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Preface

This habilitation thesis consist of several research papers, an introductory chapter, comments and complements. The references used in the introduction and in the commentaries are collected in the bibliography list at the end of the thesis.

The results contained in the papers belong to functional analysis and general topology. They concern topological aspects of Banach spaces and functional-analytic aspects of topological spaces, especially compact ones. In fact, for sake of consistency, the papers chosen for the thesis are devoted to a more narrow area – to Valdivia compact spaces and their role in Banach space theory.

I am grateful to my colleagues for asking interesting questions, for fruitful discussions etc. Thanks are due to Luděk Zajíček who informed me about a paper of M.Valdivia and in this way inspired me to investigate Valdivia compact spaces. Further thanks are due to Marián Fabian and Václav Zizler for encouraging me to study Banach space theory. For interesting and fruitful discussions I am grateful to Bernardo Cascales, Petr Holický, José Orihuela, Matias Raja, Jan Rychtář and many others.

I also wish to thank my wife Jana for her understanding and patience when I was doing mathematics.

Finally I give thanks to God for it's He who gave me the life and the ability to enjoy mathematics.

CHAPTER 1

Introduction

The aim of this introductory chapter is to give a brief survey of a wider area including recent results and open problems.

Banach spaces

A *Banach space* is a (real or complex) normed linear space which is complete in the metric generated by the norm. Simple examples are finite-dimensional spaces (i.e. $\mathbb{R}^n$ or $\mathbb{C}^n$) equipped with the euclidean norm. Classical examples of infinite-dimensional Banach spaces include sequence spaces ℓ_p (for $p \in [1, \infty]$), the space c_0 of sequences converging to 0, Lebesgue function spaces $L_p[0, 1]$ (for $p \in [1, \infty]$) or the space $C[0, 1]$ of continuous functions on the interval $[0, 1]$.

Banach spaces are used for example as a framework for differential calculus and solving differential equations. Their strength is in the abstraction, in the possibility to consider functions as points of a space. That's why it is important and interesting to study the structure of the class of Banach spaces itself.

The general theory of Banach spaces began by the three basic theorems of functional analysis – Hahn-Banach extension theorem [**44**, Theorem 31], Banach open mapping theorem [**44**, Theorem 83] and uniform boundedness principle [**44**, Theorem 60]. Later the theory developped in many directions which we are not able to name all. A part of their variety is mentioned and discussed below. The elements of the theory are explained in several books by Diestel [**25**], Dunford and Schwartz [**26**], Semadeni [**91**] and others, for references we use namely nice lecture notes by Habala, Hájek and Zizler [**44**].

Let us recall few basic properties of Banach spaces and some notation. A linear subspace of a Banach space is again a Banach space if and only if it is closed [**44**, Fact 2]. Therefore, by subspace of a Banach space we mean always a closed linear subspace. The linear quotient by a closed linear subspace of a Banach space is again a Banach space when equipped with a natural norm [**44**, Definition 12 and Theorem 13]. The space of all continuous linear operators from a Banach space X to a Banach space Y form a linear space. If we equip this space with the operator norm defined by

$$\|L\| = \sup\{\|Lx\| \colon \|x\| \leq 1, x \in X\},$$

we obtain a Banach space which is denoted by $L(X, Y)$ [**44**, page 9]. The space $L(X, \mathbb{R})$ if X is real (or $L(X, \mathbb{C})$ if X is complex) is called *dual space* to X and is denoted by X^*.

Banach spaces admit several structures. We can consider them as linear spaces, normed spaces, metric spaces or topological spaces. Banach spaces X and Y are called *isometric* if there is $T \in L(X, Y)$ which is onto and preserves the norm. In this case X and Y are indistinguishable as Banach spaces. The spaces X and Y are called *isomorphic* if there is $T \in L(X, Y)$ which is one-to-one and onto. Then, by the open mapping theorem, necessarily T^{-1} is continuous. Large number of properties of Banach spaces (including great part of the properties discussed below) are preserved by isomorphisms.

On a Banach space there are several natural topologies. The first one is the topology generated by the metric induced by the norm, called *norm topology* or *strong topology*. The second important topology is the *weak* one. It is the weakest topology having the same continuous linear functionals as the norm topology. Dual Banach spaces admit a further natural topology – the *weak** one. If X is a Banach space and X^* its dual, the weak* topology on X^* is the weakest one in which all evaluation maps $x^* \mapsto x^*(x)$ $(x \in X)$ are continuous. These two topologies are special cases of general weak topologies [**44**, Definition 205]. Further examples of such topologies on Banach spaces will be mentioned and used later.

Compact spaces

A topological space is *compact* if any cover of the space by open sets admits a finite subcover. A classical theorem of Borel says that the interval $[0, 1]$ is compact [**30**, page 107]. More generally, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded [**30**, Theorem 3.2.7]. Investigation of compact spaces is a large part of general topology. It is easy to check that compact spaces are stable to taking closed subsets and continuous images [**30**, Theorems 3.1.2 and 3.1.8]. The deep Tychonoff theorem [**30**, Theorem 3.2.4] says that an arbitrary product of compact spaces is compact. Hence, in particular, Tychonoff cubes $[0, 1]^I$ and Cantor cubes $\{0, 1\}^I$ are compact for arbitrarily large set I. Another canonical examples of compact spaces are ordinal intervals – the interval $[0, \alpha]$ is compact when equipped with the order topology for any ordinal α.

By a topological space we will mean a Hausdorff topological space, unless the converse is explicitely stated. Hausdorff compact spaces are automatically normal [**30**, Theorem 3.1.7], in particular completely regular. Further, any completely regular space admits a compactification, i.e. it can be homeomorphically embedded as a dense subset to a compact space [**30**, Theorem 3.4.1]. These compactifications, their topological, combinatorial and measure-theoretic structure are widely studied by topologiests. Let us mention only one consequence of this theory. Any compact space can be homeomorphically embedded into the Tychonoff cube $[0, 1]^I$ for a sufficiently large set I [**30**, Theorem 3.2.5].

Compact spaces appear naturally in functional analysis. For example, a Banach space X is finite-dimensional if and only if its closed unit ball

$$B_X = \{x \in X : \|x\| \leq 1\}$$

is compact (in the norm topology) [**44**, Theorem 16]. Further, the study of weak compact sets (i.e. subsets of a Banach space which are compact in the weak topology) led to many deep results on the structure of Banach spaces. Some of these results are discussed below.

Duality of Banach spaces and compact spaces

To each compact Hausdorff space K we can associate the space $C(K)$ of real continous functions on K equipped with the max-norm, i.e.

$$\|f\| = \max\{|f(k)|\colon k \in K\}.$$

This is a real Banach space. If we consider complex continuous functions, we obtain a complex Banach space which we may denote by $C(K, \mathbb{C})$. This space is usually denoted also by $C(K)$ and it should be clear from the context whether we consider real or complex spaces. On a space of the form $C(K)$ there is one more natural topology, in addition to the norm and weak ones. This is the topology τ_p of pointwise convergence, i.e. the weakest topology making the evaluation map $f \mapsto f(k)$ continuous for each $k \in K$. It is usual to write $C_p(K)$ to denote $(C(K), \tau_p)$.

Conversely, to each Banach space we can associate a compact space. If X is a Banach space, then the dual unit ball (i.e. the unit ball of the dual space X^*) B_{X^*} is compact when equipped with the weak* topology. This is the assertion of Alaoglu theorem [**44**, Theorem 61] which is a consequence of Tychonoff theorem.

Hence, we really have a kind of duality in view of the following embeddings.

If K is a compact space, $C(K)$ is a Banach space and $(B_{C(K)^*}, w^*)$ is again compact. Further, there is a natural embedding of K into $(B_{C(K)^*}, w^*)$. To a point $k \in K$ we assign the evaluation functional $\delta_k\colon f \mapsto f(k)$. It is clearly a linear functional of norm one. Moreover, the embedding $k \mapsto \delta_k$ is homeomorphic, hence we can consider K as a topological subspace of $(B_{C(K)^*}, w^*)$. By Riesz theorem [**26**, Theorem IV.6.3] we can identify $C(K)^*$ with the space of finite signed Radon measures on K. The norm of an element of $C(K)^*$ is then equal to the total variation of the representing measure. In this identification δ_k is represented by the Dirac measure supported at k.

If X is a Banach space and we denote by K the compact space (B_{X^*}, w^*), there is a natural embedding of X into $C(K)$. To any $x \in X$ we assign the function $h_x \in C(K)$ defined by $h_x(k) = k(x)$, $k \in K$. The embedding $x \mapsto h_x$ is clearly linear. By a consequence of Hahn-Banach theorem [**44**, Corolary 36] it is isometric when we consider X with its given norm and $C(K)$ with the max-norm. This mapping is, moreover, a homeomorphism from the weak topology of X into the topology of pointwise convergence on $C(K)$. Hence, identifying x and h_x we may consider X as a subspace of $C(K)$. In this identification the norm of X is the restriction of the max-norm on $C(K)$ and X is a norm-closed subset of $C(K)$. The space X equipped with the weak topology is then a topological subspace of $C_p(K)$. It is quite important that X is closed also in $C_p(K)$. This is a consequence of Banach-Dieudonné theorem [**44**, Theorem 222].

A large part of the investigation of Banach spaces is devoted to study the described duality, namely the questions of the following kind. Which topological properties of

K assure a given property of $C(K)$ and conversely? Which topological properties of (B_{X^*}, w^*) imply a given property of X and conversely?

Structure of Banach spaces and applications

The strength of Banach spaces, considered from the point of view of applications, is the possibility to consider complicated objects (sequences, functions, operators) as points of a space with some structure. This is useful, for example, in the study of differential equations (mainly partial ones) or in the search of good or best approximations. As it is usual in mathematics, the notions introduced as an auxiliary tools became interesting in themselves. This gave born to several branches of modern Banach space theory.

The questions from partial differential equations inspired, among others, developement of the operator theory. This theory contains the study of linear operators, continuous (bounded) and discontinuous (unbounded), their spectral analysis, functional caluculus etc. (see e. g. [**19, 69, 1**]). By a further abstraction it also led to theories of Banach algebras, C^* algebras, operator spaces etc. (see e. g. [**21, 80, 18, 28**]).

The questions from approximation theory inspired for example the study of geometrical and topological properties of Banach spaces and differentiability of functions on Banach spaces.

Geometry of Banach space studies namely the geometrical properties of the unit ball – rotundity, flatness, corners etc. – and its implications (see e. g. [**24, 13**]). To this end many natural kinds of norms were introduced and studied. For example, the norm is called rotund if the unit sphere contains no line segments. Also, several uniform versions of rotundity (uniformly rotund, locally uniformly rotund, weakly locally uniformly rotund etc.) are widely studied.

Another part of the area is investigation of differentiability of functions on Banach spaces. Mainly real convex or Lipschitz functions are considered. Spaces on which any real convex function is differentiable at many points are studied (see e. g. [**81, 31**]). This makes arise a natural hierarchy of classes of Banach spaces which we describe below in more detail ("differentiability hierarchy").

In both cases some properties of the norm play an important role. However, many properties do not depend on the particular norm. Hence, there is a large area of renorming – finding conditions on a space in order it admits an equivalent norm with some better properties than the original one. Many results from this area are surveyed in the monograph [**23**].

It turns out that geometrical properties of the space are related to topological properties of the weak topology. This can be seen from large amount of results on "descriptive hierarchy" of Banach spaces. We describe this hierarchy in more detail below.

In the following two sections we describe the two mentioned hierarchies of Banach spaces and some relations between them.

Descriptive hierarchy of Banach spaces

In this section we give a brief survey of a hierarchy of Banach spaces which can be described using topological properties of the weak topology. We will concetrate on characterizations of the classes and their inclusions. Some of the applications will be mentioned in the next section, in relation with the differentiability hierarchy.

Finite-dimensional spaces. For the sake of completeness we begin with finite-dimensional spaces although from our point of view it is the trivial case. There are many characterizations of finite-dimensional spaces. Let us name few of them.

THEOREM 1. *Let X be a Banach space. The following assertions are equivalent.*

(1) $\dim X < \infty$.
(2) *The closed unit ball B_X is norm-compact.*
(3) *The norm and weak topologies on X coincide.*
(4) *The weak topology on X is metrizable.*
(5) *Any linear functional defined on X is continuous.*
(6) *Any linear operator from X into any Banach space is continuous.*
(7) $\dim X^* < \infty$.

All these conditions are now considered as folklore and form part of basic courses of functional analysis. Let us remark that the fourth condition is a topological property of the weak topology. Hence the class of finite-dimensional spaces is a natural first class of the hierarchy. Next we give theorems on duality with compact spaces.

THEOREM 2. *Let K be a compact space and n be a positive integer. Then $\dim C(K) = n$ if and only if K has exactly n points.*

THEOREM 3.

- *Let X be a real Banach space and n a non-negative integer. Then the following assertions are equivalent.*
  (1) $\dim X = n$.
  (2) *(B_{X^*}, w^*) is a metrizable compact of dimension n.*
  (3) *(B_{X^*}, w^*) has dimension n.*
- *Let X be a complex Banach space and n a non-negative integer. Then the following assertions are equivalent.*
  (1) $\dim X = n$.
  (2) *(B_{X^*}, w^*) is a metrizable compact of dimension $2n$.*
  (3) *(B_{X^*}, w^*) has dimension $2n$.*

By the dimension of a topological space in the previous theorem we mean one of the three standard dimensions (small inductive, great inductive, covering – see [**30**, Chapter 7, §1]). The theorem is valid with any choice. It is an easy consequence of several facts – that all the three dimensions coincide for separable metrizable spaces [**30**, Theorem 7.3.2], that the dimension of the space $[0, 1]^n$ is equal to n [**30**, Corollary 1 to Theorem 7.3.13], and that the dimension of a closed subspace of a compact space is at most equal to the dimension of the space [**30**, Theorems 7.1.1, 7.1.3 and 7.1.7].

Separable spaces. A Banach space is *separable* if it is separable as a metric space equipped with the metric induced by the norm; i.e. if it admits a countable dense subset. The class of separable spaces contains all finite-dimensional spaces. It also contains great number of classical infinite-dimensional sequence and function spaces – for example the spaces $C[0,1]$, c_0, ℓ_p and $L_p[0,1]$ for $p \in [1,\infty)$, while spaces ℓ_∞ and $L_\infty[0,1]$ are not separable (see [**44**, Propositions 18 and 19]). Some characterizations of separable spaces are contained in the following theorem.

Theorem 4. *Let X be a Banach space. The following assertions are equivalent.*

(1) *X is separable.*
(2) *$(X, \|\cdot\|)$ has a countable basis.*
(3) *(X, w) has a countable network.*
(4) *(X, w) is separable.*

Recall that a *network* of a topological space X is a family $\mathcal{N}$ of subsets of X such that each open subset of X is the union of a subfamily of $\mathcal{N}$. I.e., it is a family which has all the properties of a basis except it need not consist of open set.

Notice that the last two conditions are topological properties of the weak topology. Except for application of elementary topological results the proof of this theorem requires Mazur theorem saying that norm-closed convex subsets of a Banach space are also weakly closed [**44**, Theorem 56] which is a consequence of Hahn-Banach theorem.

The dual class of compact spaces is that of metrizable compacta. It is witnessed by the following results (see [**44**, Proposition 62, Exercises 3.47 and 3.48]). Their proof requires Stone-Weierstrass theorem [**30**, Theorem 3.2.12].

Theorem 5.

- *A Banach space X is separable if and only if (B_{X^*}, w^*) is metrizable.*
- *A compact space K is metrizable if and only if $C(K)$ is separable.*

Let us name moreover the following embedding characterization.

Theorem 6.

- *A Banach space X is separable if and only if there is a linear operator $T\colon X^* \to c_0$ which is one-to-one and continuous from the weak* topology to the topology of pointwise convergence.*
- *A compact K is metrizable if and only if it is homeomorphic to a subset of $[0,1]^{\mathbb{N}}$.*

The first assertion is an easy consequence of the definition of separability, Hahn-Banach theorem and [**44**, Theorem 55]. Remark that the operator T can be chosen to be moreover continuous (norm to norm) and weak* to weak continuous. This easily follows from the observation that for any separable space X there is a continuous linear operator from ℓ_2 to X with dense range. The second assertion follows from [**30**, Theorem 3.2.5].

To explain its meaning recall that any compact space is homeomorphic to a subset of $[0,1]^I$ for a set I. Hence the result claims that K is metrizable if and only if the set I can be chosen to be countable. Later we will see analogous characterizations of more general classes.

Reflexive spaces. If X is a Banach space, X^* denotes the dual space and X^{**} the second dual (i.e. the dual to X^*). There is a natural embedding $\kappa\colon X \to X^{**}$ defined by the formula

$$\kappa(x)(f) = f(x), \quad f \in X^*, x \in X.$$

It is an isometric embedding which is moreover homeomorphism from the weak topology to the weak* one. A space X is called *reflexive* if $\kappa(X) = X^{**}$.

Finite-dimensional spaces are reflexive. Spaces ℓ_p or $L_p[0,1]$ for $p \in (1,\infty)$ are examples of infinite-dimensional separable reflexive spaces while spaces ℓ_1, c_0, $L_1[0,1]$ or $C[0,1]$ are separable non-reflexive spaces (see [**44**, page 46]). There are also non-separable reflexive spaces. As an example we can use a non-separable Hilbert space (i.e., a Hilbert space with uncountable orthornormal basis).

The following theorem contains some characterizations of reflexive spaces. The proof uses, in addition to elementary observations, Hahn-Banach theorem, Alaoglu theorem, Baire category theorem and Goldstine theorem [**44**, Theorem 64]. Notice that the last condition is a topological property of the weak topology.

THEOREM 7. *Let X be a Banach space. The following assertions are equivalent.*

(1) *X is reflexive.*
(2) *X^* is reflexive.*
(3) *The weak and weak* topologies on X^* coincide.*
(4) *(B_X, w) is compact.*
(5) *(X, w) is K_σ (i.e. a countable union of compact subsets).*

Spaces of the form $C(K)$ are reflexive if and only if they are finite-dimensional. I.e., we have the following elementary theorem.

THEOREM 8. *Let X be a compact space. Then $C(K)$ is reflexive if and only if K is finite.*

There is no characterization of reflexive space using a topological property of the dual unit ball equipped with the weak* topology. The reason is the following. If X is any infinite-dimensional separable Banach space, by Keller's theorem [**108**, Theorem 8.2.4] the dual unit ball (B_{X^*}, w^*) is homeomorphic to $[0,1]^{\mathbb{N}}$. Hence, for example ℓ_2 is reflexive and ℓ_1 is not reflexive while their dual unit balls are weak* homeomorphic.

Weakly compactly generated spaces. A Banach space X is called *weakly compactly generated* (or shortly *WCG*) if there is a subset $K \subset X$ compact in the weak topology such that $\overline{\operatorname{span} K} = X$. The class of WCG spaces contains both separable and reflexive spaces (see [**44**, page 217], where one can find also the examples named below).

Indeed, a space X is separable if and only if there is a norm-compact set K such that $\overline{\operatorname{span} K} = X$. If X is reflexive, then B_X is weakly compact and $\operatorname{span} B_X = X$. In fact, a converse holds true. If there is a weakly compact $K \subset X$ such that $\operatorname{span} K = X$, then X is reflexive.

As an example of a WCG space which is neither separable nor reflexive we can name the space $c_0(\Gamma)$ for an uncountable set Γ. This space is defined as the linear space

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma \colon (\forall \varepsilon > 0)(\{\gamma \colon |x(\gamma)| > \varepsilon\} \text{ is finite})\}$$

equipped with the supremum norm

$$\|x\| = \sup\{|x(\gamma)| \colon \gamma \in \Gamma\}.$$

It is clear that the supremum is finite and that it is attained. Spaces ℓ_∞ and $L_\infty[0,1]$ are not WCG. Further WCG spaces can be found among general L_1 spaces. The space $L_1(\mu)$ (where μ is a σ-additive measure on a measure space) is WCG if and only if μ is σ-finite.

We continue with some elementary characterizations of WCG spaces. The last condition is again a topological property of the weak topology.

Theorem 9. *Let X be a Banach space. The following assertions are equivalent.*

1. *X is WCG.*
2. *There is a weakly K_σ subset $K \subset X$ with $\overline{\operatorname{span} K} = X$.*
3. *There is a weakly K_σ subset $K \subset X$ which is norm-dense in X.*
4. *The space (X, w) has a dense K_σ subset.*

The class of compact spaces dual to the class of WCG spaces is that of Eberlein compact spaces. A compact space K is *Eberlein* if it is homeomorphic to a subset of (X, w) for some Banach space X. We have the following duality.

Theorem 10.

- *A compact K is Eberlein if and only if $C(K)$ is WCG.*
- *If a Banach space X is WCG, then (B_{X^*}, w^*) is Eberlein.*

The proof can be done using Stone-Weierstrass theorem and the fact that any norm-bounded τ_p-compact subset of $C(K)$ is even weakly compact (see e.g. [**36**, Theorem 12.1]).

The converse to the second statement is not true. We will discuss it in the next paragraph. Deeper results on WCG spaces and Eberlein compacta follow from the fundamental paper by Amir and Lindenstrauss [**3**]. They constructed a *projectional resolution of the identity* in any WCG space. This is a powerfull tool to study non-separable Banach spaces. We will describe these notions and results in Chapter 4. Now we are going to name embedding characterizations of WCG spaces and Eberlein compacta which are consequences of these results.

Theorem 11.

- *For a compact space K the following assertions are equivalent.*

(1) *K is Eberlein.*
(2) *K is homeomorphic to a subset of $(c_0(\Gamma), w)$ for a set Γ.*
(3) *K is homeomorphic to a subset of $(c_0(\Gamma), \tau_p)$ for a set Γ (where τ_p is the topology of pointwise convergence).*

- *For a Banach space X the following assertions are equivalent.*
  (1) *X is WCG.*
  (2) *There is, for a set Γ, a one-to-one continuous linear operator $T: X^* \to c_0(\Gamma)$ which is also weak* to weak continuous.*
  (3) *There is, for a set Γ, a one-to-one linear operator $T: X^* \to c_0(\Gamma)$ which is weak* to τ_p continuous.*

Subspaces of WCG spaces. WCG spaces are stable to quotients but not to subspaces. First example was constructed by Rosenthal [**90**] who showed that for a suitable probability measure μ the WCG space $L_1(\mu)$ has a subspace which is not WCG. Another example is due to Argyros [**4**] (see [**31**, Section 1.6]) who found an Eberlein compact K such that $C(K)$ has a subspace which is not WCG. These examples inspire the following definition.

A Banach space X is called *subspace of WCG* if it is isomorphic (or, equivalently, isometric) to a subspace of a WCG space. Subspaces of WCG share many properties of WCG spaces. However, we do not know any characterization of subspaces of WCG in terms of the weak topology. This leads us to formulate the following problem.

Problem 1. *Let X and Y be Banach spaces such that (X, w) and (Y, w) are homeomorphic. Is Y subspace of WCG whenever X has this property?*

On the other hand, duality for subspaces of WCG is complete. This is due to the result of Benyamini, Rudin and Wage [**14**] that Eberlein compacta are stable to continuous images. We have the following.

Theorem 12.

- *A Banach space X is subspace of WCG if and only if (B_{X^*}, w^*) is Eberlein.*
- *A compact space K is Eberlein if and only if $C(K)$ is subspace of WCG.*

Notice that a space of the form $C(K)$ is WCG whenever it is subspace of WCG. There is also a characterization of subspaces of WCG in terms of the embedding of the dual. It is a recent result of [**37**].

Theorem 13. *A Banach space X is subspace of WCG if and only if there is a one-to-one linear weak* continuous mapping $T: X^* \to \mathbb{R}^\Gamma$ for a set Γ satisfying the following condition:*

> *For every $\varepsilon > 0$ there are sets $\Gamma^\varepsilon_m \subset \Gamma$, $m \in \mathbb{N}$, with $\bigcup_{m=1}^\infty \Gamma^\varepsilon_m = \Gamma$, such that for every $x^* \in B_{X^*}$ and for every $m \in \mathbb{N}$ the set $\{\gamma \in \Gamma^\varepsilon_m : |T(x^*)(\gamma)| > \varepsilon\}$ is finite.*

However, this characterization is not completely analogous to the similar characterizations of separable or WCG spaces (or the spaces investigated in the following paragraph).

Weakly K-analytic and weakly countably determined spaces. In this paragraph we consider two classes of Banach spaces explicitely defined by a topological property of the weak topology. Let us first give definitions of these topological properties.

Let S and T be topological spaces and φ a set-valued mapping of S into T (i.e., the domain of φ is S and the values are subsets of T). The mapping φ is called *upper semi-continuous* (or, shortly, *usc*) if $\{s \in S \colon \varphi(s) \subset U\}$ is open in S whenever U is open in T. The mapping is called *usc-K* if it is usc and all the values are compact. The mapping φ is caled *onto* if for any $t \in T$ there is $s \in S$ with $t \in \varphi(s)$.

A topological space T is called *K-analytic* (*K-countably determined*) if there is a separable complete metric (separable metric, respectively) space S and a usc-K mapping of S onto T.

Investigation of K-analytic and K-countably determined topological spaces form a part of descriptive topology (see e.g. [**86, 92, 15, 85**]). Let us mention only few properties. Both classes are stable to taking closed subsets, countable products and continuous images. Any K-countably determined space is Lindelöf.

A Banach space X is called *weakly K-analytic* if (X, w) is K-analytic. The space X is called *weakly countably determined* (or, shortly *WCD*) if (X, w) is K-countably determined.

The following result is quite important for understanding the structure of these classes.

Theorem 14. *Let X be a Banach space and $A \subset X$ be such that $\overline{\operatorname{span} A} = X$.*

- *If (A, w) is K-analytic then X is weakly K-analytic.*
- *If (A, w) is K-countably determined then X is WCD.*

This theorem is a special case of an abstract result (see e.g. [**5**, Theorem 2.2.3.19]). The analogous statement holds for any class of topological spaces having enough stability properties. As an immediate consequence we get that WCG spaces (and hence also subspaces of WCG) are weakly K-analytic. This was shown by Talagrand [**96**]. Later Talagrand [**98**], [**99**] constructed two compact spaces K_1 and K_2 such that $C(K_1)$ is weakly K-analytic but not subspace of WCG and $C(K_2)$ is WCD but not weakly K-analytic.

Dual classes of compact spaces can be defined as follows. A compact space K is called *Talagrand* (*Gul'ko*) if the space $C_p(K)$ is K-analytic (K-countably determined, respectively). In this setting we have the following complete duality (see [**31**, Theorems 7.1.8 and 7.1.9] for the case of WCD spaces and Gul'ko compacta, the remaining case is completely analogous).

Theorem 15.

- *A Banach space X is weakly K-analytic if and only if (B_{X^*}, w^*) is Talagrand.*
- *A compact space K is Talagrand if and only if $C(K)$ is weakly K-analytic.*
- *A Banach space X is WCD if and only if (B_{X^*}, w^*) is Gul'ko.*
- *A compact space K is Gul'ko if and only if $C(K)$ is WCD.*

Deeper results on WCD spaces and Gul'ko compacta are due to Vašák [**109**] and Gul'ko [**43**]. They independently generalized the above mentioned Amir-Lindenstrauss theorem for WCD spaces. (Notice that WCD spaces are called *Vašák spaces* by some authors.) One of the corollaries are embedding characterizations which we are going to describe.

If G is a topological space with at most one non-isolated point, we denote by $C_0(G)$ the following subspace of $\mathbb{R}^G$.

$$C_0(G) = \{f \in \mathbb{R}^G \colon f \text{ is continuous} \\ \text{and } f(x) = 0 \text{ for each non-isolated } x \in G\}.$$

Then we have the following.

THEOREM 16.

- *A Banach space X is weakly K-analytic if and only if there is a K-analytic topological space G with at most one non-isolated point and a continuous one-to-one linear operator $T\colon (X^*, w^*) \to C_0(G)$.*
- *A compact space K is Talagrand if and only if it is homeomorphic to a subset of $C_0(G)$ for a K-analytic topological space G with at most one non-isolated point.*
- *A Banach space X is WCD if and only if there is a K-countably determined topological space G with at most one non-isolated point and a continuous one-to-one linear operator $T\colon (X^*, w^*) \to C_0(G)$.*
- *A compact space K is Gul'ko if and only if it is homeomorphic to a subset of $C_0(G)$ for a K-countably determined topological space G with at most one non-isolated point.*

An equivalent characterization was first obtained by Mercourakis [**72**]. Instead of spaces $C_0(G)$ he used other natural spaces.

Weakly Lindelöf determined spaces. Investigation of this class of Banach spaces was preceded by the study of the dual class of compact spaces. It is the class of *Corson* compact spaces. A compact space K is called Corson if it is homeomorphic to a subset of

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma \colon \{\gamma \in \Gamma \colon x(\gamma) \neq 0\} \text{ is countable}\}$$

for a set Γ. Notice that we consider $\Sigma(\Gamma)$ with the topology inherited from $\mathbb{R}^\Gamma$, i.e. the topology of pointwise convergence on Γ. It should be also mentioned that the space $\Sigma(\Gamma)$ can be expressed in the form $C_0(G)$ in terms of the previous paragraph. Indeed, denote by L_Γ the set $\Gamma \cup \{\infty\}$ (where $\infty \notin \Gamma$) and let any point of Γ be isolated while neighborhoods of ∞ are $L_\Gamma \setminus C$ for $C \subset \Gamma$ countable. Then L_Γ is a topological space with at most one non-isolated point and $\Sigma(\Gamma)$ can be naturally identified with $C_0(L_\Gamma)$.

Notice that the space L_Γ is Lindelöf. Moreover, its topology is the finest one among Lindelöf topologies on the set $\Gamma \cup \{\infty\}$ with all the points of Γ isolated. Therefore, a compact space K is Corson if and only if it is homeomorphic to a subset of $C_0(G)$ for a Lindelöf space G with at most one non-isolated point. Hence, using Theorem 16 we can see that any Gul'ko compact space is Corson.

Properties of the spaces $\Sigma(\Gamma)$ and more general spaces (called *Σ-products*) and their subspaces were studied already by Corson [**20**] and Glicksberg [**39**].

A Banach space is called *weakly Lindelöf determined* (or, shortly *WLD*) if there is a continuous one-to-one linear operator $T\colon (X^*, w^*) \to \Sigma(\Gamma)$ for a set Γ. These spaces were probably first studied by Valdivia [**104**] using another (equivalent) definition – he did not require T be linear. The given definition and name is due to Argyros and Mercourakis [**7**].

Using again Theorem 16 it follows that any WCD space is WLD.

To formulate the duality of WLD spaces and Corson compacta we need to recall one more notion. A compact space K is said to have *property (M)* if any Radon probability measure on K has separable support. Then we have:

Theorem 17.

- *A Banach space X is WLD if and only if (B_{X^*}, w^*) is Corson.*
- *Let K be a compact space. Then $C(K)$ is WLD if and only if K is a Corson compact with property (M). In this case $(B_{C(K)^*}, w^*)$ has property (M), too.*

The first part was first proved by Orihuela, Schachermayer and Valdivia [**79**, Proposition 4.1], the second part is due to Argyros, Mercourakis and Negrepontis [**8**, Theorem 3.5]. Remark that any Gul'ko compact has property (M). This can be proved in several ways – for example it follows from the above theorem and Theorem 15. The answer to the natural question whether the assumptions on property (M) can be dropped depends on the axioms of the set theory. These results are summed up in the following theorem.

Theorem 18.

- *Under Martin's axiom and negation of continuum hypothesis any Corson compact space has property (M).*
- *Under continuum hypothesis there is a Corson compact space without property (M).*
- *Under continuum hypothesis there is a WLD space X such that (B_{X^*}, w^*) does not have property (M).*

The first assertion follows from the facts that any support of a Radon probability measure satisfies the countable chain condition (ccc) and that under Martin's Axiom and the negation of the continuum hypothesis any Corson compact space satisfying ccc is metrizable (see e.g. [**17**, page 205]). First example confirming the second assertion is due to Kunen, Haydon and Talagrand [**77**, Theorem 5.9], another one is in [**8**, Theorem 3.12]. As for the third assertion, the history is more interesting. Section 3 of the paper [**8**] contains the claim that any convex compact subset of $\Sigma(\Gamma)$ has property (M). However, the proof of this claim contains a gap, as observed by the author who then asked whether the claim is true at all. This question was answered by Plebanek [**82**] who constructed an example confirming the third assertion.

First examples of Corson compact spaces with property (M) which are not Gul'ko (and hence examples of WLD spaces which are not WCD) are due to Alster and Pol [**2**] and to Leiderman and Sokolov [**71**].

WLD spaces can be also characterized by a topological property of the weak topology. A topological space T is called *primarily Lindelöf* if it is a continuous image of a closed subspace of $(L_\Gamma)^\mathbb{N}$ for a set Γ. Then the following holds.

THEOREM 19.

- *A compact space K is Corson if and only if $C_p(K)$ is primarily Lindelöf.*
- *A Banach space X is WLD if and only if (X, w) is primarily Lindelöf.*

The first assertion was proved by Pol [**83**] using deep results of Gul'ko. The proof is reproduced in [**6**, Section IV.3]. The second assertion is a consequence of the first one explicitely observed in [**7**, Proposition 1.2].

Valdivia type Banach spaces. A compact space K is Valdivia if, for a set Γ, it is homeomorphic to a subset $K' \subset \mathbb{R}^\Gamma$ such that $K' \cap \Sigma(\Gamma)$ is dense in K'. This class of compact spaces and related Banach spaces were first studied by Argyros, Mercourakis and Negrepontis [**8**], later by Valdivia [**105**], [**106**]. Although, as witnessed by the results in the just named papers, many properties of previous classes are shared by this one, its structure is quite different. This structure was investigated by the author and related results form main part of this thesis. Detailed exposition on this class is given in the paper [**59**] by the author.

It is clear from the definition that any Corson compact space is Valdivia. Canonical examples of Valdivia non-Corson compacta are the ordinal interval $[0, \omega_1]$ and Tychonoff cube $[0, 1]^\Gamma$ for Γ uncountable.

A key auxiliary notion is that of Σ-subset. A set $A \subset K$ is called a Σ-*subset* of K if there is, for a set Γ, a homeomorphic embedding $h : K \to \mathbb{R}^\Gamma$ with $A = h^{-1}(\Sigma(\Gamma))$. Hence K is Valdivia if and only if it has a dense Σ-subset.

There are several sets of results and problems concerning Valdivia compact spaces. Let us name some of them, main ones from the point of view of the author.

- *Topological properties of Valdivia compacta.* This area contains namely questions on stability and non-stability to topological operations (subsets, products, continuous images etc.) and characterizations of Corson compacta among Valdivia ones (and among continuous images of Valdivia compacta). Such problems are discussed in more detail in Chapter 2 and in the paper [**57**] contained in Chapter 3.
- *Study of dual classes of Banach spaces and projectional resolutions of the identity.* It is natural to define the dual class (called the class of 1-*Plichko spaces*) as those Banach spaces whose dual unit ball (equipped with the weak* topology) has a convex symmetric dense Σ-subset. It turns out that this class is not identical with the class of Banach spaces whose dual unit ball is a Valdivia compact. Neither of these classes is stable to isomorphisms or subspaces (spaces isomorphic

to 1-Plichko spaces are called *Plichko*). Further, any 1-Plichko space admits a projectional resolution of the identity and WLD spaces can be characterized in terms of projectional resolutions. These and related results are discussed in Chapter 4.
- *Exact form of the duality.* If K is Valdivia then $C(K)$ is 1-Plichko. It is an open problem whether the converse is true in general. A partial answer is given in the paper [**57**] contained in Chapter 3.
- *Relationships with other classes of Banach spaces.* Valdivia property of the unit ball of the second dual is related to the Asplund property of the space. This area is investigated in [**60**].
- *Study of the class of continuous images of Valdivia compacta.* This class has a structure somewhat similar and somewhat different from the structure of the class of Valdivia compacta. It has a stable subclass of *weakly Corson* compacta. It is unknown which stability properties do enjoy dual classes of Banach spaces. These classes are studied for example in [**55, 64, 65**].

Classes defined by renorming. One of the consequences of structure theorems on mentioned classes of Banach spaces is that any such space admit a locally uniformly rotund equivalent norm. Recall that a norm $\|\cdot\|$ on a Banach space X is called *locally uniformly rotund* (or, shortly, *LUR*) if $\|x_n - x\| \to 0$ whenever $x, x_1, x_2, \dots \in S_X$ are such that $\|x_n + x\| \to 2$. It is well known and easy to see that, if the norm on X is LUR, then the norm and weak topologies coincide on the unit sphere, i.e. the norm is *Kadec*.

LUR and Kadec renormable spaces were thoroughly studied. The first result is due to Kadec [**51**] who proved that separable spaces admit an equivalent LUR norm. This was later extended by Troyanski [**102**] to WCG spaces. Further extensions (up to 1-Plichko spaces) were enabled by results on projectional resolutions of the identity (mentioned in the previous paragraphs) and a result of Zizler [**110**] on LUR renorming using projectional resolutions. A characterization of LUR renormable spaces using martingales was given by Troyanski [**103**]. His results were essentially simplified by Raja [**87**].

The class of LUR renormable spaces is much larger than the previous classes (cf. e.g. [**23**, Chapter VII, Sections 4–7]), there is also a Kadec renormable space which is not LUR renormable [**23**, Theorem VII.6.8].

Kadec renorming has also topological implications. If X admits an equivalent Kadec norm, the Borel σ-algebras of the norm and weak topologies coincide. Further, X is a Borel set when canonically embedded into (X^{**}, w^*). Both results are due to Edgar [**27**]

That not all spaces have these property can be illustrated by the space ℓ_∞ which is not Borel when embedded into the bidual and there is a norm-closed subset of ℓ_∞ which is not weakly Borel. This was shown by Talagrand [**97**].

General descriptive classes. There is a large area of descriptive topology studying various generalizations of the classical Borel and analytic spaces. K-analytic spaces mentioned above are such a class. Another classes are Čech-analytic or scattered-K-analytic spaces. They can be defined using the classical Suslin operation. A completely regular space T is *Čech-analytic* (*scattered-K-analytic*) if it can be represented as a result of Suslin operation on Borel sets (on resolvable sets, respectively) in a compactification. For detailed description and properties of these classes we refer to [**46**], [**45**] and [**49**].

It follows from the above quoted result of [**27**] that if a Banach space X admits an equivalent Kadec norm, then (X, w) is Čech-analytic. Further, any Čech-analytic space is clearly scattered-K-analytic. Banach spaces which are weakly scattered-K-analytic are exactly those spaces whose weak topology is σ-fragmented by the norm [**48**, Theorem 6]. (In this paper scattered-K-analytic spaces are called almost-K-descriptive and another definition is used. The two definitions are equivalent by [**48**, Theorem 3].)

Also some other intermediate classes were defined and studied (see [**45**] or [**49**]). We will not discuss them in detail as it is not clear whether these classes, different within topological spaces, coincide in the framework of Banach spaces. Let us mention the main open problem in this area.

PROBLEM 2. *Is there a weakly scattered-K-analytic Banach space which does not admit equivalent Kadec norm?*

Differentiability hierarchy

Let X be a (real) Banach space, $a \in X$ and f a real function defined on a neighborhood of X. The directional derivative of f at a in the direction $h \in X$ is defined in the same way as for functions of several real variables, i.e.

$$\partial_h f(a) = \lim_{t \to 0} \frac{f(a+th) - f(a)}{t}$$

provided the limit exists. If the mapping $h \mapsto \partial_h f(a)$ is a continuous linear functional, the function f is said to be *Gâteaux differentiable* at a and the functional is called *Gâteaux derivative* of f at a. If the limit is, moreover, uniform for h from the unit sphere, f is said to be *Fréchet differentiable* at a and the respective functional is called *Fréchet derivative* of f at a.

Using these kinds of differentiability the following classes of Banach spaces were defined by Asplund [**10**] (using another names). A space X is *Asplund* (*weak Asplund*) if any real continuous convex function defined on a open convex set of X is Fréchet-differentiable (Gâteaux differentiable, respectively) at points of a dense G_δ subset of X.

While the set of points of Fréchet differentiability of a continuous convex function is always G_δ (and hence it does not matter whether we say 'dense' or 'dense G_δ' in the definition of Asplund spaces) for Gâteaux differentiability it is not the case. Hence it is natural to define one more notion. A space X is called *Gâteaux differentiability space* (or, shortly, *GDS*) if any real continuous convex function defined on a open convex set of X is Gâteaux differentiable at points of a dense subset of X.

It was a longstanding open problem whether any GDS is weak Asplund. This was recently answered in the negative by [**75**].

Asplund spaces. Asplund spaces admit many equivalent characterizations and enjoy nice stability properties. Many of such results are collected in the book by Phelps [**81**]. Asplund spaces are stable to subspaces and quotients and have so-called three-space property (i.e. X is Asplund provided there is a subspace $Y \subset X$ such that both Y and X/Y are Asplund). Some of the characterizations are named in the following theorem.

Theorem 20. *Let X be a Banach space. The following assertions are equivalent.*

(1) *X is Asplund.*
(2) *Y^* is separable for any $Y \subset X$ separable.*
(3) *For any nonempty bounded set $M \subset X^*$ and any $\varepsilon > 0$ there is $x \in X$ and $c \in \mathbb{R}$ such that $\{\xi \in M : \xi(x) > c\}$ is nonempty and has diameter less than ε.*
(4) *(B_{X^*}, w^*) is fragmented by the norm metric.*
(5) *X^* has the Radon-Nikodým property.*
(6) *For any $A \subset X$ weakly separable we can write $A = \bigcup_{n\in\mathbb{N}} A_n$ where (A_n, w) is metrizable and closed in (A, w) for each $n \in \mathbb{N}$.*

A topological space (T, τ) is *fragmented* by a metric d on the set T if for each nonempty $A \subset T$ and each $\varepsilon > 0$ there is a nonempty relatively τ-open $U \subset A$ with d-diameter less than ε. A Banach space X has Radon-Nikodým property if any vector-valued measure defined on the σ-algebra of Lebesgue-measurable subsets of $[0, 1]$, which is absolutely continuous with respect to Lebesgue mesure, admits a Radon-Nikodým derivative. The equivalence of the first five conditions can be found in [**81**]. The last condition seems a bit technical but we mention it as it is a topological property of the weak topology. The equivalence $2 \Leftrightarrow 6$ follows easily from [**44**, Exercise 3.50].

As for examples, it follows easily from the equivalence $1 \Leftrightarrow 2$ that reflexive spaces are Asplund. Further, space c_0 or, more generally, $c_0(\Gamma)$ for any Γ, is Asplund. On the other hand, spaces ℓ_1, ℓ_∞, $L_1[0, 1]$, $L_\infty[0, 1]$ and $C[0, 1]$ are not Asplund.

The duality with some compact spaces is described in the following theorem [**44**, Theorem 296]

Theorem 21. *The space $C(K)$ is Asplund if and only if K is scattered.*

There is no characterization of Asplund spaces by a topological property of the dual unit ball. The space ℓ_2 is Asplund, the space ℓ_1 is not Asplund and their dual unit balls are weak* homeomorphic (cf. the comments after Theorem 8). However, such balls share some nice properties. This led to the following definition.

A compact space K is called *Radon-Nikodým* if it is homeomorphic to a subset of (X^*, w^*) for an Asplund space X. Radon-Nikodým compacta can be characterized using the notion of fragmentability – a compact space is Radon-Nikodým if and only if it is fragmented by a lower semicontinuous metric. This is a result of Namioka [**76**].

The class of Radon-Nikodým compacta contains all scattered compacta. It also contains all Eberlein compact spaces. This follow from the fact that any Eberlein compact can be found as a weakly compact subset in a reflexive space (see [**22**]). Radon-Nikodým compacta are stable to closed subsets and countable products. It is a longstanding open problem whether they are preserved by continuous images.

There are also several results of the relationship of Asplund spaces to the descriptive hierarchy. Let us name some of them.

• An Asplund WLD space is already WCG. This follows from a more general result which we mention in the next paragraph.

• If X^* is weakly Lindelöf, then X is Asplund and X^* is WLD. The validity of the first part was observed [**27**, Proposition 1.8] using a deep result of Stegall [**93**]. The second part is due to Orihuela [**78**, Corollary 8].

• If X is Asplund, then X^* admits an equivalent LUR norm. This was shown by Fabian and Godefroy [**32**].

• If X is an Asplund space of density $\aleph_1$, then $(B_{X^{**}}, w^*)$ is a Valdivia compact space. This follows again from [**32**].

Some related results on relationship of Asplund spaces and Valdivia compacta obtained by the author are contained in [**60**]. Let us mention only one of them – an example of an Asplund space of density $\aleph_2$ whose bidual unit ball is not a Valdivia compact space.

Asplund spaces are also related to smoothness of norms. It was proved already in [**10**] that X is Asplund provided it admits an equivalent norm such that the respective dual norm on X^* is LUR. Any such norm is Fréchet smooth (i.e. it is Fréchet differentiable at all points except for the origin), see [**44**, p. 98]. In fact, any Banach space which admits an equivalent Fréchet smooth norm is Asplund. An even weaker sufficient condition is the existence of a Fréchet smooth bump function [**29**]. A bump function is a function with bounded support which is not identically 0.

There are spaces with Fréchet smooth norm which admit no norm with LUR dual [**100**] and also spaces with a Fréchet smooth bump function without Fréchet smooth norm [**47**]. Hoewever, the following is a long-standing open problem.

PROBLEM 3. *Does every Asplund space admit a Fréchet smooth bump function?*

Weak Asplund spaces. Unlike Asplund spaces, weak Asplund ones admit up to now no reasonable characterization. On the other hand, it is quite a large class of spaces. It contains all Asplund spaces and also all WCD spaces. The properties of this class and various subclasses are described in Fabian's book [**31**]. Let us name basic subclasses and their relationships.

The largest known subclass with reasonable stability properties is Stegall's one. This class was introduced by Stegall [**94**]. We do not use the original definition but the one which became usual later and is used e.g. in [**31**, Chapter 3]. Let us first give the definition of Stegall's class of topological spaces. A usc-K mapping $\varphi\colon S \to T$ is called *minimal usco* if it is nonempty-valued and any nonempty-valued usc-K mapping $\psi\colon S \to T$ satisfying $\psi(s) \subset \varphi(s)$ for each $s \in S$ coincides with φ. A topological space T is said to be *Stegall* if for any Baire topological space B and any minimal usco mapping $\varphi\colon B \to T$ there is a point of B such that $\varphi(b)$ is a singleton. A Banach space X belongs to *Stegall's class* if (X^*, w^*) is Stegall.

The key observation is the following. If any minimal usco $\varphi\colon (X, \|\cdot\|) \to (X^*, w^*)$ is singlevalued at points of a dense G_δ set, then X is weak Asplund. From this observation it easily follows, using Banach localization principle [**70**, Chapter 1, §10.V], that any Banach space from Stegall's class is weak Asplund.

A further subclass is the class of spaces with weak* fragmentable dual. A topological space is said to be *fragmentable* if it is fragmented by some metric. It is not hard to see that any fragmentable topological space is Stegall. Classes of fragmentable and Stegall compacta are stable to taking closed subsets, continuous images and countable products. Stegall's class of Banach spaces and the class of Banach spaces with weak* fragmentable dual are stable to subspaces, quotients and finite products. These results can be found in [**31**, Chapters 3 and 5].

We have the following duality.

THEOREM 22.
- *A Banach space X is in Stegall's class if and only if (B_{X^*}, w^*) is Stegall.*
- *A Banach space X has weak* fragmentable dual if and only if (B_{X^*}, w^*) is fragmentable.*
- *A compact space K is fragmentable if and only if $C(K)$ has weak* fragmentable dual.*

The first two assertions are easy consequences of definitions. The third one is due to Ribarska [**88**]. Notice, that it is not known whether the analogue of the third assertion holds for Stegall spaces. We formulate this question in the following problem, together with open problems concerning stability of weak Asplund spaces.

PROBLEM 4.
- *Is $C(K)$ in Stegall's class whenever K is a Stegall compact space?*
- *Is a subspace of weak Asplund space again weak Asplund?*
- *Is the product of two weak Asplund spaces weak Asplund? Is $X \times \mathbb{R}$ weak Asplund provided X has this property?*

To distinguish these classes of spaces we can use compact spaces K_A introduced by the author in [**53**]. For $A \subset (0,1)$ put $K_A = ((0,1] \times \{0\}) \cup ((\{0\} \cup A) \times \{1\})$ and equip it with the order topology generated by the lexicographic order.

In [**53**] the author showed that, under some additional axioms of the set theory, there is a set $A \subset (0,1)$ such that the compact space K_A is Stegall but not fragmentable. Kenderov, Moors and Sciffer [**66**] later showed that the respective space $C(K_A)$ belongs to Stegall's class (and hence it is an example of a Banach space from Stegall's class whose dual is not weak* fragmentable). After that the author [**62**] showed that, under another additional axioms, there is a set $A \subset (0,1)$ such that $C(K_A)$ is weak Asplund but does not belong to Stegall's class. This example inspires the following concrete question. *Is $C(K_A) \times H$ weak Asplund if H is a large Hilbert space?*

Similarly as Asplund spaces are related to Fréchet smoothness of norms and bumps, weak Asplund spaces are related to Gâteaux smoothness. It was proved already in [**10**] that any Banach space admitting an equivalent norm such that the respective dual norm is strictly convex is weak Asplund. Any such norm is necessarily Gâteaux smooth (i.e. Gâteaux differentiable at all points except the origin). The question whether any space with Gâteux smooth norm is weak Asplund was positively answered by Preiss, Phelps and Namioka [**84**]. In fact, they proved that any such space belongs to Stegall's class of Banach spaces. After that Ribarska [**89**] showed that any such space has weak* fragmentable dual. Later it was shown in [**38**] (see [**67**] for a more general result) that even any Banach space with a Lipschitz Gâteaux differentiable bump function has weak* fragmentable dual. These generalizations are strict ones as there is a space with Gâteaux smooth norm with no equivalent norm with strictly convex dual [**100**] and another space having a Lipschitz Gâteaux smooth bump function but no equivalent Gâteaux smooth norm [**47**]???

It seems that the following problem is open.

Problem 5. *Let X be a Banach space with weak* fragmentable dual. Does X admit a Lipschitz Gâteaux smooth bump function?*

Another subclass of spaces with weak* fragmentable dual is formed by Asplund spaces (by Theorem 20). In fact, there are natural wider subclasses. A Banach space X is called *Asplund generated* (or, shortly, AG) if there is an Asplund space Y and a bounded linear operator $T\colon Y \to X$ with TY dense in X. Then T^* is a one-to-one weak*-to-weak* continuous mapping, thus (B_{X^*}, w^*) is a Radon-Nikodým compact and hence fragmentable. The class of AG spaces is a non-trivial extension of Asplund spaces as it contains all WCG spaces (by [**22**] any WCG space is "reflexive-generated" and reflexive spaces are Asplund). We have the following duality.

Theorem 23.

- *A compact space K is Radon-Nikodým if and only if $C(K)$ is an AG space.*
- *If X is an AG space, then (B_{X^*}, w^*) is a Radon-Nikodým compact.*

The converse to the second assertion does not hold. Let X be a subspace of WCG which is not WCG. Then (B_{X^*}, w^*) is Eberlein and hence Radon-Nikodým but X is not AG. The last statement is a corollary to [**79**]. This example shows that the following definition is natural. A Banach space is called *subspace of AG* if it is isomorphic (or, equivalently, isometric) to a subspace of an AG space. Any example of a subspace of WCG which is not WCG is an example of a subspace of AG which is not AG. We have the following duality.

Theorem 24.

- *A compact space K is a continuous image of a Radon-Nikodým compact space if and only if $C(K)$ is subspace of AG.*
- *A Banach space X is AG if and only if (B_{X^*}, w^*) is a continuous image of a Radon-Nikodým compact space.*

Recall that it is an open problem whether Radon-Nikodým compacta are preserved by continuous images. However, fragmentable compacta are preserved by continuous images, hence any subspace of AG has weak* fragmentable dual.

Let us now collect results on relationship of weak Asplund spaces to the descriptive hierarchy of Banach spaces.

As already remarked above, any WCG space is AG and any subspace of WCG is subspace of AG. On the other hand, there are even Asplund spaces which are not subspaces of WCG. As an example we can use $C[0,\omega_1]$.

Further, Mercourakis [**72**] showed that any WCD space admits an equivalent norm with strictly convex dual. There are non-WCD Banach space with strictly convex dual. A WLD such space can be found in [**7**, Theorem 3.3], an Asplund such space due to Johnson and Lindenstrauss [**50**]. Further, there are WLD spaces which are not weak Asplund [**7**, Theorem 3.6].

It was proved in [**79**] that any Radon-Nikodým Corson compact is Eberlein and that any Banach space which is simultaneously WLD and AG is already WCG. In [**95**] it is shown that even any Corson compact space which is a continuous image of a Radon-Nikodým compact is already Eberlein. Hence, any WLD space which is subspace of AG is already subspace of WCG.

Gâteaux differentiability spaces. Gâteaux differentiability spaces form a wider class than weak Asplund spaces. This is a recent result of Moors and Somasundaram [**75**] who constructed a set $A \subset (0,1)$ such that the space $C(K_A)$ is a GDS but not weak Asplund.

As said above, almost nothing is known about the structure of the class of weak Asplund spaces. In case of GDS the situation is better. A Banach space X is GDS if and only if any nonempty convex weak* compact subset of X^* has a weak* exposed point [**81**, Theorem 6.2]. Moreover, the product $X \times Y$ is GDS whenever X is GDS and Y is separable [**16**]. However, it is not known whether subspace of GDS is GDS or whether product of two GDS is GDS.

There is, similarly as in case of weak Asplund spaces, a natural subclass of GDS defined via minimal usco mappings. A topological space T is called *weakly Stegall* if for any complete metric space M and any minimal usco $\varphi\colon M \to T$ there is $m \in M$ with $\varphi(m)$ being singleton. This class was introduced by the author in an unpublished manuscript [**52**]. The motivation of this definition is, again, the fact that if (X^*, w^*) is weakly Stegall, then X is GDS. In fact, in this case X is *almost weak Asplund* (i.e. any real valued convex continuous function on X is Gâteaux differentiable at points of an everywhere second category set, see [**74**]). The example of [**75**] is in fact almost weak Asplund (a space with weakly Stegall dual).

We finish this section by recalling the following open problems.

Problem 6.

- *Is any WLD space GDS?*

- *Is the space $C(K_{(0,1)})$ a GDS?*

Some other classes of compact spaces

There are also other classes of compact spaces which either do not fit into the two hierarchies or refine them even more. We will briefly mention some of them.

Uniform Eberlein compacta. A compact space is *uniform Eberlein* if it is homeomorphic to a subset of a Hilbert space equipped with the weak topology. Any metrizable compact space is uniform Eberlein and any uniform Eberlein compact space is Eberlein. This class was studied together with Eberlein compacta. Recently Fabian, Godefroy and Zizler [**35**] showed that (B_{X^*}, w^*) is uniform Eberlein if (and only if) X admits an equivalent uniformly Gâteaux smooth norm. A hierarchy of these spaces is discussed in [**33**].

Rosenthal compacta. A compact space is called *Rosenthal* if it is homeomorphic to a subset of the space of functions of the first Baire class on a separable complete metric space equipped with the topology of pointwise convergence. This class contains all metrizable compacta but is in no relation with respect to the other classes. Properties of this class are studied for example in [**40**].

Dyadic compacta. A compact space is called *dyadic* if it is a continuous image of the Cantor cube $\{0,1\}^I$. As Cantor cubes are Valdivia compacta, any dyadic compact space is a continuous image of a Valdivia compact space. Further, any Abelian compact group is dyadic (see e.g. [**101**]), and hence a continuous image of a Valdivia compact space. In fact, compact groups are even open continuous images of Valdivia compact spaces [**68**].

Polyadic compacta. A compact space is polyadic if it is a continuous image of a products M^I where M is a one-point compactification of a discrete space and I is a set. This class clearly contains all dyadic compact spaces and was investigated for example in [**11, 12**]. Again, polyadic compacta are continuous images of Valdivia compacta.

CHAPTER 2

Valdivia compacta and continuous images

In study of various classes of compact spaces one encounters questions on stability of the class with respect to standard topological operations (subsets, products, continuous images etc.). We review some known results on the classes mentioned in the introduction with focus on continuous images.

Most of the mentioned classes are stable to taking closed subsets. This observation is trivial for metrizable, Eberlein, Talagrand, Gul'ko, Corson, scattered, Radon-Nikodým, Stegall and fragmentable compacta. On the other hand, Valdivia compacta are not stable to closed subsets. The reason is that any compact space is homeomorphic to a closed subset of the Valdivia compactum $[0,1]^I$ for a set I.

Further, scattered compacta are stable to finite products and all the other classes are preserved by countable products. Valdivia compacta are preserved by arbitrary products.

The results on continuous images are more complicated.

Easier results use the following argument. Let K be a compact space and L a continuous image of K. Then the space $C(L)$ is isometric to a subspace of $C(K)$. Hence, if a class of compact spaces is characterized by a Banach space property of the space of continuous functions which is inherited by subspaces, then this class is stable to continuous images. In this way we can see that metrizable, Talagrand and Gul'ko compacta are preserved by continuous images.

Eberlein and Corson compacta are stable to continuous images, too. However, these results are essentially deeper. The result for Eberlein compacta is due to Benyamini, Rudin and Wage [**14**], as remarked in the introduction. A simplification of this proof is due to Michael and Rudin [**73**]. The proof of [**73**] is purely topological. It is based on the following characterization of Eberlein compacta.

Theorem 25. *Let K be a compact space. Then the following are equivalent.*

1. *K is Eberlein.*
2. *There is a σ-point finite family $\mathcal{U}$ of open F_σ sets that separates points of K (i.e. whenever x and y are two distinct points of K then there is $U \in \mathcal{U}$ containing exactly one of the points x and y).*
3. *There is a σ-point finite family $\mathcal{U}$ of open sets that F-separates points of K (i.e. whenever $x, y \in K$ are two distinct points, there is $U \in \mathcal{U}$ such that $x \in U$ and $y \notin \overline{U}$ or vice versa).*

They claim that in the virtually the same way one can prove that Corson compact spaces are stable to continuous images and that the previous theorem remains valid if we replace 'Eberlein' by 'Corson' and 'σ-point finite' by 'point-countable'. The results are valid but the assertion that the proof is *virtually the same* seems not to be appropriate. The reason is that they used an intermediate equivalent condition on K:

> *There is a σ-point finite family $\mathcal{U}$ of open sets such that whenever $A, B \subset K$ are closed and disjoint there is a finite subfamily $\mathcal{F} \subset \mathcal{U}$ such that for any $a \in A$ and $b \in B$ there is $U \in \mathcal{F}$ containing exactly one of the points a, b.*

This condition is another equivalent characterization of Eberlein compacta. The difference with Corson compacta is that the analogous condition (with 'point countable' instead of 'σ-point finite') is not a characterization of Corson compact spaces. Indeed, the space $K = [0, \omega_1]$ is not Corson and the family $\mathcal{U} = \{[0, \omega_1)\} \cup \{(\alpha, \omega_1)\colon \alpha < \omega_1\}$ is point countable and satisfies the required condition.

Another proof of the stability of Corson compacta to continuous images is due to Gul'ko [**42**]. He independently proved a more general result using properties of retractions.

Further proofs are due to Gruenhage [**41**]. He characterized Eberlein and Corson compacta K using covering properties of $K \times K$ and showed that these covering properties are preserved by continuous images.

The main result of the following paper [**54**] concerns continuous images of Valdivia compacta. The question whether Valdivia compacta are preserved by continuous images was asked in [**23**]. A first counterexample was given by Valdivia [**107**] who showed that the quotient space arising from the Valdivia compact space $[0, \omega_1]$ by identifying points ω and ω_1 is not Valdivia. In the following paper it is shown that any non-Corson Valdivia compact space can be continuously mapped onto a non-Valdivia compactum.

In view of this non-stability it is natural to study the class of continuous images of Valdivia compacta. It is done in papers [**55, 64, 65**], first of which forms part of this thesis.

Another related natural question is the stability of Valdivia compacta to images by open continuous mappings. In [**57**] (this paper is included in Chaper 3) it is proved that an open continuous image of a Valdivia compact space is Valdivia provided it has a dense set of G_δ points.

An example of a non-Valdivia open continuous image of a Valdivia compact space was recently given by Kubiś and Uspenskij [**68**]. They observed that any compact Abelian group is an open continuous image of a Valdivia compact space and found such a group which is not Valdivia.

Scattered, fragmentable and Stegall compacta are preserved by taking continuous images. The main tool used in proving this is the fact that for any continuous surjection $f\colon K \to L$ between compact spaces there is a minimal closed subset $F \subset K$ with $f(F) = L$.

Finally, the question whether Radon-Nikodým compacta are stable to continuous images is a long-standing open problem. The answer is known to be positive if the image is Corson [**95**] or zero-dimensional (for a common generalization see [**9**]).

CHAPTER 3

Markushevich bases and primarily Lindelöf spaces

Many classes of Banach spaces in the descriptive hierarchy can be characterized using some topological property of the weak topology (as discussed in the introduction). Some of them can be moreover characterized by having a certain type of Markushevich basis. Recall that a Markushevich basis of a space X is an indexed family $(x_\alpha, x_\alpha^*)_{\alpha\in I}$ of pairs from $X \times X^*$ satisfying the following conditions.

- $x_\alpha^*(x_\alpha) = 1$, $x_\alpha^*(x_\beta) = 0$ for $\alpha \neq \beta$;
- $\overline{\operatorname{span}\{x_\alpha\colon \alpha \in I\}} = X$;
- for any $x \in X \setminus \{0\}$ there is $\alpha \in I$ such that $x_\alpha^*(x) \neq 0$.

By a classical result of Markushevich any separable Banach space has a countable Markushevich basis [**44**, Theorem 272]. In fact, the converse theorem is easy and hence separable Banach spaces are exactly those having a countable Markushevich basis. It is known that any Plichko space (in particular, any WLD space) has a Markushevich basis [**59**, Theorem 4.6] and that the space ℓ_∞ has no Markushevich basis [**44**, Theorem 306].

Let us consider now a Banach space X having a Markushevich basis $(x_\alpha, x_\alpha^*)_{\alpha\in I}$ and put $G = \{x_\alpha\colon \alpha \in I\} \cup \{0\}$. It easily follows from the definition of Markushevich basis that G is a weakly closed subset of X and that any non-zero point of G is an isolated point of G. If G is not weakly compact, then (X, w) is a continuous image of a closed subset of $(G, w)^{\mathbb{N}}$. This is the motivation for the following definition.

Let G be a topological space with at most one non-isolated point. We say that a topological space T belongs to the class $M(G)$ if it can be represented as a continuous image of a closed subset of $G^{\mathbb{N}}$. A space T is called primarily Lindelöf if it belongs to the class $M(G)$ for some Lindelöf space G with at most one non-isolated point. We have the following results.

Theorem 26. *Let X be a Banach space and K a compact space. All spaces denoted by G are supposed to have at most one non-isolated point.*

1. *X is separable if and only if $(X, w) \in M(G)$ for a countable G. K is metrizable if and only if $C_p(K) \in M(G)$ for a countable G.*
2. *If X is subspace of WCG, then $(X, w) \in M(G)$ for a σ-compact G. If K is Eberlein, then $C_p(K) \in M(G)$ for a σ-compact G.*
3. *X is weakly K-analytic if and only if $(X, w) \in M(G)$ for a K-analytic G. K is Talagrand if and only if $C_p(K) \in M(G)$ for a K-analytic G.*

(4) *X is WCD if and only if $(X,w)\in M(G)$ for a K-countably determined G. K is Gul'ko if and only if $C_p(K)\in M(G)$ for a K-countably determined G.*
(5) *X is WLD if and only if (X,w) is primarily Lindelöf. K is Corson if and only if $C_p(K)$ is primarily Lindelöf.*

The first assertion is an easy consequence of the existence of countable Markushevich basis in separable Banach spaces and Stone-Weierstrass theorem. The second assertion follows from the results of Amir and Lindenstrauss [**3**]. It seems to be unknown whether the converse implications hold.

The third and fourth assertions follow from results of Mercourakis [**72**]. The last one for the $C_p(K)$ case is due to Pol [**83**] (see [**6**,]).

The following paper [**57**] contains similar results for Valdivia compact spaces and its application to the questions on stability of the class of Valdivia compacta having a dense set of G_δ points.

The results on Valdivia compact spaces and associated Banach spaces are similar but not completely analogous. The difference is that they do not use weak topology and the topology of pointwise convergence but weaker topologies.

Namely, let K be a compact space and A a dense subset of K. Then A is a Σ-subset of K if and only if A is countably compact and $C(K)$ equipped with the topology of pointwise convergence on A is primarily Lindelöf. The Banach space counterpart is analogous.

That the results use weaker topologies is essential. Let $K=[0,\omega_1]$ and L be the quotient space made from K by identifying points ω and ω_1. Then K is Valdivia, L is not Valdivia and the spaces $C_p(K)$ and $C_p(L)$ are homeomorphic. Similarly, neither having Valdivia dual unit ball nor being 1-Plichko is an isomorphic property.

However, the following question seems to be open.

PROBLEM 7. *Let X and Y be Banach spaces such that (X,w) and (Y,w) are homeomorphic and X is Plichko. Is Y Plichko, too?*

CHAPTER 4

Valdivia type Banach spaces and projectional resolutions of the identity

Projectional resolutions of the identity are an important tool to study non-separable Banach spaces. If X is a Banach space of density character $\kappa > \omega$, a PRI on X is an indexed family of projections $(P_\alpha)_{\omega\le\alpha\le\kappa}$ satisfying the following conditions.

(1) $P_\omega = 0$, $P_\kappa = \mathrm{Id}_X$;
(2) $\|P_\alpha\| = 1$ for $\alpha \in (\omega, \kappa]$;
(3) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ whenever $\omega \le \alpha \le \beta \le \kappa$;
(4) $\mathrm{dens} P_\alpha X \le \mathrm{card}\alpha$;
(5) $P_\mu X = \overline{\bigcup_{\alpha<\mu} P_\alpha X}$ for $\mu \in (\omega, \kappa]$ limit.

Existence of PRI enables to prove some properties of Banach spaces by transfinite induction (see e.g. [**110**] or [**31**, Chapter 6]).

Amir and Lindenstrauss [**3**] constructed a PRI in any WCG space. This result was later extended by Vašák [**109**] to WCD spaces. Using PRI's one can show that any WCD space is WLD (see [**72**]). Any WLD space admits a PRI and WLD spaces are exactly those with Corson dual unit ball (see [**104**], [**8**] and [**7**]). A further extension was done by Valdivia who showed that the space $C(K)$ admits a PRI if K is a Valdivia compact space [**105**] and, more generally, that any Banach space X whose dual unit ball has a dense convex symmetric Σ-subset has a PRI [**106**]. The spaces with the latter property are called 1-Plichko spaces.

The results of Valdivia are in a way final positive results. In [**34**] it was observed that any Banach space with density $\aleph_1$ is 1-Plichko if it admits a PRI. For higher densities this converse does not hold.

Let us now describe some converse theorems obtained by the author contained in papers [**58, 63, 56, 61**]. Three of these papers are included in this final chapter of the thesis.

In [**58**] it is shown that a Banach space X is WLD if (and only if) for any equivalent norm on X the respective dual unit ball is a Valdivia compact space. In view of the above mentioned result of [**34**] it folows that a space with density $\aleph_1$ is WLD if (and only if) it has a PRI with respect to each equivalent norm.

In [**63**] it is showh that a Banach space X is WLD if (and only if) any non-separable Banach space isomorphic to a complemented subspace of X admits a PRI. This result may be viewed as a real converse to Amir-Lindenstrauss theorem.

The paper [**56**] studies the following question. *Let X be a Banach space such that any non-separable subspace of X admits a PRI. Is then X WLD?* A partial positive answer is given but the general question remains open.

Finally, [**61**] contains an example of a Banach space with Valdivia dual unit ball which does not admit PRI. This shows that in this case a topological structure of the dual unit ball does not assure the linear structure.

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