Spatial semidiscretizations and time integration of 2D parabolic singularly perturbed problems

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Abstract In this work we deal with a different technique from the considered one in [1,3], to analyze the uniform convergence of some numerical methods which have been used to solve successfully two dimensional parabolic singularly perturbed problems of convection-diffusion type. For getting this, we split the discretization methods in a two stage procedure where, firstly, we semidiscretize in space, using the classical upwind scheme on a piecewise uniform Shishkin mesh, and, secondly, we integrate in time the Initial Value Problems derived from the first stage, by using the implicit Euler method. The analysis combines a suitable maximum semidiscrete principle joint to some well known techniques used in the proof of the uniform convergence of numerical schemes for elliptic singularly perturbed problems. We prove that the stiff initial value problems resulting of the spatial semidiscretization processes, have a unique solution which converges uniformly with respect to the singular perturbation parameter. Using this technique, some restrictions among the discretization parameters, which appeared in the uniform convergence analysis in [3], can be removed. Some numerical results corroborate in practice the robustness of the numerical method, according to the theoretical results.

1 Introduction

Let us consider 2D time-dependent convection-diffusion problems modeled by the PDE

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$$\frac{\partial u}{\partial t} - \varepsilon \Delta u + \overrightarrow{v} \cdot \overrightarrow{\nabla} u + ku = f(x,y,t),$$

where the diffusion parameter ε, $0 < \varepsilon \leq 1$, can be very small and let us deal with the efficient numerical integration of them. Although the proposed technique can be extended to several interesting situations, here we consider only the case

$$\overrightarrow{v} \equiv (v_1(x,y), v_2(x,y)), \text{ with } v_i(x,y) \geq \overline{v} > 0, \ i = 1,2, k \equiv k(x,y) \geq 0. \tag{1}$$

We assume that sufficient smoothness and compatibility conditions between data hold in order to the solution is four times derivable in space and twice in time.

It is well known that, in general, the solution of these problems has a multiscale character even for smooth data and, usually, regular boundary layers at $x = 1, y = 1$ of width $O(\varepsilon)$ appear in the outflow boundary; as well, other types of layers can appear depending on the geometry of the domain (see [5, 7, 9]). In such cases, standard finite difference or finite element methods, defined on uniform meshes, are shown inappropriate to solve the problem unless a large number (ε-dependent) of mesh points is considered. To find precise approximations, even inside the layers, using meshes with a number of grid points which does not depend of the size of ε, it is necessary to use uniformly convergent methods. For them, the rates of convergence and the error constants of the methods are independent of ε. To construct a uniformly convergent scheme, we use a fitted mesh method (see [7, 9]), which concentrates appropriately the grid points in the boundary layer regions.

Numerical methods for 2D singularly perturbed elliptic problems have been developed and analyzed in many papers (see [2, 6, 8] and references therein). For 2D time dependent problems, in [1, 3] the fully discrete numerical scheme is defined and analyzed as a two step discretization process, discretizing firstly only in time and later on in space. Nevertheless, in the proof of the uniform convergence of the method there are some drawbacks related to the uniform stability of the spatial discretization process and, sometimes, a ratio between the discretization parameters, which does not appear in the numerical experiments, is needed in the theoretical analysis. To avoid these difficulties, here we consider an alternative technique of analysis, discretizing first in space and, later on, integrating in time the resulting stiff Initial Value Problems.

The paper is structured as follows: in section 2, we introduce the spatial discretization of the continuous problem on a special nonuniform mesh of Shishkin type. In section 3 we prove almost first order uniform convergence for the spatial semidiscretization. In section 4 we introduce the time discretization and, consequently, define the numerical algorithm, proving also its uniform convergence. Finally, in section 5 we include the numerical results obtained in two different examples, showing the influence of the compatibility conditions between data.

Henceforth, C denotes a generic positive constant independent of the diffusion parameter ε and also of the discretization parameters N and M.

2 Spatial semidiscretization

Let us consider the following initial-boundary value problem

$$\begin{aligned} &\mathscr{L}u \equiv \frac{\partial u}{\partial t} + \mathscr{L}_\varepsilon u = f, \text{ in } \Omega \times (0,T], \\ &u(x,y,0) = \varphi(x,y), \text{ in } \Omega, \\ &u(x,y,t) = g(x,y,t), \text{ in } \partial\Omega \times [0,T], \end{aligned} \tag{2}$$

where $\Omega \equiv (0,1)\times(0,1)$, and the spatial differential operator $\mathscr{L}_\varepsilon$ is given by

$$\mathscr{L}_\varepsilon u \equiv -\varepsilon\Delta u + \overrightarrow{v} \cdot \overrightarrow{\nabla} u + ku, \tag{3}$$

together with the asumptions given in (1)

For this case, we will assume the following smoothness and compatibility conditions:

$$\begin{aligned} &f \in \mathscr{C}^{2,1}(\bar{\Omega}\times[0,T]), \varphi \in \mathscr{C}^4(\bar{\Omega}), \\ &\{g(0,y,t), g(1,y,t), g(x,0,t), g(x,1,t)\} \subset \mathscr{C}^{4,2}([0,1]\times[0,T]), \end{aligned} \tag{4}$$

$$\begin{aligned} &g(x,y,0) = \varphi(x,y), \quad (x,y) \in \partial\Omega, \\ &\frac{\partial g}{\partial t}(x,y,0) = f(x,y,0) - \mathscr{L}_\varepsilon\varphi(x,y), \quad (x,y) \in \partial\Omega, \\ &\frac{\partial^2 g}{\partial t^2}(x,y,0) = \frac{\partial f}{\partial t}(x,y,0) - \mathscr{L}_\varepsilon f(x,y,0) + \mathscr{L}_\varepsilon^2\varphi(x,y), \quad (x,y) \in \partial\Omega, \end{aligned} \tag{5}$$

and

$$\frac{\partial g}{\partial t}(x,y,t) + \mathscr{L}_\varepsilon g(x,y,t) = f(x,y,t), (x,y) \in \{(0,0),(0,1),(1,0),(1,1)\}, t \in [0,T] \tag{6}$$

A finite difference spatial semidiscretizaton of (2) provides approximations of $u(x_i,y_j,t)$, where $(x_i,y_j),\ i,j = 0,\ldots,N$, are the grid points of a rectangular mesh $\overline{\Omega}_N$. For simplicity, we take the same number of mesh points in both space directions.

Let us denote $[.]_N$ the restriction of a function defined on $\overline{\Omega}$ to $\overline{\Omega}_N$ and let $u_N(t)$ be a semidiscrete function defined in $\overline{\Omega}_N$ which approaches the exact solution $u(x_i,y_j,t)$ of (2) for all $(x_i,y_j) \in \overline{\Omega}_N$. Typically, $u_N(t)$ is defined as the solution of an IVP of the form

$$\begin{aligned} &u_N'(t) + \mathscr{L}_{\varepsilon,N} u_N(t) = [\tilde{f}]_N, \\ &u_N(0) = [\varphi]_N, \end{aligned} \tag{7}$$

where $\mathscr{L}_{\varepsilon,N}$ is a finite difference approximation of the elliptic operator $\mathscr{L}_\varepsilon$ and

$$[\tilde{f}]_N(x_i,y_j,t) \equiv \begin{cases} f(x_i,y_j,t), & (x_i,y_j) \in \Omega_N, \\ g(x_i,y_j,t) + \dfrac{\partial g}{\partial t}(x_i,y_j,t), & (x_i,y_j) \in \partial\Omega_N. \end{cases}$$

The first step to define the scheme is the choice of a suitable mesh $\overline{\Omega}_N$. Such mesh will be described as a tensor product of one dimensional meshes, i.e., $\overline{\Omega}_N \equiv I_{x,\varepsilon,N} \times I_{y,\varepsilon,N} \subset \overline{\Omega}$, where $I_{x,\varepsilon,N} = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$, $I_{y,\varepsilon,N} = \{0 = y_0 < y_1 < \cdots < y_N = 1\}$. We show the details of the construction of $I_{x,\varepsilon,N}$ and analogously it can be done for $I_{y,\varepsilon,N}$. Let us take N as a multiple of four. We define the transition parameter

$$\sigma_x = \min\left(\frac{1}{2},\ m_x \varepsilon \ln N\right), \tag{8}$$

where $m_x \geq \frac{1}{\bar{v}}$; then, we construct a piecewise uniform mesh with $\frac{N}{2}+1$ equidistant points in $[0, 1-\sigma_x]$ and the same in $[1-\sigma_x, 1]$. Therefore, the mesh points are given by

$$x_i = \begin{cases} i\dfrac{2(1-\sigma_x)}{N},\ i = 0, \cdots, \dfrac{N}{2}, \\ 1-\sigma_x + (i-\dfrac{N}{2})\dfrac{2\sigma_x}{N},\ i = \dfrac{N}{2}+1, \cdots, N. \end{cases} \tag{9}$$

On this mesh we consider the simple upwind finite difference scheme, which is defined as follows:

$$\begin{aligned} &\mathscr{L}_{\varepsilon,N} u_N(t)(x_i,y_j) \equiv l_{i,j-} u_N(t)(x_i,y_{j-1}) + l_{i-,j} u_N(t)(x_{i-1},y_j) + \\ &l_{i+,j} u_N(t)(x_{i+1},y_j) + l_{i,j} u_N(t)(x_i,y_j) + l_{i,j+} u_N(t)(x_i,y_{j+1}),\ i,j = 1,\ldots,N-1, \\ &\mathscr{L}_{\varepsilon,N} u_N(t)(x_i,y_j) \equiv u_N(x_i,y_j,t),\ i = 0,N \text{ or } j = 0,N, \end{aligned} \tag{10}$$

where

$$\begin{aligned} l_{i-,j} &= \frac{-\varepsilon}{h_{x,i}\tilde{h}_{x,i}} - \frac{v_1(x_i,y_j)}{h_{x,i}},\ \ l_{i+,j} = \frac{-\varepsilon}{h_{x,i+1}\tilde{h}_{x,i}}, \\ l_{i,j-} &= \frac{-\varepsilon}{h_{y,j}\tilde{h}_{y,j}} - \frac{v_2(x_i,y_j)}{h_{y,j}},\ \ l_{i,j+} = \frac{-\varepsilon}{h_{y,j+1}\tilde{h}_{y,j}}, \\ l_{i,j} &= k(x_i,y_j) - l_{i,j-} - l_{i-,j} - l_{i+,j} - l_{i,j+}, \end{aligned} \tag{11}$$

with $h_{x,i} = x_i - x_{i-1}$, $h_{y,j} = y_j - y_{j-1}$, $\tilde{h}_{x,i} = (h_{x,i} + h_{x,i+1})/2,\ i = 1, \cdots, N$, $\tilde{h}_{y,j} = (h_{y,j} + h_{y,j+1})/2,\ j = 1, \cdots, N$.

The following result (see [4] for full details of the proof) is one of the main keys in the analysis of the uniform convergence of the spatial semisdiscretization.

Lemma 1. *Let us suppose that $\mathscr{L}_{\varepsilon,N}$ is an inverse monotone and consistent operator of the form (10). If $[f(x,y,t)]_N \leq 0$, $[\varphi(x,y)]_N \geq 0$, for any $(x,y) \in \overline{\Omega}_N$ and any $t \in [0,T]$, and $g(x,y,t) \geq 0$ for any $(x,y) \in \partial\Omega_N \equiv I_{x,\varepsilon,N} \times \{0,1\} \bigcup \{0,1\} \times I_{y,\varepsilon,N}$ and any $t \in [0,T]$, then the solution of (7) achieves its maximum value at the discrete initial condition or at the discrete boundary $\partial\Omega_N \times [0,T]$, i.e., it holds*

$$\max_{(x,y,t) \in \overline{\Omega}_N \times [0,T]} \{u_N(t)(x,y)\} \leq \max_{(x,y) \in \overline{\Omega}_N,\ (x,y,t) \in \partial\Omega_N \times [0,T]} \{\varphi(x,y), g(x,y,t)\}. \tag{12}$$

From this result, we can deduce (see [4]) the following corollary, proving the uniform stability of the semidiscretization.

Corollary 1. *(inverse monotonicity of the semidiscrete operator). If a semidiscrete function ψ defined on $\Omega_N \times [0,T]$ is less or equal to zero at the initial points $(x_i, y_j, 0)$ and also on the points belonging to $\partial\Omega_N \times [0,T]$, and it satisfies that $\psi'(t)(x,y) + (\mathscr{L}_{\varepsilon,N}\psi(t))(x,y) \leq 0, \quad \forall\ (x,y,t) \in \Omega_N \times [0,T]$, then, it holds $\psi(t)(x,y) \leq 0 \quad \forall\ (x,y,t) \in \overline{\Omega}_N \times [0,T]$.*

Moreover, there exists a constant C, depending only on $\overline{v}$, such that

$$\|u_N(t)\|_{\infty,N} \leq \max\{\|[\varphi]_N\|_{\infty,N}, G\} + C \max_{t\in[0,T]} \|[f(x,y,t)]_N\|_{\infty,N},$$

where

$$\|\psi\|_{\infty,N} \equiv \max_{(x,y)\in\overline{\Omega}_N} |\psi(x,y)|,$$

and

$$G = \max_{(x,y,t)\in\partial\Omega_n\times[0,T]} |g(x,y,t)|.$$

To obtain the uniform convergence of the scheme we must bound, uniformly in ε, the global error

$$e_N u(t) = [u(t)]_N - u_N(t).$$

For getting this, we start by analyzing the local truncation error

$$\tau_N u(t) = [(\frac{\partial}{\partial t} + \mathscr{L}_\varepsilon)u(t)]_N - (\frac{d}{dt} + \mathscr{L}_{\varepsilon,N})[u(t)]_N.$$

Note that this error admits an immediate simplification to

$$\tau_N u(t) = [\mathscr{L}_\varepsilon u(t)]_N - \mathscr{L}_{\varepsilon,N}[u(t)]_N,$$

and therefore the analysis of the consistency for the space semidiscretization of problem (2) is essentially identical to the analysis performed for its corresponding stationary version. For such problems, in [3] it was proven that u can be decomposed in the form $u = u_0 + w$, where u_0 is the regular part of u and w is the singular component, which can also be decomposed in the form $w = u_1 + u_2 + u_3$, where u_1, u_2 are the regular layer functions near $x = 1$ and $y = 1$ respectively and u_3 is the corner layer function. Also in [3] appropriate bounds of the derivatives of the regular and singular components were proved.

Imitating these decompositions of the continuous problem, we decompose the numerical solution of the upwind scheme in a similar way, i.e.,

$$u_N(t) = u_{0,N}(t) + \sum_{p=1}^{3} u_{p,N}(t),$$

where

$$\begin{cases} u'_{0,N}(t) + \mathscr{L}_{\varepsilon,N} u_{0,N}(t) = [f]_N, \\ u_{0,N}(0) = [u_0]_N, \end{cases}$$
$$\begin{cases} u'_{p,N}(t) + \mathscr{L}_{\varepsilon,N} u_{p,N}(t) = [g_p + \dfrac{\partial g}{\partial t}]_N, \quad p = 1,2,3, \\ u_{p,N}(0) = [u_p]_N, \quad p = 1,2,3 \end{cases}$$

where g_1, g_2 contain evaluations of certain boundary conditions at the two outflow sides of Ω ($x = 1$ and $y = 1$), and $\tilde{g}_3$ contains evaluations of boundary conditions at the corner $(1,1)$.

Then, (see [4, 6] for full details), using the barrier function technique and the appropriate bounds for the truncation errors associated to the components of the exact and the numerical solution introduced before, it can be deduced that

$$\|e_N u(t)\|_{\infty,N} \le C N^{-1} \ln N,$$

proving that the space semidiscretization is uniformly convergent of almost first order.

3 Time integration: the fully discrete scheme

After the spatial semidiscretization stage introduced and analyzed in previous section, to obtain a fully discrete scheme we discretize in time the IVP resulting of the first stage. To get this with a uniform behavior, both in N and ε, the simplest scheme is the backward Euler method. Let $t_m = m\tau$, where $\tau = T/M$ (for simplicity we consider a uniform mesh in time); then, using the Euler method, the fully discrete scheme is given by

$$\begin{aligned} &\frac{u_N^m - u_N^{m-1}}{\tau} + \mathscr{L}_{\varepsilon,N} u_N^m = [\tilde{f}]_N(t_m), \quad m = 1,\ldots,M, \\ &u_N^0 = [\varphi]_N. \end{aligned} \tag{13}$$

or equivalently by

$$\begin{aligned} &(I_N + \tau \mathscr{L}_{\varepsilon,N}) u_N^m = u_N^{m-1} + \tau [\tilde{f}]_N(t_m), \quad m = 1,\ldots,M, \\ &u_N^0 = [\varphi]_N. \end{aligned} \tag{14}$$

Lemma 2. *Operators $I_N + \tau \mathscr{L}_{\varepsilon,N}$ are inverse monotone and it holds*

$$\|(I_N + \tau \mathscr{L}_{\varepsilon,N})^{-1}\|_{\infty,N} \le 1. \tag{15}$$

From Lemma 2, it immediately follows the next result, proving the numerical stability as well as the contractivity of the method.

Corollary 2. *If we consider the perturbed problem*

$$\frac{\tilde{u}_N^m - \tilde{u}_N^{m-1}}{\tau} + \mathscr{L}_{\varepsilon,N}\tilde{u}_N^m = [\tilde{f}]_N(t_m) + \delta^m, m = 1,\ldots,M,$$
$$\tilde{u}_N^0 = [\varphi]_N + \tau\delta^0,$$

then it holds that

$$\|\tilde{u}_N^m - u_N^m\|_{\infty,N} \le \tau \sum_{i=o}^{m} \|\delta^i\|_{\infty,N}.$$

Moreover, if $f = 0, g = 0$ in the continuous problem (2), then it holds

$$\|u_N^m\|_{\infty,N} \ \le \|u_N^{m-1}\|_{\infty,N}.$$

Now, let us introduce the local truncation error at time t_m for method (13) by

$$e_N^m = u_N(t_m) - (I_N + \tau\mathscr{L}_{\varepsilon,N})^{-1}(u_N(t_{m-1}) + \tau[\tilde{f}]_N(t_m)).$$

Then, it can be proved the following result of uniform consistency (see [4]).

Lemma 3. *The local truncation error satisfies*

$$\|e_N^m\|_{\infty,N} \le C\tau^2, \tag{16}$$

being C a constant independent of ε and N.

Jointing the previous results of uniform consistency and stability, for the global error of the time integration process

$$E_N^m = u_N(t_m) - u_N^m,$$

we deduce the following result, proving the first order uniform convergence for it.

Lemma 4. *The global error satisfies*

$$\|E_N^m\|_{\infty,N} \le CM^{-1}, \tag{17}$$

being C a constant independent of ε and N.

Finally, combining the main results of this section and the previous one, we deduce the next uniform and unconditional convergence result for our proposal.

Theorem 1. *Assuming that the solution of the continuous problem (2) satisfies that $u \in \mathscr{C}^{4,2}(\bar{\Omega} \times [0,T])$, the global error, associated to the numerical method (14), satisfies*

$$\max_{1\le i,j\le N, 1\le m\le M} |u(x_i,y_j,t_m) - u_N^m(x_i,y_j)| \le C\left(N^{-1}\ln N + M^{-1}\right).$$

Proof. Using that

$$\begin{aligned} u(x_i,y_j,t_m) &- u_N^m(x_i,y_j) \\ &= (u(x_i,y_j,t_m) - u_N(t_m)(x_i,y_j)) + (u_N(t_m)(x_i,y_j) - u_N^m(x_i,y_j)), \end{aligned}$$

the result follows from the uniform convergence results of section 2 and Lemma 4. □

Remark 1. Note that we have proved the uniform convergence of the fully discrete scheme without any ratio between the two discretization parameters N and M, in contrast with the results previously proved in [1, 3].

4 Numerical experiments

The first example is given by

$$\begin{aligned}
&u_t - \varepsilon\Delta u + \left(1 - \frac{xy}{2}\right) u_x + \left(1 + \frac{xy}{2}\right) u_y = f(x,y,t),\ (x,y,t) \in \Omega \times [0,1],\\
&u(0,y,t) = u(1,y,t) = 0, \quad y \in [0,1],\, t \in [0,1]\\
&u(x,0,t) = u(x,1,t) = 0, \quad x \in [0,1],\, t \in [0,1]\\
&u(x,y,0) = 0, \quad x,y \in [0,1],
\end{aligned} \tag{18}$$

with $f(x,y,t) = t(1-e^t)\,(\cos(\pi xy/2) - 1 + xy)$.

To approximate the maximum pointwise errors, we use a variant of the two-mesh principle. Then, we calculate $\{\hat{u}^N\}$, the numerical solution on the mesh $\{(\hat{x}_i,\hat{y}_j,\hat{t}_n)\}$ containing the original mesh points and its midpoints, i.e.,

$$\begin{aligned}
\hat{x}_{2i} = x_i,\ \ i = 0,\ldots,N, \quad \hat{x}_{2i+1} = (x_i + x_{i+1})/2,\ \ i = 0,\ldots,N-1,\\
\hat{y}_{2j} = y_j,\ \ j = 0,\ldots,N, \quad \hat{y}_{2j+1} = (y_j + y_{j+1})/2,\ \ j = 0,\ldots,N-1,\\
\hat{t}_{2m} = t_m,\ \ m = 0,\ldots,M, \quad \hat{t}_{2m+1} = (t_m + t_{m+1})/2,\ \ m = 0,\ldots,M-1.
\end{aligned}$$

The maximum errors at the mesh points of the coarse mesh are approximated by computing the following two-mesh differences

$$d_{N,M} = \max_{0\le m\le M} \max_{0\le i,j\le N} |u^N(x_i,y_j,t_m) - \hat{u}^N(x_i,y_j,t_m)|,$$

and the orders of convergence are calculated by

$$q = \log\left(d_{N,M}/d_{2N,2M}\right)/\log 2.$$

From the double-mesh differences we obtain the uniform maximum errors by

$$d^{N,M} = \max_{\varepsilon} d_{N,M},$$

and from them, in a usual way, the corresponding numerical uniform orders of convergence by

$$q^{uni} = \log\left(d^{N,M}/d^{2N,2M}\right)/\log 2.$$

To solve the linear system associated to the full discrete method at each time level, we use the BI-CGSTAB algorithm with a relaxed incomplete LU-factorization (see [10]) applied to a pentadiagonal matrix, with a tolerance equal to $1.e-10$.

Table 1 displays the results obtained in this case; the numerical orders show a slow approaching to 1 as long as N increases, which is typical when the logarithmic factor appears. This behavior of the errors is related with the election which we have made of the parameters N and M; then, the errors associated to the spatial semidiscretization stage dominate in the global error of the numerical scheme in this experiment. Other elections of the discretization parameters, with N much larger for the same M, will cause that the error in time dominates and, consequently, values much closer to 1 for the numerical orders of convergence will be observed.

Table 1 Maximum errors and orders of convergence for example (18)

ε	N=16 M=8	N=32 M=16	N=64 M=32	N=128 M=64	N=256 M=128
2^{-6}	0.7137E-2	0.4236E-2	0.2329E-2	0.1216E-2	0.6193E-3
	0.753	0.863	0.938	0.973	
2^{-8}	0.7773E-2	0.4747E-2	0.2677E-2	0.1426E-2	0.7345E-3
	0.711	0.826	0.909	0.957	
2^{-10}	0.7964E-2	0.4933E-2	0.2803E-2	0.1495E-2	0.7721E-3
	0.691	0.816	0.906	0.954	
2^{-12}	0.8015E-2	0.4984E-2	0.2839E-2	0.1518E-2	0.7864E-3
	0.685	0.812	0.903	0.949	
2^{-14}	0.8028E-2	0.4997E-2	0.2848E-2	0.1525E-2	0.7909E-3
	0.684	0.811	0.902	0.947	
2^{-16}	0.8031E-2	0.5000E-2	0.2851E-2	0.1526E-2	0.7921E-3
	0.684	0.811	0.901	0.946	
$d^{N,M}$	0.8031E-2	0.5000E-2	0.2851E-2	0.1526E-2	0.7921E-3
q^{uni}	0.684	0.811	0.901	0.946	

The second example is given by

$$
\begin{aligned}
& u_t - \varepsilon \Delta u + \left(1 - \frac{xy}{2}\right) u_x + \left(1 + \frac{xy}{2}\right) u_y = f(x,y),\ (x,y,t) \in \Omega \times [0,1], \\
& u(0,y,t) = u(1,y,t) = \sin(\pi y), \quad y \in [0,1],\, t \in [0,1] \\
& u(x,0,t) = u(x,1,t) = \sin(\pi x), \quad x \in [0,1],\, t \in [0,1] \\
& u(x,y,0) = \sin(\pi x) + \sin(\pi y), \quad x,y \in [0,1],
\end{aligned}
\tag{19}
$$

and $f(x,y,t)$ is the same as in the first example.

Table 2 displays the results obtained in this case; again we observe a uniformly convergent behavior in them, but a reduction in the numerical orders of convergence appears due to some of the compatibility conditions (5), (6) are not fulfilled. The influence of such incompatibilities, which causes a lack of smoothness in $u(x,y,t)$ and a subsequent reduction in the order of convergence of its numerical approaches, will be the subject of future studies.

Table 2 Maximum errors and orders of convergence for example (19)

ε	N=16 M=8	N=32 M=16	N=64 M=32	N=128 M=64	N=256 M=128
2^{-6}	0.3837E+0	0.2858E+0	0.1931E+0	0.1194E+0	0.6812E-1
	0.425	0.566	0.694	0.810	
2^{-8}	0.4304E+0	0.3386E+0	0.2534E+0	0.1771E+0	0.1148E+0
	0.346	0.418	0.517	0.626	
2^{-10}	0.4419E+0	0.3546E+0	0.2760E+0	0.2036E+0	0.1424E+0
	0.318	0.362	0.439	0.516	
2^{-12}	0.4447E+0	0.3594E+0	0.2820E+0	0.2117E+0	0.1524E+0
	0.307	0.350	0.414	0.475	
2^{-14}	0.4455E+0	0.3606E+0	0.2835E+0	0.2139E+0	0.1552E+0
	0.305	0.347	0.407	0.463	
2^{-16}	0.4456E+0	0.3609E+0	0.2839E+0	0.2144E+0	0.1559E+0
	0.304	0.346	0.405	0.460	
$d^{N,M}$	0.4456E+0	0.3609E+0	0.2839E+0	0.2144E+0	0.1559E+0
q^{uni}	0.304	0.346	0.405	0.460	

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