN condition in the shorter domain coincides much better with the one in the longer configuration. In particular, the spurious recirculation is avoided with the DDN condition. We conclude that the origin of this spurious inflow is a consequence of the lack of stability of the classical do-nothing (CDN) condition.

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