A posteriori error estimation of a stabilized mixed finite element method for Darcy flow

Tomás Patricio Barrios, José Manuel Cascón, and María González

Abstract We consider the augmented mixed finite element method proposed in [3] for Darcy flow. We develop the a priori and a posteriori error analyses taking into account the approximation of the Neumann boundary condition. We derive an a posteriori error indicator that consists of two residual terms on interior elements and an additional term that accounts for the error in the boundary condition on boundary elements. We prove that the error indicator is reliable and locally efficient on interior elements. Numerical experiments illustrate the good performance of the adaptive algorithm.

1 Introduction

The problem of Darcy flow is used to describe the flow of a fluid through a porous medium. It arises in many applications in science and engineering. The natural unknowns (the fluid pressure and the fluid velocity) can be approximated simultaneously with the mixed formulation, which is the most popular approach in applications. The Galerkin scheme associated to this formulation is not always well-posed and stability is ensured only for certain combinations of finite element subspaces. In this framework, several stabilization methods have been proposed in the literature (see, for instance, [2, 5, 10, 3] and the references therein).

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In this paper, we consider the augmented variational formulation proposed in [3] for heterogeneous, possibly anisotropic, porous media flow, which is a slight generalization of a method introduced in [10]. The augmented variational formulation is obtained by adding to the classical dual-mixed variational formulation two weighted residual type terms, that are related with Darcy's law and the mass conservation equation. We provided sufficient conditions on the stabilization parameters that ensure that the augmented variational formulation is well-posed. Under these same hypotheses, we also proved that the corresponding Galerkin scheme is well-posed and a Céa-type estimate holds whatever finite-dimensional subspaces are used. In particular, we provided a priori error bounds when the fluid velocity is approximated by Raviart-Thomas or Brezzi-Douglas-Marini elements, and the pressure is approximated using continuous piecewise polynomials. We remark that in this case local mass conservation is not guaranteed. A special feature of this formulation is that the stabilization parameters can be chosen independently of the mesh size and the type of elements employed to solve the discrete problem. Further, we propose in [3] a two-term a posteriori error estimator for the total error. The two residual terms account for the error in Darcy's law and in the mass conservation equation. This a posteriori error estimator is jump-free and can be used with any conforming approximation in $\mathbb{R}^d$, for $d = 2,3$. Moreover, we proved that it is reliable and locally efficient. However, the numerical analysis presented in [3] is done under the assumption that the Neumann boundary condition is satisfied exactly.

Our aim now is to develop an a priori and a posteriori numerical analysis of the Galerkin scheme (11), that assumes the approximation of the Neumann boundary condition by an L^2-projection of the Neumann datum on an appropriate discrete space. We develop a residual-based a posteriori error analysis of the augmented discrete scheme (11) and derive a simple a posteriori error indicator that coincides with the a posteriori error estimator proposed in [3] on interior elements. On each boundary element, the new a posteriori error indicator consists of an additional term that accounts for the error in the Neumann boundary condition. We prove that the new error indicator is reliable and locally efficient on interior elements. We remark that this a posteriori error indicator does not involve the computation of any jump across the elements of the mesh. Numerical experiments illustrate the good performance of the adaptive algorithm based on the new a posteriori error indicator. Indeed, efficiency indices are close to one and the adaptive algorithm is able to localize the singularities and high-variation regions of the exact solution.

The paper is organized as follows. In Section 2 we describe the problem of Darcy's flow and recall the augmented variational formulation analyzed in [3]. In Section 3 we propose an augmented discrete scheme in which the Neumann boundary condition is approximated using an L^2-projection on an appropriate piecewise polynomial space. We analyze the stability and convergence properties of this discrete scheme, and provide the corresponding rate of convergence when the velocity is approximated by Raviart-Thomas or Brezzi-Douglas-Marini elements, and the pressure is approximated by Lagrangian finite elements. The new a posteriori error indicator is derived in Section 4, where we also prove that it is reliable and locally efficient on interior elements. Finally, some numerical experiments are reported in Section 5.

Throughout this paper we will use the standard notations for Sobolev spaces and norms (see, for instance, [1]). In particular, for a given bounded open domain $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ with a Lipschitz-continuous boundary Γ, we denote $L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q = 0\}$; $H^{-1/2}(\Gamma)$ is the dual space of the trace space $H^{1/2}(\Gamma)$, and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product. Finally, given $\zeta \in H^{-1/2}(\Gamma)$, we denote by $H_\zeta := \{\mathbf{w} \in H(\mathrm{div},\Omega) : \mathbf{w}\cdot\mathbf{n} = \zeta \quad \text{on } \Gamma\}$ (see [8]). We use C, with or without subscripts, to denote generic constants, independent of the discretization parameter, that may take different values at different occurrences.

2 The augmented variational formulation

We assume that the porous medium Ω is a bounded connected open domain of $\mathbb{R}^d$ $(d=2,3)$ with a Lipschitz-continuous boundary Γ, and denote by $\mathbf{n}$ the unit outward normal vector to Γ. Let $\mathscr{K} \in [L^\infty(\Omega)]^{d\times d}$ be the hydraulic conductivity tensor. We assume that $\mathscr{K}$ is symmetric and uniformly positive definite, that is,

$$\Big(\mathscr{K}(\mathbf{x})\mathbf{y}\Big)\cdot\mathbf{y} \geq \alpha\,||\mathbf{y}||^2, \qquad \text{a.e. } \mathbf{x}\in\Omega\,, \quad \forall\,\mathbf{y}\in\mathbb{R}^d\,, \tag{1}$$

for some $\alpha > 0$. We recall that in isotropic porous media, the hydraulic conductivity tensor is a diagonal tensor of the form $\mathscr{K} = \frac{\kappa}{\mu}\mathbf{I}$, where $\kappa > 0$ is the permeability of the porous media, $\mu > 0$ is the viscosity of the fluid and $\mathbf{I} \in \mathbb{R}^{d\times d}$ is the identity matrix.

Let $\rho > 0$ be the fluid density, $\mathbf{g}$ be the gravity acceleration vector, g_c be a conversion constant, φ be the volumetric flow rate source or sink and ψ be the normal component of the velocity field on the boundary. Then, the Darcy problem reads: find the fluid velocity $\mathbf{v} : \Omega \to \mathbb{R}^d$ and the fluid pressure $p : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \mathscr{K}^{-1}\mathbf{v} + \nabla p = \mathbf{f} \ \text{ in } \Omega\,, \\ \qquad \mathrm{div}(\mathbf{v}) = \varphi \ \text{ in } \Omega, \\ \qquad\quad \mathbf{v}\cdot\mathbf{n} = \psi \text{ on } \Gamma\,, \end{cases} \tag{2}$$

where $\mathbf{f} := -\frac{\rho}{g_c}\mathbf{g}$. In what follows, we assume that $\mathbf{f} \in [L^2(\Omega)]^d$, $\varphi \in L^2(\Omega)$ and $\psi \in H^{-1/2}(\Gamma)$. We also assume that φ and ψ satisfy the compatibility condition $\int_\Omega \varphi = \langle\psi,1\rangle_\Gamma$. Under these assumptions, problem (2) has a unique solution $(\mathbf{v},p)$ in $H_\psi \times M$, with $M := H^1(\Omega)\cap L_0^2(\Omega)$.

Let $\mathbf{H} := H(\mathrm{div},\Omega)\times M$ and let $\|\cdot\|_{\mathbf{H}}$ be the product norm of $\mathbf{H}$. We consider the following variational formulation of problem (2), that was proposed and analyzed in [3]: *find* $(\mathbf{v},p) \in H_\psi \times M$ *such that*

$$A_s((\mathbf{v},p),(\mathbf{w},q)) = F_s(\mathbf{w},q)\,, \qquad \forall\,(\mathbf{w},q) \in H_0 \times M, \tag{3}$$

where the bilinear form $A_s : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ and the linear functional $F_s : \mathbf{H} \to \mathbb{R}$ are defined by

$$\begin{aligned} A_s((\mathbf{v},p),(\mathbf{w},q)) &:= \int_\Omega \mathscr{K}^{-1}\mathbf{v}\cdot\mathbf{w} - \int_\Omega p\,\mathrm{div}(\mathbf{w}) + \int_\Omega q\,\mathrm{div}(\mathbf{v}) \\ &+ \kappa_1 \int_\Omega (\nabla p + \mathscr{K}^{-1}\mathbf{v})\cdot(\nabla q - \mathscr{K}^{-1}\mathbf{w}) + \kappa_2 \int_\Omega \mathrm{div}(\mathbf{v})\,\mathrm{div}(\mathbf{w}), \end{aligned} \tag{4}$$

and

$$F_s(\mathbf{w},q) := \int_\Omega \mathbf{f}\cdot\mathbf{w} + \int_\Omega \varphi\, q + \kappa_1 \int_\Omega \mathbf{f}\cdot(\nabla q - \mathscr{K}^{-1}\mathbf{w}) + \kappa_2 \int_\Omega \varphi\,\mathrm{div}(\mathbf{w}), \tag{5}$$

for all $(\mathbf{v},p),(\mathbf{w},q) \in \mathbf{H}$.

We remark that the variational formulation (3) is obtained by adding to the usual dual-mixed variational formulation of problem (2) two weighted residuals related with Darcy's law and the mass conservation equation. In what follows, we assume that the stabilization parameters κ_1 and κ_2 are such that

$$\kappa_1 \in \big(0, \frac{\alpha}{\|\mathscr{K}\|^2_{\infty,\Omega}\|\mathscr{K}^{-1}\|^2_{\infty,\Omega}}\big) \quad \text{and} \quad \kappa_2 > 0 \tag{6}$$

where $\|\cdot\|_{\infty,\Omega}$ denotes the usual norm in $[L^\infty(\Omega)]^{d\times d}$. Under these conditions, the bilinear form $A_s(\cdot,\cdot)$ is elliptic in $\mathbf{H}$, with ellipticity constant $C_{\mathtt{ell}}$, and problem (3) has a unique solution (cf. Lemma 1 and Theorem 1 in [3]).

3 The stabilized mixed finite element method

From now on, we assume that Ω is a polygonal or polyhedral domain. We also assume that $\psi \in L^2(\Gamma)$. Let $\{\mathscr{T}_h\}_{h>0}$ be a family of shape-regular meshes of $\bar{\Omega}$ made up of triangles if $d=2$ or tetrahedra if $d=3$. We denote by h_T the diameter of an element $T \in \mathscr{T}_h$ and define $h := \max_{T\in\mathscr{T}_h} h_T$. Hereafter, given $T \in \mathscr{T}_h$ and an integer $l \geq 0$, we denote by $\mathscr{P}_l(T)$ the space of polynomials of total degree at most l on T. Now, let $H_h \subset H(\mathrm{div};\Omega)$ be either the Raviart-Thomas space of order $r \geq 0$, $\mathscr{RT}_r(\mathscr{T}_h)$ (cf. [11]), i.e.,

$$H_h := \Big\{\mathbf{w}_h \in H(\mathrm{div};\Omega) : \mathbf{w}_{h|T} \in \Big([\mathscr{P}_r(T)]^d + \mathbf{x}\,\mathscr{P}_r(T)\Big),\ \forall T \in \mathscr{T}_h\Big\}, \tag{7}$$

where $\mathbf{x} \in \mathbb{R}^d$ is a generic vector, or the Brezzi-Douglas-Marini space of order $r+1$, $\mathscr{BDM}_{r+1}(\mathscr{T}_h)$, $r \geq 0$ (cf. [4]), i.e.,

$$H_h := \Big\{\mathbf{w}_h \in H(\mathrm{div};\Omega) : \mathbf{w}_{h|T} \in [\mathscr{P}_{r+1}(T)]^d, \quad \forall T \in \mathscr{T}_h\Big\}. \tag{8}$$

We also consider the standard Lagrange space of order $m \geq 1$:

$$M_h := \mathscr{L}_m(\mathscr{T}_h) = \left\{ q_h \in \mathscr{C}(\bar{\Omega}) \cap L^2_0(\Omega) \, : \, q_h\big|_T \in \mathscr{P}_m(T), \quad \forall T \in \mathscr{T}_h \right\}. \tag{9}$$

Let $\mathscr{F}_h := \{e_1, \ldots, e_n\}$ be the partition of Γ inherited from $\mathscr{T}_h$. Given an integer $l \geq 0$, we denote $\mathscr{P}_l(\mathscr{F}_h) := \{p \in L^2(\Gamma) \, : \, p_{|e_i} \in \mathscr{P}_l(e_i), \quad \forall i = 1, \ldots, n\}$, where $\mathscr{P}_l(e_i)$ denotes the space of polynomials of total degree at most l on e_i. We consider the L^2-projection operator $\pi_l : L^2(\Gamma) \to \mathscr{P}_l(\mathscr{F}_h)$, defined by

$$\int_\Gamma \zeta \, q = \int_\Gamma \pi_l(\zeta) \, q \, , \quad \forall q \in \mathscr{P}_l(\mathscr{F}_h) \, . \tag{10}$$

Then, for any $\zeta \in L^2(\Gamma)$, we denote $H_{\zeta,h} := H_{\pi_l(\zeta)} \cap H_h$, with $l = r$ if $H_h = \mathscr{RT}_r(\mathscr{T}_h)$ and $l = r+1$ if $H_h = \mathscr{BDM}_{r+1}(\mathscr{T}_h)$. We remark that $H_{0,h} = H_0 \cap H_h \subset H_0$. However, $H_{\psi,h}$ is not contained in H_ψ in general.

We consider the following Galerkin scheme associated to problem (3): *find* $(\mathbf{v}_h, p_h) \in H_{\psi,h} \times M_h$ *such that*

$$A_s((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) = F_s(\mathbf{w}_h, q_h) \, , \qquad \forall (\mathbf{w}_h, q_h) \in H_{0,h} \times M_h \, . \tag{11}$$

Since the bilinear form $A_s(\cdot,\cdot)$ is elliptic in $\mathbf{H}$, it follows that problem (11) has a unique solution $(\mathbf{v}_h, p_h) \in H_{\psi,h} \times M_h$. Moreover, there exists a constant $C > 0$, independent of h, such that

$$||(\mathbf{v} - \mathbf{v}_h, p - p_h)||_{\mathbf{H}} \leq C \inf_{(\mathbf{w}_h, q_h) \in H_{\psi,h} \times M_h} ||(\mathbf{v} - \mathbf{w}_h, p - q_h)||_{\mathbf{H}} \, . \tag{12}$$

The corresponding a priori error bound is given in the next theorem.

Theorem 1. *Assume that the stabilization parameters* κ_1 *and* κ_2 *satisfy* (6). *Then, if* $\mathbf{v} \in [H^t(\Omega)]^d$, $\mathrm{div}(\mathbf{v}) \in H^t(\Omega)$ *and* $p \in H^{t+1}(\Omega)$, *there exists* $C > 0$, *independent of* h, *such that*

$$||(\mathbf{v} - \mathbf{v}_h, p - p_h)||_{\mathbf{H}} \leq C h^\beta \Big(||\mathbf{v}||_{[H^t(\Omega)]^d} + ||\mathrm{div}(\mathbf{v})||_{H^t(\Omega)} + ||p||_{H^{t+1}(\Omega)} \Big) . \tag{13}$$

with $\beta := \min\{t, m, r+1\}$.

Proof. It follows straightforwardly from the Céa estimate (12) and the approximation properties of the corresponding finite element subspaces (cf. [4]).

4 A posteriori error analysis

In this section, we develop a residual-based a posteriori error analysis of the augmented discrete scheme (11). As compared to the analysis presented in [3], here we take into account the error in the approximation of the Neumann boundary condition. We derive a simple a posteriori error indicator that requires the computation of two residuals per interior element and three residuals on each boundary element. We show that this error indicator is reliable and locally efficient on interior elements.

Let $H_h \times M_h$ be one of the finite element subspaces of $H(\mathrm{div},\Omega)\times M$ considered in the previous section. We assume that the stabilization parameters κ_1 and κ_2 satisfy (6), and we let $(\mathbf{v},p)\in H_\psi \times M$ and $(\mathbf{v}_h,p_h)\in H_{\psi,h}\times M_h$ be the unique solutions to problems (3) and (11), respectively.

Now, let $(\bar{\mathbf{v}},\bar{p})$ be the solution of a Darcy problem with boundary data $\pi_l\psi$, that is,

$$\begin{cases} \mathscr{K}^{-1}\bar{\mathbf{v}} + \nabla\bar{p} = \mathbf{f} & \text{in } \Omega\,, \\ \mathrm{div}(\bar{\mathbf{v}}) = \varphi & \text{in } \Omega\,, \\ \bar{\mathbf{v}}\cdot\mathbf{n} = \pi_l\psi & \text{on } \Gamma\,. \end{cases} \tag{14}$$

Then, by the triangle inequality,

$$\|(\mathbf{v}-\mathbf{v}_h, p-p_h)\|_{\mathbf{H}} \le \|(\mathbf{v}-\bar{\mathbf{v}}, p-\bar{p})\|_{\mathbf{H}} + \|(\bar{\mathbf{v}}-\mathbf{v}_h, \bar{p}-p_h)\|_{\mathbf{H}}\,. \tag{15}$$

Let η_h be the a posteriori error estimator proposed in [3]:

$$\eta_h^2 := \sum_{T\in\mathscr{T}_h} \eta_h(T)^2\,, \tag{16}$$

with

$$\eta_h(T)^2 := \|\mathbf{f}-\nabla p_h - \mathscr{K}^{-1}\mathbf{v}_h\|^2_{[L^2(T)]^d} + \|\varphi - \mathrm{div}(\mathbf{v}_h)\|^2_{L^2(T)}\,. \tag{17}$$

Then, from the analysis in [3], we have that

$$\|(\bar{\mathbf{v}}-\mathbf{v}_h, \bar{p}-p_h)\|_{\mathbf{H}} \le C_{\mathtt{rel}}\,\eta_h, \tag{18}$$

with $C_{\mathtt{rel}} := \sqrt{2}C_{\mathtt{ell}}^{-1}\max(1+\kappa_1(1+\|\mathscr{K}^{-1}\|_{\infty,\Omega}), 1+\kappa_2)$.

On the other hand, the pair $(\mathbf{v}-\bar{\mathbf{v}}, p-\bar{p})$ satisfies

$$\begin{cases} \mathscr{K}^{-1}(\mathbf{v}-\bar{\mathbf{v}}) + \nabla(p-\bar{p}) = \mathbf{0} & \text{in } \Omega\,, \\ \mathrm{div}(\mathbf{v}-\bar{\mathbf{v}}) = 0 & \text{in } \Omega, \\ (\mathbf{v}-\bar{\mathbf{v}})\cdot\mathbf{n} = \psi - \pi_l\psi & \text{on } \Gamma\,. \end{cases} \tag{19}$$

Therefore, using the continuity of the solution with respect to the data, we have that

$$\|(\mathbf{v}-\bar{\mathbf{v}}, p-\bar{p})\|_{\mathbf{H}} \le C\|\psi-\pi_l\psi\|_{H^{-1/2}(\Gamma)}\,. \tag{20}$$

The $H^{-1/2}(\Gamma)$-norm in (20) can be estimated using a duality argument. Indeed, since $\psi-\pi_l\psi$ is orthogonal to $\mathscr{P}_0(\mathscr{F}_h)$,

$$\begin{aligned} \|\psi-\pi_l\psi\|_{H^{-1/2}(\Gamma)} &= \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \phi\neq 0}} \frac{\displaystyle\int_\Gamma (\psi-\pi_l\psi)(\phi-\pi_0\phi)}{\|\phi\|_{H^{1/2}(\Gamma)}} \\ &= \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \phi\neq 0}} \frac{\displaystyle\sum_{e\in\mathscr{F}_h}\int_e (\psi-\pi_l\psi)(\phi-\pi_0\phi)}{\|\phi\|_{H^{1/2}(\Gamma)}}\,, \end{aligned} \tag{21}$$

where $\pi_0 : L^2(\Gamma) \to \mathscr{P}_0(\mathscr{F}_h)$ denotes the L^2-projection operator onto $\mathscr{P}_0(\mathscr{F}_h)$. Then, applying the Cauchy-Schwarz inequality and using that (cf. equation (20) in [7])

$$\|\phi - \pi_0 \phi\|_{L^2(e)} \leq C h_e^{1/2} \|\phi\|_{H^{1/2}(e)}\,, \tag{22}$$

where h_e is the measure of the boundary element $e \in \mathscr{F}_h$, we have

$$\|\psi - \pi_l \psi\|_{H^{-1/2}(\Gamma)} \leq C \sup_{\substack{\phi \in H^{1/2}(\Gamma) \\ \phi \neq 0}} \frac{\displaystyle\sum_{e \in \mathscr{F}_h} h_e^{1/2} \|\psi - \pi_l \psi\|_{L^2(e)} \|\phi\|_{H^{1/2}(e)}}{\|\phi\|_{H^{1/2}(\Gamma)}}\,. \tag{23}$$

Then, using the Cauchy-Schwarz inequality, taking into account that (see [7])

$$\sup_{\substack{\phi \in H^{1/2}(\Gamma) \\ \phi \neq 0}} \frac{\left(\displaystyle\sum_{e \in \mathscr{F}_h} \|\phi\|^2_{H^{1/2}(e)}\right)^{1/2}}{\|\phi\|_{H^{1/2}(\Gamma)}} \leq C\,, \tag{24}$$

where C can be bounded independently of h, and that $\mathbf{v}_h \cdot \mathbf{n} = \pi_l \psi$ on Γ, we deduce that

$$\|\psi - \pi_l \psi\|_{H^{-1/2}(\Gamma)} \leq C \left(\sum_{e \in \mathscr{F}_h} h_e \|\psi - \mathbf{v}_h \cdot \mathbf{n}\|^2_{L^2(e)}\right)^{1/2}\,. \tag{25}$$

Motivated by the previous analysis, we define the error indicator ζ_h by

$$\zeta_h^2 := \sum_{T \in \mathscr{T}_h} \zeta_h(T)^2\,, \tag{26}$$

where

$$\zeta_h(T)^2 := \eta_h(T)^2 + \mathrm{osc}_h(T)^2\,, \tag{27}$$

with

$$\mathrm{osc}_h(T)^2 := \sum_{e \in E(T) \cap \mathscr{F}_h} h_e \|\psi - \mathbf{v}_h \cdot \mathbf{n}\|^2_{L^2(e)}\,, \tag{28}$$

where $E(T)$ denotes the set of edges ($d = 2$) or faces ($d = 3$) of an element $T \in \mathscr{T}_h$.

We remark that when T is an interior element, the local error indicator $\zeta_h(T)$ consists of two residual terms, namely, the local residual in Darcy's law and the local residual in the mass conservation equation. In case T is a boundary element, then $\zeta_h(T)$ contains an additional term that accounts for the error in the Neumann boundary condition. We also notice that the global a posteriori error indicator ζ_h does not involve the computation of any jump across the elements of the mesh. This fact, besides the properties of the estimator stated in the next theorem, makes ζ_h well-suited for numerical computations.

Theorem 2. *There exists a positive constant $C_{\mathtt{r}}$, independent of h, such that*

$$\|(\mathbf{v}-\mathbf{v}_h, p-p_h)\|_{\mathbf{H}} \leq C_{\mathtt{r}}\, \zeta_h\,. \tag{29}$$

Moreover, if $T \in \mathscr{T}_h$ is an interior element of $\mathscr{T}_h$, there exists a positive constant $C_{\mathtt{eff}}$, independent of h and T, such that

$$C_{\mathtt{eff}}\, \zeta_h(T) \,\leq\, \|(\mathbf{v}-\mathbf{v}_h, p-p_h)\|_{H(\mathrm{div},T)\times H^1(T)}\,, \qquad \forall\, T \in \mathscr{T}_h\,. \tag{30}$$

In fact, $C_{\mathtt{eff}}^{-1} := \max(1, \sqrt{2}\, \|\mathscr{K}^{-1}\|_{\infty,\Omega})$.

Proof. Inequality (29) follows from inequalities (15), (18), (20), (25) and the definition of ζ_h. On the other hand, let T be an interior element of $\mathscr{T}_h$. Then, $\zeta_h(T) = \eta_h(T)$ and inequality (30) follows using that $(\mathbf{v}, p)$ satisfies the two first equations in (2) and applying the triangle inequality. $\square$

5 Numerical results

In this section we present some numerical experiments that illustrate the reliability and efficiency of the a posteriori error estimator ζ_h. The experiments have been performed with the finite element toolbox ALBERTA (cf. [12]) using refinement by recursive bisection [9]. The solutions of the corresponding linear systems have been computed using the SuperLU library [6]. We use the standard adaptive finite element method (AFEM) based on the loop:

$$\mathsf{SOLVE} \rightarrow \mathsf{ESTIMATE} \rightarrow \mathsf{MARK} \rightarrow \mathsf{REFINE}.$$

Hereafter, we replace the subscript h by k, where k is the counter of the adaptive loop. Then, given a mesh $\mathscr{T}_k$, the procedure SOLVE is an efficient direct solver for computing the discrete solution $(\mathbf{v}_k, p_k)$, ESTIMATE calculates the error indicators $\zeta_k(T)$, for all $T \in \mathscr{T}_k$, using the computed solution and the data. Based on the values of $\{\zeta_k(T)\}_{T\in\mathscr{T}_k}$, the procedure MARK generates a set of marked elements subject to refinement. For the elements selection, we rely on the maximum strategy with a threshold $\sigma = 0.6$. Finally, the procedure REFINE creates a conforming refinement $\mathscr{T}_{k+1}$ of $\mathscr{T}_k$, bisecting d times all marked elements (where $d = 2,3$ is the space dimension).

The robustness of the augmented scheme (11) with respect to the stabilization parameters and the sensitivity of the stabilized formulation to the ratio of the permeability to the viscosity were tested in [3] for the finite element pair $(\mathscr{R}\mathscr{T}_0, \mathscr{L}_1)$. Here we compare the performance of a finite element method based on uniform refinement (in each step, all elements of the actual mesh are bisected twice), with the adaptive algorithm described above.

Let $\Omega = (0,1)^2$ be the unit square and let $\mathscr{K} = \varepsilon\, \mathbf{I}$, with $\varepsilon > 0$. We take the data $\mathbf{f}$, φ and ψ so that the exact solution of problem (2) is given by the pair $(\mathbf{v}, p)$, with

$$p(x,y) = x\,y\left(1 - e^{\frac{x-1}{\varepsilon}}\right)\left(1 - e^{\frac{y-1}{\varepsilon}}\right), \quad (x,y) \in \Omega\,, \tag{31}$$

and $\mathbf{v} = -\mathscr{K}\,\nabla p$. We remark that the solution has a boundary layer around the point $(1,1)$.

We solve the problem using the finite element pair $(\mathscr{RT}_0, \mathscr{L}_1)$ with uniform refinement (FEM algorithm) and the adaptive refinement algorithm (AFEM) described above. We choose $\kappa_1 = \frac{\varepsilon}{2}$ and $\kappa_2 = 1.0$. We remark that these values of the stabilization parameters are consistent with the theory and ensure that the bilinear form $A_s(\cdot,\cdot)$ is elliptic in the whole space. For implementation purposes, instead of imposing the null media condition required to the elements of M_h, we fix to zero the value of the pressure in a corner of the domain. Finally, the non-homogeneous Neumann boundary condition is imposed by interpolation.

In Figure 1 we show the decay of the total error and the a posteriori error indicator ζ_h versus the degrees of freedom (DOFs) for the uniform (FEM) and adaptive (AFEM) refinements for $\varepsilon = 10^{-2}$ (left) and $\varepsilon = 10^{-3}$ (right). We observe that for $\varepsilon = 10^{-2}$, the uniform FEM algorithm attains the theoretical convergence rates predicted by the theory. However, the AFEM algorithm converges faster. For $\varepsilon = 10^{-3}$, the uniform refinement procedure does not attain the optimal convergence rate, whereas the AFEM algorithm is able to attain linear convergence, revealing itself as a competitive algorithm.

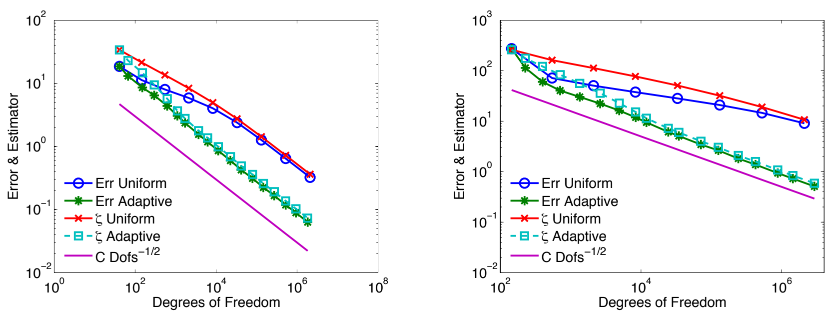

Fig. 1 Decays of total error and estimator vs. DOFs for $\varepsilon = 10^{-2}$ (left) and $\varepsilon = 10^{-3}$ (right).

Efficiency indices (which are defined as the ratio of the estimated error to the total error) are reported in Figure 2. We observe there that the efficiency indices are bounded from above and below by positive constants, independently of the mesh size, which confirms that error and estimator are equivalent. In fact, the efficiency indices for the AFEM tend to stabilize around 1.

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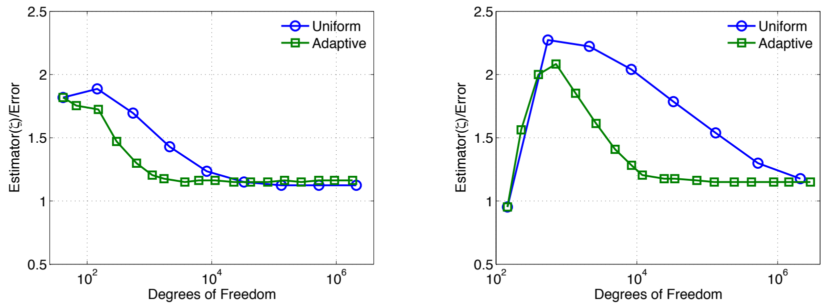

Fig. 2 Efficiency indices vs. DOFs for $\varepsilon = 10^{-2}$ (left) and $\varepsilon = 10^{-3}$ (right).

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