# Difference schemes of high accuracy order on uniform grids for a singularly perturbed parabolic reaction-diffusion equation

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**Abstract** For a singularly perturbed parabolic reaction-diffusion equation with a perturbation parameter $\varepsilon$ ($\varepsilon \in (0,1]$) multiplying the highest-order derivative, we consider a technique to construct $\varepsilon$-uniformly convergent in the maximum norm difference schemes of higher accuracy order on uniform grids. In constructing such schemes, we use the solution decomposition method, in which grid approximations of the regular and singular components in the solution are considered on uniform grids. Increasing of the convergence rate of the scheme constructed with improved accuracy of order $\mathscr{O}\left(N^{-4}\ln^4 N + N_0^{-2}\right)$, where $N$ and $N_0$ are the number of nodes in the meshes in $x$ and $t$, respectively, is achieved using a Richardson extrapolation technique applied to the regular and singular components. In the proposed Richardson technique, when constructing embedded grids we use most dense grids as main grids. This approach allows us to construct schemes that converge $\varepsilon$-uniformly in the maximum norm at the rate $\mathscr{O}\left(N^{-6}\ln^6 N + N_0^{-3}\right)$ and higher.

## 1 Introduction

Efficiency of numerical methods for solving regular boundary and initial-boundary value problems is largely determined by order of their convergence rate (see, e.g., [2, 3]; there also high-order schemes are considered). For a number of singularly perturbed problems (with a perturbation parameter $\varepsilon$ ($\varepsilon \in (0,1]$) multiplying the highest-order derivative), difference schemes with improved accuracy order which converge uniformly with respect to the perturbation parameter $\varepsilon$, i.e., $\varepsilon$-uniformly, were constructed on the basis of piecewise-uniform grids condensing in the boundary layer using a Richardson extrapolation technique (see, e.g., [5, 6, 7, 9, 11], and also [10], Ch. 10, and the bibliography therein). Note that serious difficulties arise

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when constructing $\varepsilon$-uniformly convergent finite difference schemes of high accuracy order with using difference schemes on non-uniform grids. Thus, in the case of a parabolic reaction-diffusion equation, there exist no Richardson schemes on piecewise-uniform grids convergent $\varepsilon$-uniformly with accuracy order higher than three in $x$ (see [7]), and in the case of an elliptic convection-diffusion equation such accuracy order is not higher than two (see [11]). In [12] for a singularly perturbed ordinary differential reaction-diffusion equation, a new approach was developed in order to construct special $\varepsilon$-uniform difference schemes based on the asymptotic construction technique, namely, the solution decomposition method. The main in this approach is the use of classical approximations to subproblems for the regular and singular components of the solution on uniform grids. In the same paper, using the Richardson technique, an improved scheme of the solution decomposition method was first constructed which converges $\varepsilon$-uniformly in the maximum norm at the rate $\mathscr{O}\left(N^{-4}\ln^4 N\right)$ where $N+1$ is the number of nodes in the spatial grid.

In [8], for a Dirichlet problem for the one-dimensional singularly perturbed parabolic reaction-diffusion equation, a scheme of the solution decomposition method was first constructed which converges $\varepsilon$-uniformly in the maximum norm at the rate $\mathscr{O}\left(N^{-2}\ln^2 N+N_0^{-1}\right)$, where $N+1$ and $N_0+1$ are the numbers of nodes in the spatial and temporal grids, respectively

In [13] for the same problem as in [8], using the Richardson technique, an improved scheme of the solution decomposition method was constructed which converges $\varepsilon$-uniformly in the maximum norm at the rate $\mathscr{O}\left(N^{-4}\ln^4 N+N_0^{-2}\right)$. When constructing the improved scheme, embedded refined meshes were used, where the main grid was coarse grid; this Richardson technique turned out to be little used (because of its cumbersome) to create schemes of the solution decomposition method convergent with order of the convergence rate higher than the fourth at $x$.

On the other hand, when constructing an improved Richardson scheme it is possible to apply the alternative approach proposed in[13], i.e., to use embedded grids where the main grid is the most finest (with smallest step-size) grid. Such an alternative approach allows us to construct improved difference schemes convergent $\varepsilon$-uniformly at the rate $\mathscr{O}\left(N^{-6}\ln^6 N+N_0^{-3}\right)$ when the number of embedded grids equals three, and with a higher accuracy order when the number of embedded grids more than three.

In the present paper, for an initial-boundary value problem for a singularly perturbed parabolic reaction-diffusion equation, we developed a technique (an alternative to approach in [13]) for constructing an improved difference scheme when the number of embedded grids is two. On the basis of the solution decomposition method and the Richardson extrapolation method, we constructed a difference scheme of high accuracy order convergent $\varepsilon$-uniformly at the rate $\mathscr{O}\left(N^{-4}\ln^4 N+N_0^{-2}\right)$.

Detailed results of the research will be presented in the journal publication.

## 2 Problem formulation and aim of research

On the set $\overline{G}$
$$\overline{G} = G \cup S, \quad G = D \times (0,T], \quad D = (0,d), \tag{1}$$
we consider a boundary value problem for the one-dimensional singularly perturbed parabolic reaction-diffusion equation[1]

$$L_{(2)} u(x,t) \equiv \left\{ \varepsilon^2 a(x,t) \frac{\partial^2}{\partial x^2} - c(x,t) - p(x,t) \frac{\partial}{\partial t} \right\} u(x,t) = f(x,t), \quad (x,t) \in G, \tag{2}$$
$$u(x,t) = \varphi(x,t), \quad (x,t) \in S.$$

The functions $a(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$ and $\varphi(x,t)$ are assumed to be sufficiently smooth on the set $\overline{G}$ and on the lower and lateral sides of the set $S$, respectively, moreover,[2]

$$a(x,t),\ c(x,t),\ p(x,t) > m, \quad |f(x,t)| \le M, \quad (x,t) \in \overline{G};$$
$$|\varphi(x,t)| \le M, \quad (x,t) \in S;$$

the parameter $\varepsilon$ takes arbitrary values in $(0,1]$. Here $S = S_0 \cup S^L$, $S_0$ and $S^L$ are the lower and lateral sides of the boundary $S$, $S^L = S_1^L \cup S_2^L$, $S_1^L$ and $S_2^L$ are the left and right parts of the lateral boundary, and $S_0 = \overline{S}_0$.

We assume that the data of problem (2), (1) on the set of corner points $S^* = S_0 \cap \overline{S}^L$ satisfy the compatibility conditions ensuring the required smoothness of the solution on $\overline{G}$.

For small values of the parameter $\varepsilon$, a parabolic boundary layer appears in a neighborhood of the set $S^L$ [4, 10].

Our aim is for initial-boundary value problem (2), (1), on the basis of the solution decomposition method (using standard grid approximations of the regular and singular components of the solution on uniform grids) and the Richardson extrapolation technique, to construct a difference scheme that converges $\varepsilon$-uniformly in the maximum norm with an improved accuracy order (two with respect to $t$ and four with respect to $x$ up to a logarithmic factor).

## 3 Difference scheme of the solution decomposition method

In this section, we first consider a decomposition of the solution to problem (2), (1) using an asymptotic construction technique. Further, on the basis of this solu-

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[1] The notation $L_{(j)}$ ($M_{(j)}$, $G_{h(j)}$) means that these operators (constants, grids) are introduced in formula $(j)$.

[2] By $M$ (or $m$), we denote sufficiently large (small) positive constants independent of the parameter $\varepsilon$ and of the discretization parameters.

tion decomposition of the differential problem, we construct a difference scheme (scheme of the solution decomposition method), in which the regular and singular components of the discrete solution are computed on uniform meshes.

### 3.1 Solution decomposition of the differential problem

For the solution $u(x,t)$ of differential problem (2), (1), we construct the following decomposition:

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}, \tag{3a}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular components of the solution, respectively. For the regular component, we use the following expansion with respect to the parameter $\varepsilon$ from three members:

$$U(x,t) = U_0(x,t) + \varepsilon^2 U_1(x,t) + v_U(x,t), \quad (x,t) \in \overline{G}, \tag{3b}$$

where $U_0(x,t)$ is the main term, $U_1(x,t)$ is the first term and $v_U(x,t)$ is the remainder term. We represent the singular component $V(x,t)$ as the sum of the functions

$$V(x,t) = V_1(x,t) + V_2(x,t), \quad (x,t) \in \overline{G}, \tag{3c}$$

where $V_i(x,t)$ is the singular component of the solution in a neighbourhood of the lateral side to the boundary $S_i^L$, $i = 1,2$. The components in the representation (3) are solutions of corresponding differential problems (see, e.g., [8, 13, 14]).

### 3.2 Basic scheme of the solution decomposition method

Now, we construct difference schemes for boundary value problem (2), (1) by approximating corresponding problems for the solution components. We consider two cases depending on the value of the parameter $\varepsilon$.

For not too small values of the parameter $\varepsilon$, namely, provided

$$\varepsilon \geq \varepsilon_0(N), \quad \varepsilon_0(N) = m\,\ell^{-1}\,d\,\ln^{-1} N, \tag{4}$$

where $m$ is an arbitrary number in $(0, m_0)$, $m_0 = \min_{\overline{G}}^{1/2}\,[a^{-1}(x,t)\,c(x,t)]$, $\ell = 2$, we approximate problem (2), (1) by the standard difference scheme on a uniform grid $\overline{G}_h$

$$\Lambda z(x,t) \equiv \{\varepsilon^2 a(x,t)\,\delta_{x\overline{x}} - c(x,t) - p(x,t)\,\delta_{\overline{t}}\}z(x,t) = f(x,t), \quad (x,t) \in G_h, \tag{5}$$

$$z(x,t) = \varphi(x,t), \quad (x,t) \in S_h.$$

Here

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \quad \overline{G}_h = G_h \cup S_h, \tag{6}$$

is the uniform grid with the numbers of nodes $N+1$ and $N_0+1$ in the meshes $\overline{\omega}$ in $x$ and $\overline{\omega}_0$ in $t$, respectively. Using the solution of difference scheme (5), (6), we construct the linear interpolant (see, e.g., [1])

$$\bar{z}_u(x,t), \quad (x,t) \in \overline{G}, \ \ \textit{under conditon} \ \ (4), \tag{7a}$$

which is the solution of the difference scheme $\{(5), (6); (4)\}$ approximating the differential problem (2), (1) under the condition (4).

Further, for approximation of the regular and singular components of the solution, we construct difference schemes for sufficiently small values of the parameter $\varepsilon$, namely, provided

$$\varepsilon < \varepsilon_{0\,(4)}(N). \tag{8}$$

We approximate the function $U(x,t)$ and its components in the representation (3b) on the uniform grid (6). We find solutions of the corresponding difference schemes (see, e.g., [8, 13, 14]) and obtain the function

$$z_U(x,t) = z_{U_0}(x,t) + \varepsilon^2 z_{U_1}(x,t) + z_{v_U}(x,t), \quad (x,t) \in \overline{G}_h.$$

Using the values of the function $z_U(x,t)$ at the nodes of the grid $\overline{G}_h$ on the elementary partitions of the set $\overline{G}$, generated by the grid $\overline{G}_h$, we construct the bilinear interpolant $\bar{z}_U(x,t), (x,t) \in \overline{G}$. The function $z_U(x,t)$, $(x,t) \in \overline{G}_h$, and also its interpolant $\bar{z}_U(x,t), (x,t) \in \overline{G}$, are called solutions of difference schemes approximating differential problems for the regular components under condition (8).

We approximate the singular components $V_j(x,t)$ in (3c) on uniform grids which are constructed on subdomains $\overline{G}^{\sigma}_j$ in $\overline{G}$, adjacent to the boundaries $S^L_j$, $j = 1,2$:

$$\overline{G}^{\sigma}_j = G^{\sigma}_j \cup S^{\sigma}_j, \quad G^{\sigma}_j = D^{\sigma}_j \times (0,T], \quad j = 1,2, \tag{9}$$

$$D^{\sigma}_1 = (0,\sigma), \quad D^{\sigma}_2 = (d-\sigma, d), \ \ \sigma = \sigma(\varepsilon, N, \ell) = \min\left[d, m^{-1}\ell\,\varepsilon \ln N\right].$$

Solving discrete problems on the uniform grids

$$\overline{G}^{\sigma}_{jh} = \overline{\omega}^{\sigma}_j \times \overline{\omega}_0, \quad \overline{G}^{\sigma}_{jh} = G^{\sigma}_{jh} \cup S^{\sigma}_{jh}, \quad j = 1,2,$$

where $\overline{\omega}_0 = \overline{\omega}_{0\,(6)}$, $\overline{\omega}^{\sigma}_j$ is the mesh on $\overline{D}^{\sigma}_j$ with the step-size $h^{\sigma} = \sigma N^{-1}$ and the number of nodes $N+1$, we find the functions $z_{V_j}(x,t)$, $(x,t) \in \overline{G}^{\sigma}_{jh}$, and their interpolants $\bar{z}_{V_j}(x,t)$, $(x,t) \in \overline{G}^{\sigma}_j$. We assume that, outside the set $\overline{G}^{\sigma}_j$, the functions $z_{V_j}(x,t)$ and $\bar{z}_{V_j}(x,t)$ vanish. We set

$$\bar{z}_V(x,t) = \bar{z}_{V_1}(x,t) + \bar{z}_{V_2}(x,t), \quad (x,t) \in \overline{G}.$$

The function $\bar{z}_V(x,t)$, $(x,t) \in \overline{G}^{\sigma}$ is the solution of discrete problems approximating the differential problems for the singular components under condition (8).

We call the following function:

$$\overline{z}_u(x,t) = \overline{z}_U(x,t) + \overline{z}_V(x,t), \quad (x,t) \in \overline{G}, \ \textit{under condition} \ (8), \tag{7b}$$

the solution of difference schemes approximating differential problem (2), (1) under condition (8).

Totality of the difference schemes used above forms the basic scheme of the solution decomposition method. On its basis, the function $\overline{z}_{u\,(7\mathrm{a,b})}(x,t)$, $(x,t) \in \overline{G}$, is constructed which approximates the solution of problem (2), (1). In [13] for the solution $\overline{z}_u(x,t)$ to the basic scheme of the solution decomposition method we have obtained the following $\varepsilon$-uniform estimate:

$$|u(x,t) - \overline{z}_{u\,(7\mathrm{a,b})}(x,t)| \le M\,[N^{-2}\ln^2 N + N_0^{-1}], \quad (x,t) \in \overline{G}. \tag{10}$$

## 4 Richardson extrapolation on the basis of classical scheme

We describe the Richardson extrapolation method, which is used for improving the accuracy of the solution to difference scheme (5). On the set $\overline{G}$ we construct grids uniform in $x$ and $t$

$$\overline{G}_h^i = \overline{\omega}^i \times \overline{\omega}_0^i, \quad i = 1,2,3. \tag{11a}$$

Here $\overline{G}_h^1$ is $\overline{G}_{h(6)}$, in which $h_x^1 = dN^{-1}$ is the step-size in the mesh $\overline{\omega}^1$ with the number of nodes $N+1$, and $h_t^1 = TN_0^{-1}$ is the step-size in the mesh $\overline{\omega}_0^1$ with the number of nodes $N_0+1$; $\overline{G}_h^2$ and $\overline{G}_h^3$ are "coarsened" grids. The step-size $h_x^2$ in the mesh $\overline{\omega}^2$ is $k$ times larger than the step-size $h_x^1$ in the mesh $\overline{\omega}^1$, i.e., $h_x^2 = k\,d\,N^{-1}$ and $k^{-1}N+1$ is the number of nodes in the mesh $\overline{\omega}^2$. The step-size $h_t^2$ in the mesh $\overline{\omega}_0^2$ is $k^2$ times larger than the step-size $h_t^1$ in the mesh $\overline{\omega}_0^1$, i.e., $h_t^2 = k^2\,T\,N_0^{-1}$ and $k^{-2}N_0+1$ is the number of nodes in the mesh $\overline{\omega}_0^2$. The step-size $h_x^3$ in the mesh $\overline{\omega}^3$ is $k^2$ times larger than the step-size $h_x^2$ in the mesh $\overline{\omega}^2$, i.e., $h_x^{=}k^2\,d\,N^{-1}$ and $k^{-2}N+1$ is the number of nodes in the mesh $\overline{\omega}^3$. The step-size $h_t^3$ in the mesh $\overline{\omega}_0^3$ is $k^2$ times larger than the step-size $h_t^2$ in the mesh $\overline{\omega}_0^2$, i.e., $h_t^3 = k^4\,T\,N_0^{-1}$ and $k^{-4}N_0+1$ is the number of nodes in the mesh $\overline{\omega}_0^3$. For simplicity, we consider the case with two embedded uniform grids. Let

$$\overline{G}_h^0 = \overline{G}_h^1 \cap \overline{G}_h^2 \tag{11b}$$

$\overline{G}_h^0 = \overline{G}_h^1$ if $k$ is an integer ($k \ge 2$), and $\overline{G}_h^0 \ne \overline{G}_h^1$ if $k$ is a noninteger; $\overline{G}_h^0 = \overline{\omega}^0 \times \overline{\omega}_0^0$. Let $z^i(x,t)$, $(x,t) \in \overline{G}_h^i$, $i = 1, 2$ be solutions of the difference schemes

$$\Lambda_{(5)} z^i(x,t) = f(x,t), \quad (x,t) \in G_h^i, \tag{12a}$$
$$z^i(x,t) = \varphi(x,t), \quad (x,t) \in S_h^i, \quad i = 1,2.$$

We set

$$z^0(x,t) = \gamma_1 \, z^1(x,t) + \gamma_2 \, z^2(x,t), \quad (x,t) \in \overline{G}_h^0, \tag{12b}$$

where

$$\gamma_i = \gamma_i(k), \quad i = 1,2, \quad \gamma_1 = -(k^2-1)^{-1}, \quad \gamma_2 = 1 - \gamma_1 = k^2 \, (k^2-1)^{-1}.$$

Difference scheme (12), (11) constructed on the basis of scheme (5), (6) is called the Richardson scheme on two embedded grids. The function $z^0_{(12)}(x,t)$, $(x,t) \in \overline{G}_h^0$, is called the solution to Richardson scheme (12), (11); the functions $z^1_{(12)}(x,t)$, $(x,t) \in \overline{G}_h^1$, and $z^2_{(12)}(x,t)$, $(x,t) \in \overline{G}_h^2$, are called the components generating the solution of scheme (12), (11). The solution $z^0(x,t)$ of the Richardson scheme converges to the solution $u(x,t)$ of boundary value problem (2), (1) with the estimate

$$|u(x,t) - z^0(x,t)| \le M \left[ \varepsilon^{-4} N^{-4} + N_0^{-2} \right], \quad (x,t) \in \overline{G}_h^0, \tag{13}$$

i.e., with the fourth accuracy order in $x$ but for fixed values of $\varepsilon$ and under the sufficiently restrictive condition $N^{-1} = o(\varepsilon)$, $N_0^{-1} = o(1)$ (see [13]).

## 5 Richardson extrapolation for solution decomposition scheme

To improve accuracy order of discrete solutions obtained on the basis of the solution decomposition method, we will apply the Richardson extrapolation technique described in section 4. We repeat constructions from Subsection 3.2, applying in each case two, instead of one grid, (or three) embedded grids.

We consider the construction of schemes with improved accuracy for not too small values of $\varepsilon$

$$\varepsilon \ge \varepsilon_0(N), \quad \varepsilon_0(N) = m \, \ell^{-1} \, d \, \ln^{-1} N, \tag{14}$$

and for sufficiently small values of the parameter $\varepsilon$

$$\varepsilon < \varepsilon_0(N), \quad \varepsilon_0(N) = \varepsilon_{0\,(14)}(N), \tag{15}$$

where $m = m_{(4)}$, and $\ell = 4$ unlike from Subsection 3.2.

Under the condition (14), using the solution of the Richardson difference scheme (12), (11) on two uniform embedded grids, we construct the interpolant

$$\widehat{z}_u(x,t), \quad (x,t) \in \overline{G} \;\; \textit{under condition} \;\; (14), \tag{16a}$$

which we call the solution of the scheme $\{$(12), (11), (14)$\}$, approximating differential problem (2), (1) under condition (14).

Let the condition (15) be fulfilled. We construct the grid approximation of the regular component $U(x,t)$, using the Richardson extrapolation on the embedded uniform grids

$$\overline{G}_h^i = \overline{G}_{h(11)}^i = \overline{\omega}^i \times \overline{\omega}_0^i, \quad i = 1,2; \quad \overline{G}_h^0 = \overline{G}_{h(11)}^0. \tag{17}$$

Unlike from Subsection 3.1, under approximating the problem for $U_0(x,t)$, we use its "extension" to the set

$$\overline{G}^e = \overline{D}^e \times [0,T], \quad \overline{D}^e = [-h^0, d + h^0],$$

where $h^0$ is the step-size of the "common" mesh $\overline{\omega}^1$. On the set $\overline{G}^e$, we construct embedded grids

$$\overline{G}_h^{ei} = \overline{\omega}^{ei} \times \overline{\omega}_0^i, \quad i = 1,2; \quad \overline{G}_h^{e0} = \overline{G}_h^{e1} \cap \overline{G}_h^{e2},$$

where $\overline{\omega}^{ei}$ are "extended" uniform meshes, $\overline{\omega}^{ei} \cap \overline{D} = \overline{\omega}^i$, $i = 1,2$. We find discrete solutions for the components $z_{U_0}^{ei}(x,t)$ and $z_{U_1}^i(x,t)$, $z_{v_U}^i(x,t)$ on the grid $\overline{G}_h^{ei}$ and $\overline{G}_h^i$, respectively. Set

$$z_U^i(x,t) = z_{U_0}^{ei}(x,t) + \varepsilon^2 z_{U_1}^i(x,t) + z_{v_U}^i(x,t), \quad i = 1,2.$$

On the set $\overline{G}_h^0$, we define the function $z_U^0(x,t)$

$$z_U^0(x,t) = \gamma_1 z_U^1(x,t) + \gamma_2 z_U^2(x,t), \quad (x,t) \in \overline{G}_h^0, \quad \gamma_i = \gamma_{i(12)}(k), \tag{18}$$

which is the grid approximation of the function $U(x,t)$ constructed on the basis of the Richardson technique. Using the function $z_U^0(x,t)$, $(x,t) \in \overline{G}_h^0$, we construct its interpolant

$$\widehat{z}_U^0(x,t), \quad (x,t) \in \overline{G}, \tag{19}$$

which is the continual approximation of the function $U(x,t)$.

Under the condition (15), using the Richardson technique, we construct the grid approximation of the singular component $V(x,t)$. On $\overline{G}$ we introduce the sets $\overline{G}_j^\sigma$

$$\overline{G}_j^\sigma = \overline{G}_{j(9)}^\sigma = G_j^\sigma \cup S_j^\sigma, \quad \sigma = \sigma_{(9)}(\varepsilon, N, \ell) \ \ for \ \ \ell = 4, \ \ j = 1,2. \tag{20}$$

On the sets $\overline{G}_j^\sigma$, we construct the embedded grids (similar to grids $\overline{G}_{h(17)}^i$, $\overline{G}_{h(17)}^0$)

$$\overline{G}_{jh}^{\sigma i} = \overline{G}_{jh(21)}^{\sigma i} = \overline{\omega}_j^{\sigma i} \times \overline{\omega}_0^i, \quad i = 1,2; \tag{21}$$

$$\overline{G}_{jh}^{\sigma 0} = \overline{G}_{jh(21)}^{\sigma 0} = \overline{G}_{jh}^{\sigma 1} \cap \overline{G}_{jh}^{\sigma 2}, \quad j = 1,2.$$

Solving discrete problems on $G_{jh}^{\sigma i}$, we find $z_{V_j}^i(x,t)$, $i = 1,2$. On the set $G_{jh}^{\sigma 0}$ we define the function

$$z_{V_j}^0(x,t) = \gamma_1 z_{V_j}^1(x,t) + \gamma_2 z_{V_j}^2(x,t), \quad (x,t) \in \overline{G}_{jh}^{\sigma 0}, \tag{22}$$

$$j = 1,2, \quad \gamma_i = \gamma_{i(12)}, \quad i = 1,2.$$

The function $z^0_{V_j}(x,t)$, $(x,t) \in \overline{G}^{\sigma 0}_{jh}$, is the grid approximation of the function $V_j(x,t)$, constructed using the Richardson technique. We construct its interpolant:

$$\widehat{z}^0_{V_j}(x,t), \quad (x,t) \in \overline{G}^{\sigma 0}_j, \quad j = 1,2; \tag{23}$$

outside the set $\overline{G}^{\sigma 0}_j$, the function $\widehat{z}^0_{V_j}(x,t)$ is assumed to be zero. Set

$$\widehat{z}^0_V(x,t) = \widehat{z}^0_{V_1}(x,t) + \widehat{z}^0_{V_2}(x,t), \quad (x,t) \in \overline{G}.$$

We call the function

$$\widehat{z}_u(x,t) = \widehat{z}^0_U(x,t) + \widehat{z}^0_V(x,t), \quad (x,t) \in \overline{G}, \ \textit{under condition} \ (15). \tag{16b}$$

the solution of the Richardson difference scheme, which approximates differential problem (2), (1) under condition (15).

Thus, we have constructed the function $\widehat{z}_{u\,(16\text{a,b})}(x,t)$, $(x,t) \in \overline{G}$, approximating the solution of the differential problem (2), (1). This function and the grid functions $z^{e0}_{U_0}(x,t)$,$z^0_{U_1}(x,t)$, $z^0_{v_U}(x,t)$, $(x,t) \in \overline{G}^0_h$, and $z^0_{V_j}(x,t)$, $(x,t) \in \overline{G}^{\sigma 0}_{jh}$, $j = 1,2$, are called the continual and grid solutions, respectively, of the Richardson difference scheme of the solution decomposition method.

For the solution to the Richardson scheme of the solution decomposition method, in [13] we have obtained the following $\varepsilon$-uniform estimate:

$$|u(x,t) - \widehat{z}_u(x,t)| \le M\,[N^{-4} \ln^4 N + N_0^{-2}], \quad (x,t) \in \overline{G}. \tag{24}$$

## 6 On the higher accuracy order schemes

The technique described above allows us to construct a Richardson scheme of type (12) on the three embedded grids $\overline{G}^1_h$, $\overline{G}^2_h$ and $\overline{G}^3_h$ with the solution $z^0(x,t)$ on the set $\overline{G}^0_h$, which is the intersection of these sets,

$$\overline{G}^0_h = \overline{G}^1_h \cap \overline{G}^2_h \cap \overline{G}^3_h, \quad \overline{G}^i_h = \overline{G}^i_{h\,(11\text{a})}, \quad i = 1,2,3.$$

Application of this Richardson scheme to the basic scheme of the solution decomposition (similar constructions presented here) leads to the scheme of higher accuracy order whose solution $\widehat{z}_u(x,t)$ converges $\varepsilon$-uniformly in the maximum norm at the rate $\mathscr{O}\left(N^{-6} \ln^6 N + N_0^{-3}\right)$ on the set $\overline{G}$.

**Acknowledgements** This research was partially supported by the Russian Foundation for Basic Research under grant No.13-01-00618.

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