Blow-up of solutions and interior layers in a Caputo two-point boundary value problem

Martin Stynes and José Luis Gracia

Abstract A two-point boundary value problem whose highest-order term is a Caputo fractional derivative of order $\delta \in (1,2)$ is considered on the interval $[0,1]$. It is asserted in the published literature that the solutions of such problems can exhibit a boundary layer at $x = 1$ as $\delta \to 1^+$ when the convection coefficient b satisfies $\max_{x\in[0,1]} b(x) \geq 1$. It will be shown here that for constant b a boundary layer can appear in the case $b > 1$ but in the case $b = 1$ the behaviour of the solution is substantially different. Furthermore, a numerical example is given to show that for certain $b(x)$ the solution can exhibit an interior layer—a phenomenon that has not previously been reported in the research literature.

1 Introduction

Let $\delta \in (1,2)$. Let $g \in C^1[0,1]$ with g' absolutely continuous on $[0,1]$. Then the Caputo fractional derivative $D_*^\delta g$ associated with the point $x = 0$ is defined by

$$D_*^\delta g(x) := \frac{1}{\Gamma(2-\delta)} \int_{t=0}^{x} (x-t)^{1-\delta} g''(t)\, dt \quad \text{for } 0 < x \leq 1;$$

see [1, 4]. In the present paper we shall consider the two-point boundary value problem

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$$-D_*^\delta u(x)+b(x)u'(x)+c(x)u(x)=f(x) \quad \text{for } x\in(0,1), \tag{1a}$$

$$u(0)-\alpha_0 u'(0)=\gamma_0, \quad u(1)+\alpha_1 u'(1)=\gamma_1. \tag{1b}$$

The constants $\alpha_0,\alpha_1,\gamma_0,\gamma_1$ and the functions b,c and f are given. We assume that $b,c,f\in C^1[0,1]$ with $c\geq 0$ in $[0,1]$. We assume also that

$$\alpha_0=\frac{1}{\delta-1} \quad \text{and} \quad \alpha_1\geq 0. \tag{2}$$

With the exception of α_0, all data in (1) is independent of δ. The problem (1) has been investigated in [4, 7, 8] where it is shown that the conditions on c,α_0 and α_1 ensure satisfaction of a comparison/maximum principle, and hence that (1) has a solution $u\in C^1[0,1]\cap C^2(0,1]$ and this solution is unique.

The problem (1) is also examined in [3], where its use in modelling anomalous diffusion is motivated analytically and various applications of it are listed. In that paper one has $b\equiv 0$, so we have generalised the problem studied in [3] to one that includes convective processes.

Remark 1. The analysis of [7, 8] assumes that $\alpha_0\geq 1/(\delta-1)$ and $\alpha_1\geq 0$, which is more general than (2). Here we take $\alpha_0=1/(\delta-1)$ since an examination of the more general case would force us to consider further possibilities in the sections below. We will address these in a later paper.

Remark 2. In a sister paper [2] we examine boundary layers in solutions to a problem resembling (1) except that the Caputo derivative is replaced by a Riemann-Liouville fractional derivative and $\alpha_0=\gamma_0=0$.

In [7] we investigated the behaviour of the solution u as δ varied between 1 and 2, and found that in certain cases u exhibited a boundary layer at $x=1$ as $\delta\to 1^+$. In Sections 2 and 3 we shall discuss some aspects of this boundary layer that were not revealed in [7]. Furthermore, in Section 4 we demonstrate that an interior layer can appear in u; this possibility has not previously been reported in the research literature.

Notation. We use the "big O" notation in its sharp form. Thus when we write for example $g=O(1/(\delta-1))$ as $\delta\to 1^+$, we mean that $\lim_{\delta\to 1^+}[(\delta-1)g]$ exists and is non-zero.

2 Background material for the constant-coefficient case

Throughout Sections 2 and 3 assume that b and f are non-zero constants, and $c\equiv 0$. (If $c>0$ then the solution of (1) is much better behaved; see [7, Theorem 3.3].)

In Figure 1 we take $b=1.1$, $f=1$, $\alpha_1=0$, $\gamma_0=0.4$ and $\gamma_1=1.7$, and plot the solution u of (1) for $\delta=1.1, 1.075, 1.05, 1.025$. It is clear that as $\delta\to 1^+$, a boundary layer develops in u at $x=1$. The values of γ_0 and γ_1 were chosen arbitrarily; one can see easily from the analysis in Section 3.1 that for $b > 1$, $f \neq 0$ and any values of γ_0 and γ_1, one will always have a boundary layer at $x = 1$ as $\delta \to 1^+$.

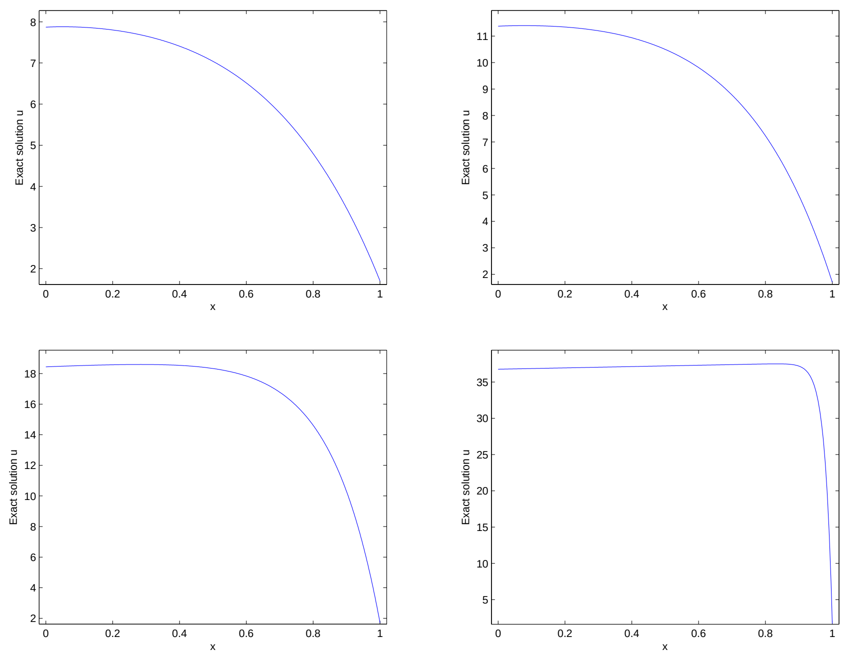

Fig. 1 Exact solution of (1) for $b \equiv 1.1$, $c \equiv 0$, $f \equiv 1$, $\alpha_1 = 0$, $\gamma_0 = 0.4$ and $\gamma_1 = 1.7$, with $\delta = 1.1$ (1st row, left), $\delta = 1.075$ (1st row, right), $\delta = 1.05$ (2nd row, left) and $\delta = 1.025$ (2nd row, right), showing development of a boundary layer as $\delta \to 1^+$.

The analysis in [7] (see Section 2.2 and Theorem 3.5) shows that as $\delta \to 1^+$ one has $|u'(1)| \to \infty$ if and only if $b \geq 1$; in this regime, the problem (1) is singularly perturbed. If in Figure 1 one changes the value of b to any other number greater than 1, then for δ close to 1, the graph of u will resemble qualitatively the final graph in Figure 1, i.e., a boundary layer at $x = 1$ will be evident.

But if $b = 1$, although the theory of [7, Section 2.2] still predicts that $|u'(1)| \to \infty$ as $\delta \to 1^+$, the graph of u is fundamentally different. Figure 2 shows that for $b = 1$ and the same values of f, α_1, γ_0 and γ_1 as in Figure 1, as $\delta \to 1^+$ the solution u of (1) *does not exhibit a standard boundary layer at* $x = 1$. (The scaling of the vertical axis in Figure 2 implies that $|u'(x)| \to \infty$ at *each* point in $[0, 1]$ as $\delta \to 1^+$; this surprising property will be proved analytically in Section 3.2.)

This difference in the case $b = 1$ passed unnoticed in [7]; our main aim in the present paper is to explain what Figure 2 displays.

We begin by listing some known results that will be needed later in our analysis. Define the two-parameter Mittag-Leffler function by

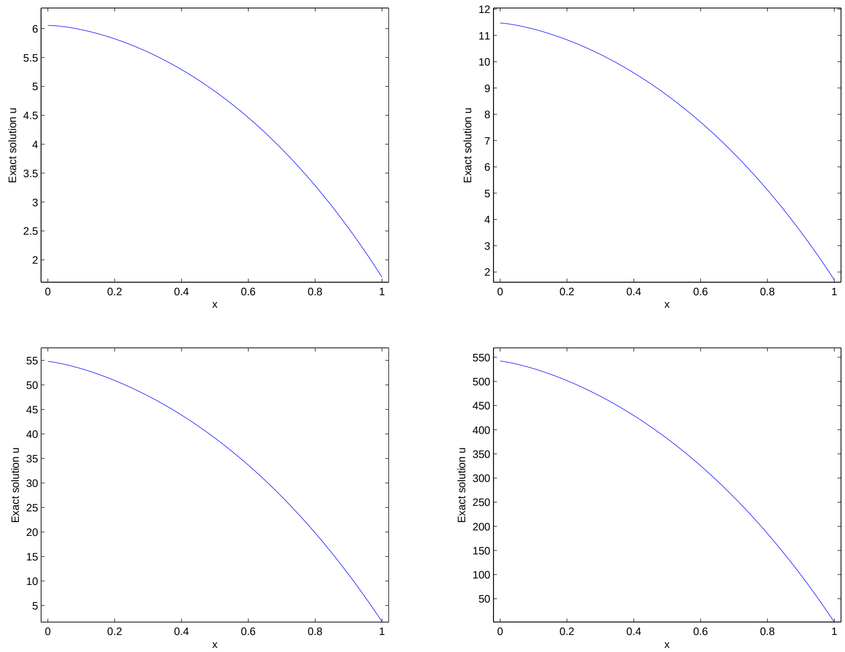

Fig. 2 Exact solution of (1) for $b \equiv 1$, $c \equiv 0$, $f \equiv 1$, $\alpha_1 = 0$, $\gamma_0 = 0.4$ and $\gamma_1 = 1.7$, with $\delta = 1.1$ (1st row, left), $\delta = 1.05$ (1st row, right), $\delta = 1.01$ (2nd row, left) and $\delta = 1.001$ (2nd row, right).

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \text{for } \alpha, \beta, z \in \mathbb{R} \text{ with } \alpha > 0. \tag{3}$$

See [1, 5] for the analysis of this function.

Lemma 1. *(Properties of the Mittag-Leffler function)*

$$(i)\quad \frac{d}{dx}\big(xE_{\delta-1,2}(bx^{\delta-1})\big) = E_{\delta-1,1}(bx^{\delta-1}) \tag{4a}$$

$$(ii)\quad \frac{(2-\delta)(e-i)}{\delta-1} \le E_{\delta-1,i}(1) \le \frac{\delta(e+1-i)}{\delta-1} \quad \textit{for } i=1,2 \tag{4b}$$

(iii) *For* $y \in (0,1)$, *one has*

$$\frac{1}{2(1-y)} \le E_{\delta-1,1}(y) \le \frac{1}{\theta(1-y)} \tag{4c}$$

where $\theta = \min\{\Gamma(x) : 1 \le x \le 2\} \approx 0.8856$

(iv) *Fix* $y > 1$, $x \in (0,1]$ *and* $n > 0$. *Then as* $\delta \to 1^+$ *one has*

$$E_{\delta-1,n}\left(yx^{\delta-1}\right) = \frac{1}{\delta-1}\left(yx^{\delta-1}\right)^{(1-n)/(\delta-1)} \exp\left(\left(yx^{\delta-1}\right)^{1/(\delta-1)}\right) + O\left(\frac{1}{(\delta-1)^2}\right). \tag{4d}$$

Proof. For (4a) see [5, (1.82)]; for (4c) see [7, (2.28)]. Bounds slightly stronger than (4b) are proved in [7, (2.23)–(2.25)] under the assumption that $1/(\delta-1)$ is an integer; when this assumption is removed, a straightforward modification of these calculations (as in the proof of [7, Lemma 3.6]) yields (4b). The estimate (4d) is proved in [7, (2.19)] for the case $x = 1$. The same argument will work when $x \in (0,1)$ because $yx^{\delta-1} \to y > 1$ as $\delta \to 1^+$. $\square$

By [7, (2.13)], the solution of (1) is

$$u(x) = \gamma_0 + \frac{(\alpha_0+x)f}{b} + \frac{\left[\alpha_0 + xE_{\delta-1,2}(bx^{\delta-1})\right]\left[\gamma_1 - \gamma_0 - (1+\alpha_0+\alpha_1)f/b\right]}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)} \tag{5}$$

for $0 \le x \le 1$. Hence, using (4a),

$$u'(x) = \frac{f}{b} + \frac{\left[\gamma_1 - \gamma_0 - (1+\alpha_0+\alpha_1)f/b\right]E_{\delta-1,1}(bx^{\delta-1})}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)} \quad \text{for } 0 \le x \le 1. \tag{6}$$

3 Behaviour of solutions as $\delta \to 1^+$

In this section we give a definition of *layer* that is suitable for the typical boundary layers that one encounters in solutions of singularly perturbed two-point boundary value problems. This definition will be seen to be suitable also for the layers encountered when $b > 1$, as in Figure 1. But the behaviour of the solution of (1) when $b = 1$, which is exhibited in Figure 2, is different.

Let us start from familiar territory: boundary layers in convection-diffusion boundary value problems. As is well known (see, e.g., [6, Section I.1.1]), the solution of such problems can usually be expressed as a sum of smooth and layer components; as a typical example we take $\phi(x) = 5x + e^{-2(1-x)/\varepsilon}$ for $0 \le x \le 1$.

Here $\varepsilon \in (0,1]$ is the singular perturbation parameter and $\phi(x)$ develops a boundary layer at $x = 1$ as $\varepsilon \to 0^+$ because of the component $e^{-2(1-x)/\varepsilon}$.

With a view to what comes later, we now define what is meant by a layer.

Definition 1. Let $v \in C^1[0,1]$, with v dependent on some parameter $\varepsilon \in (0,1]$. We say that v has a *layer* at a point $z \in [0,1]$ as $\varepsilon \to 0^+$ if

(i) $\lim_{\varepsilon\to 0^+} v'(z)$ is ∞ or $-\infty$,
(ii) $\lim_{\varepsilon\to 0^+} v'(x)$ is finite at each point $x \in [0,1]$ satisfying $0 < |x-z| < k$ for some positive constant k. (Here k can depend on z but not on ε.)

Our example above satisfies this definition with $v = \phi$ and $z = 1$; for one has $\lim_{\varepsilon\to 0^+} \phi'(1) = \infty$ and $\lim_{\varepsilon\to 0^+} \phi'(x) = 5$ for each fixed $x \in [0,1)$. Definition 1 is adequate for typical singularly perturbed convection-diffusion and reaction-diffusion problems in one dimension, including those whose solutions have interior layers; see [6, Chapter I].

One can use Definition 1 *mutatis mutandis* to define a layer at $x = 1$ for the solution u of (1) as $\delta \to 1^+$.

3.1 The case $b > 1$

Assume in Section 3.1 that $b > 1$. From (6) and [7, Section 2.2.1] it follows that

$$\lim_{\delta\to 1^+} u'(1) = \pm\infty, \tag{7}$$

with sign equal to the sign of $-f/b$. Furthermore, by inspection, as $\delta \to 1^+$ the dominant term in the numerator of the second fraction of (6) is $-\alpha_0 (f/b) E_{\delta-1,1}(bx^{\delta-1})$. But by (4d), for each fixed x in $(0,1)$ one has

$$\begin{aligned}
\lim_{\delta\to 1^+} \frac{\alpha_0 E_{\delta-1,1}(bx^{\delta-1})}{E_{\delta-1,2}(b)} &= \lim_{\delta\to 1^+} \frac{\frac{1}{\delta-1}\left[\frac{1}{\delta-1}\exp\left((bx^{\delta-1})^{1/(\delta-1)}\right) + O\left(\frac{1}{(\delta-1)^2}\right)\right]}{\frac{1}{\delta-1}\, b^{-1/(\delta-1)} \exp\left(b^{1/(\delta-1)}\right) + O\left(\frac{1}{(\delta-1)^2}\right)} \\
&= \lim_{\delta\to 1^+} \frac{\frac{1}{\delta-1}\exp\left((bx^{\delta-1})^{1/(\delta-1)}\right)}{b^{-1/(\delta-1)}\exp\left(b^{1/(\delta-1)}\right)} \\
&= \lim_{\delta\to 1^+} \frac{1}{\delta-1}\, b^{1/(\delta-1)} \exp\left((x-1)b^{1/(\delta-1)}\right) \\
&= 0.
\end{aligned} \tag{8}$$

From (6) and (8) it follows that

$$\lim_{\delta\to 1^+} u'(x) = \frac{f}{b} \quad \text{for each } x \in (0,1). \tag{9}$$

Taking (7) and (9) together, we have shown analytically that when $b > 1$, the solution of (1) has a layer (in the sense of Definition 1) at $x = 1$ as $\delta \to 1^+$.

3.2 The case $b = 1$

Assume in Section 3.2 that $b = 1$. In [7] it was shown that $u'(1)$ blows up as $\delta \to 1^+$, and from this fact it was inferred that "u exhibits a boundary layer at $x = 1$" [7, Section 2.2.2], but as we shall see below, the behaviour of u as $\delta \to 1^+$ does not satisfy Definition 1 at $x = 1$.

Invoking (4b), from equation (6) one sees that

$$\lim_{\delta \to 1^+} u'(1) = \pm\infty \tag{10}$$

with sign equal to the sign of $-f/b$.

Next, (2) and (4b) imply that

$$\frac{1}{\delta - 1} \le \alpha_0 + E_{\delta-1,2}(1) + \alpha_1 E_{\delta-1,1}(1) \le \frac{(1+\alpha_1)e\delta}{\delta - 1}. \tag{11}$$

Consider an arbitrary but fixed point $x \in (0,1)$. By (11) we have

$$\lim_{\delta \to 1^+} \frac{\alpha_0 E_{\delta-1,1}(x^{\delta-1})}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)} \ge \lim_{\delta \to 1^+} \frac{E_{\delta-1,1}(x^{\delta-1})}{(1+\alpha_1)e\delta}. \tag{12}$$

Writing $\lfloor 1/(\delta-1) \rfloor$ for the integer part of $1/(\delta-1)$, from the definition (3) one obtains

$$\begin{aligned} E_{\delta-1,1}(x^{\delta-1}) &\ge \sum_{k=0}^{\lfloor 1/(\delta-1) \rfloor} \frac{(x^{\delta-1})^k}{\Gamma(2)} \\ &= \frac{1-(x^{\delta-1})^{1+\lfloor 1/(\delta-1) \rfloor}}{1-x^{\delta-1}} \\ &> \frac{1-x}{1-x^{\delta-1}} \end{aligned} \tag{13}$$

because $(x^{\delta-1})^{1+\lfloor 1/(\delta-1) \rfloor} < (x^{\delta-1})^{1/(\delta-1)} = x$. Combining (12) and (13) yields

$$\lim_{\delta \to 1^+} \frac{\alpha_0 E_{\delta-1,1}(x^{\delta-1})}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)} \ge \lim_{\delta \to 1^+} \frac{1-x}{(1+\alpha_1)e\delta(1-x^{\delta-1})} = \infty \tag{14}$$

since $0 < x < 1$. It now follows from (6) that

$$\lim_{\delta \to 1^+} u'(x) = \pm\infty \quad \text{for each } x \in (0,1), \tag{15}$$

with sign equal to the sign of $-f/b$.

That is: unlike the case $b > 1$ where $u'(x)$ blows up only at $x = 1$ as $\delta \to 1^+$, the limits (10) and (15) show that when $b = 1$ the derivative $u'(x)$ of the solution of (1) *blows up at every point x in* $(0,1]$ as δ approaches its limiting value!

Figure 2 demonstrates this behaviour clearly—note the increasing compression of the y-axis scales in the graphs as $\delta \to 1^+$.

Heuristically, the essential difference between the cases $b = 1$ and $b > 1$ lies in the denominator $\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)$ of (5) and (6): when $b = 1$ this denominator is $O(1/(\delta-1))$ by (4b) but when $b > 1$ it is much larger, as can be seen from (4d).

4 An interior layer

Our discussions above and the analysis in [7] have focused on the presence or absence of boundary layers in the solution u of (1). These depend on the value(s) taken by b in (1a). We close our presentation by alerting the reader to the fact that when $b(x)$ is non-constant and varies in a certain way, it can engender an *interior layer* in the solution u. This possibility has not previously been mentioned in the research literature. It will be demonstrated here by means of an example.

Suppose that $b(x) = -8x^2 + 6x + 1$ for $0 \le x \le 1$. The key property of this particular function is that $b(x) > 1$ on part of the interval but $b < 1$ near $x = 1$. Its graph is shown in Figure 3.

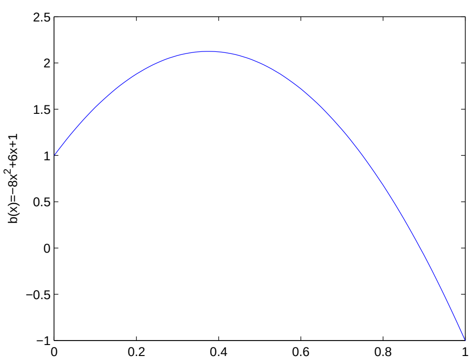

Fig. 3 Plot of $b(x) = -8x^2 + 6x + 1$

Take $f \equiv 1$, $c \equiv 0$, $\alpha_1 = 0$, $\gamma_0 = 0.4$ and $\gamma_1 = 1.7$ in (1). Graphs of the solution u, which are computed using the numerical method of [8], are shown in Figure 4 for $\delta = 1.1, 1.01, 1.001, 1.0001$. They exhibit an interior layer at the point where $b(x)$ switches from $b > 1$ to $b < 1$. No qualitative change in the solution is evident at the point where $b(x)$ changes sign.

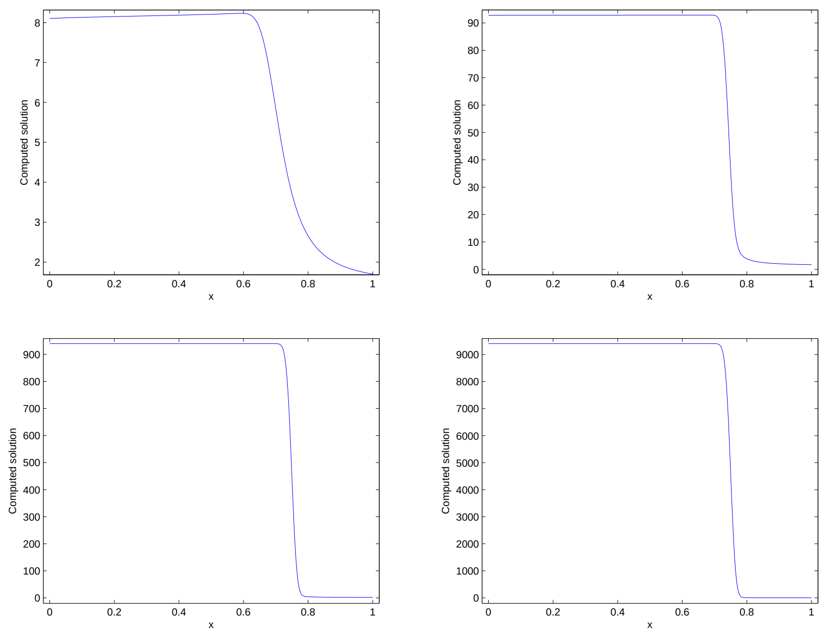

Fig. 4 Test problem with $b(x) = -8x^2 + 6x + 1$, $c \equiv 0$, $f \equiv 1$, $\alpha_1 = 0$, $\gamma_0 = 0.4$, $\gamma_1 = 1.7$. Solutions computed for $\delta = 1.1$ (1st row, left), $\delta = 1.01$ (1st row, right), $\delta = 1.001$ (2nd row, left) and $\delta = 1.0001$ (2nd row, right)

Note that in Figure 4 it is apparent that $\|u\|_\infty := \max_{[0,1]} |u(x)| = O(1/(\delta - 1))$. This behaviour agrees with the bound

$$\|u\|_\infty \le C \min\{\alpha_0, E_{\delta-1,\delta+1}(M)\} = C\alpha_0$$

of [7, Theorem 3.10(i)], where $M := \max_{[0,1]} b(x)$ and we used (4d).

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