Preface

Functional analysis is a wide field of mathematics which has been developing for more than sixty years. Its main objects of interest are normed linear space and their generalizations, together with operators between such spaces. Classical examples of normed spaces are function spaces and sequence spaces. They include, for example, spaces of continuous, differentiable or integrable functions, spaces of convergent, summable or bounded sequences.

The strength of functional analysis is in the abstraction. More precisely, it enables us to consider some complicated objects – for example continuous or integrable functions – as points of a space with some geometrical structure. This approach has a lot of applications – for example for solving various kinds of differential equations.

Further, as it is usual in mathematics, it appears that it is interesting to study the structure of these spaces themselves. This is the subject of Banach space theory – it investigates the structure of Banach spaces. Banach spaces admit several structures, for example geometrical and topological ones. Moreover, they can have some additional structures – this leads to study of Banach lattices, Banach algebras, C^* algebras and many other type of spaces.

The subject of the present thesis concerns the interplay of Banach spaces and compact topological spaces. Compact spaces are closely related to Banach spaces. They are in a kind of duality, so understanding the structure of some compact spaces helps in understanding properties of Banach spaces. For sake of consistency, the papers chosen for the thesis are devoted to two more narrow areas contained in Chapters 2 and 3.

The first chapter contains a more detailed explanation of the relationship of Banach spaces and compact spaces. It continues by a brief description of the two areas addressed in the following two chapters and by a summary of the results contained in the thesis. It also contains the list of articles contained in the thesis and their citations.

Chapter 2 is devoted to differentiability of convex functions on Banach spaces and related classes of compact spaces. We focus on distinguishing of certain classes of Banach spaces defined via differentiability.

Chapter 3 concerns nonseparable Banach spaces which may be in a nice way decomposed to separable pieces. It turns out that this is closely related to a class of compact spaces called Valdivia compacta. We include many results on the structure of this class and applications to Banach spaces.

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Contents

CHAPTER 1

Introduction

The aim of this introductory chapter is to give a brief survey of a wider area including recent results and open problems. It also contains a brief summary of the results included in the thesis

1.1. Banach spaces, compact spaces and their duality

A *Banach space* is a (real or complex) normed linear space which is complete in the metric generated by the norm. Simple examples are finite-dimensional spaces (i.e. $\mathbb{R}^n$ or $\mathbb{C}^n$) equipped with the euclidean norm. Classical examples of infinite-dimensional Banach spaces include sequence spaces ℓ_p (for $p \in [1, \infty]$), the space c_0 of sequences converging to 0, Lebesgue function spaces $L_p[0,1]$ (for $p \in [1, \infty]$) or the space $C[0,1]$ of continuous functions on the interval $[0,1]$.

The general theory of Banach spaces began by the three basic theorems of functional analysis – Hahn-Banach extension theorem, Banach open mapping theorem and uniform boundedness principle (see [**HaHaZi96**, Theorems 31, 83 and 60]). Later the theory developed in many directions which we are not able to name all. The elements of the theory are explained in several books – for example by Diestel [**Di84**], Dunford and Schwartz [**DuSch67**], Semadeni [**Se71**]. For references we use namely nice lecture notes by Habala, Hájek and Zizler [**HaHaZi96**].

An important notion is also that of a linear operator, i.e. a linear mapping from one Banach space into another one. If the target space is the respective field ($\mathbb{R}$ or $\mathbb{C}$), we speak about linear functionals. It is clear that continuous linear functionals on a Banach space X form a linear space. Further, if we equip this space by the operator norm, i.e. if we set

$$\|f\| = \sup\{|f(x)| : x \in X, \|x\| \leq 1\}$$

for each continuous linear functional f on X we obtain a Banach space. This space is denoted by X^* and called *dual space* of X.

Banach spaces admit several structures. We can consider them as linear spaces, normed spaces, metric spaces or topological spaces. Banach spaces X and Y are called *isometric* if there is a linear bijection $L : X \to Y$ which preserves the norm. In this case X and Y are indistinguishable as Banach spaces. The spaces X and Y are called *isomorphic* if there is a continuous linear bijection $L : X \to Y$. Then, by the open mapping theorem, necessarily L^{-1} is continuous. Large number of properties of Banach spaces are preserved by isomorphisms.

On a Banach space there are several natural topologies. The first one is the topology generated by the metric induced by the norm, called *norm topology* or *strong topology*. The second important topology is the *weak* one. It is the weakest topology having the same continuous linear functionals as the norm topology.

Dual Banach spaces admit a further natural topology – the *weak** one. If X is a Banach space and X^* its dual, the weak* topology on X^* is the weakest one in which all evaluation maps $x^* \mapsto x^*(x)$ $(x \in X)$ are continuous. These two topologies are special cases of general weak topologies [**HaHaZi96**, Definition 205].

There are some important relationship of Banach spaces to compact spaces. Let us recall that a topological space is *compact* if any cover of the space by open sets admits a finite subcover. A classical theorem of Borel says that the interval $[0,1]$ is compact [**En89**, page 107]. More generally, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded [**En89**, Theorem 3.2.7]. Investigation of compact spaces forms a large part of general topology. It is easy to check that compact spaces are stable to taking closed subsets and continuous images [**En89**, Theorems 3.1.2 and 3.1.8]. The deep Tychonoff theorem [**En89**, Theorem 3.2.4] says that an arbitrary product of compact spaces is compact. Hence, in particular, Tychonoff cubes $[0,1]^I$ and Cantor cubes $\{0,1\}^I$ are compact for arbitrarily large set I. Tychonoff cube is in fact a universal object for compact Hausdorff spaces, as any compact Hausdorff space can be homeomorphically embedded into the Tychonoff cube $[0,1]^I$ for a sufficiently large set I [**En89**, Theorem 3.2.5].

The first result connecting Banach spaces with compact spaces says that the closed unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ is compact in the norm topology if and only if X has finite dimension ([**HaHaZi96**, Theorem 16]). Another one is Alaoglu theorem [**HaHaZi96**, Theorem 61] which is a consequence of Tychonoff theorem. It asserts that whenever X is a Banach space, then the dual unit ball (i.e. the unit ball of the dual space X^*) B_{X^*} is compact when equipped with the weak* topology.

This result is useful in the investigation of weak compactness, i.e. of subsets of Banach spaces which are compact in the weak topology. The relationship appears due to the canonical embedding of any space X to its second dual X^{**}. The second dual is of course the dual space of the dual space of X and the canonical embedding $\varepsilon : X \to X^{**}$ is defined as the evaluation mapping, i.e. $\varepsilon(x)(f) = f(x)$ for each $x \in X$ and $f \in X^*$. Now, it follows easily from Alaoglu theorem that $K \subset X$ is relatively weakly compact if and only if $\overline{\varepsilon(K)}^{w^*} \subset \varepsilon(X)$. In particular, X is *reflexive*, i.e. $\varepsilon(X) = X^{**}$, if and only if B_X is weakly compact.

To each compact Hausdorff space K we can associate the space $C(K)$ of real continuous functions on K equipped with the max-norm, i.e.

$$\|f\| = \max\{|f(k)| \colon k \in K\}.$$

This is a real Banach space. If we consider complex continuous functions, we obtain a complex Banach space which we may denote by $C(K,\mathbb{C})$. This space is usually denoted also by $C(K)$ and it should be clear from the context whether we consider real or complex spaces. On a space of the form $C(K)$ there is one more natural topology, in addition to the norm and weak ones. This is the topology τ_p of pointwise convergence, i.e. the weakest topology making the evaluation map $f \mapsto f(k)$ continuous for each $k \in K$. It is usual to write $C_p(K)$ to denote $(C(K), \tau_p)$.

Now we can see that we really have a kind of duality between Banach spaces and compact spaces. It is witnessed by the following embeddings.

If K is a compact space, $C(K)$ is a Banach space and $(B_{C(K)^*}, w^*)$ is again compact. Further, there is a natural embedding of K into $(B_{C(K)^*}, w^*)$. To a point $k \in K$ we assign the evaluation functional $\delta_k \colon f \mapsto f(k)$. It is clearly a linear functional of norm one. Moreover, the embedding $k \mapsto \delta_k$ is homeomorphic, hence we can consider K as a topological subspace of $(B_{C(K)^*}, w^*)$. By Riesz theorem [**DuSch67**, Theorem IV.6.3] we can identify $C(K)^*$ with the space of finite signed Radon measures on K. The norm of an element of $C(K)^*$ is then equal to the total variation of the representing measure. In this identification δ_k is represented by the Dirac measure supported at k.

If X is a Banach space and we denote by K the compact space (B_{X^*}, w^*), there is a natural embedding of X into $C(K)$. To any $x \in X$ we assign the function $h_x \in C(K)$ defined by $h_x(k) = k(x)$, $k \in K$. The embedding $x \mapsto h_x$ is clearly linear. By a consequence of Hahn-Banach theorem [**HaHaZi96**, Corollary 36] it is isometric when we consider X with its given norm and $C(K)$ with the max-norm. This mapping is, moreover, a homeomorphism from the weak topology of X into the topology of pointwise convergence on $C(K)$. Hence, identifying x and h_x we may consider X as a subspace of $C(K)$. In this identification the norm of X is the restriction of the max-norm on $C(K)$ and X is a norm-closed subset of $C(K)$. The space X equipped with the weak topology is then a topological subspace of $C_p(K)$. It is quite important that X is closed also in $C_p(K)$. This is a consequence of Banach-Dieudonné theorem [**HaHaZi96**, Theorem 222].

A large part of the investigation of Banach spaces is devoted to study the described duality, namely the questions of the following kind. Which topological properties of K assure a given property of $C(K)$ and conversely? Which topological properties of (B_{X^*}, w^*) imply a given property of X and conversely?

In the following two sections we describe two directions of such an investigation. These two directions correspond to the following two chapters of the thesis.

1.2. Differentiability of convex functions

1.2.1. Fréchet and Gâteaux derivatives. Computing derivatives and differentials plays an important role in mathematical analysis – starting from real-valued functions of one or several real variables and continuing until non-linear operators between Banach spaces. The importance of derivative is in the possibility to approximate a complicated function by a simple one, namely by an affine one. There are different types of derivatives, differentials (and also subderivatives and subdifferentials) which yield different types of approximation. We recall the two basic types of derivatives.

We restrict ourselves to real-valued functions defined on real Banach spaces. So, suppose that X is a real Banach space and f is a real-valued function defined on an open subset $D \subset X$. If $x \in D$ and $h \in X$ is arbitrary, the *directional derivative* of f at x in the direction h is defined by

$$\partial_h f(x) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

provided this limit exists and is finite. I.e., it is just the derivative of the function $t \mapsto f(x+th)$ at the point 0. This quantity has the real sense of a directional derivative if h is a unit vector. But it is useful to define it for any h.

If the assignment $h \mapsto \partial_h f(x)$ defines a bounded linear functional on X, f is said to be *Gâteaux differentiable* at x and the respective functional is called *Gâteaux derivative* of f at x and is denoted by $f'_G(x)$.

A function f is *Fréchet differentiable* at x if there is a linear functional $L \in X^*$ with

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - L(h)}{\|h\|} = 0.$$

The functional L is then called *Fréchet derivative* of f at x and denoted by $f'_F(x)$. It is easy to see that Fréchet differentiable function is also Gâteaux differentiable and that in this case $f'_F(x) = f'_G(x)$.

If X has dimension 1, i.e. if $X = \mathbb{R}$, then both Fréchet differentiability and Gâteaux differentiability coincide with the ordinary differentiability. However, already in case $X = \mathbb{R}^2$ there are continuous functions which are Gâteaux differentiable but not Fréchet differentiable. Nonetheless, for locally Lipschitz functions on a finite-dimensional space Gâteaux differentiability coincides with Fréchet differentiability. On infinite dimensional spaces the situation is even more complicated.

An important issue is the investigation of convex continuous functions defined on open convex sets. In this case the situation is easier. Let us mention that in the finite-dimensional case convex functions are automatically continuous, and that continuous convex function are automatically locally Lipschitz. Canonical examples of continuous convex functions are equivalent norms or, more generally, Minkowski functionals of convex neighborhoods of the origin.

Although convex continuous functions may look quite special, the results on their differentiability may be applied for larger classes of functions – for example for *d.c.-functions* (i.e., differences of convex continuous functions), or, more generally for locally d.c.-functions.

1.2.2. Asplund spaces, weak Asplund spaces and Gâteaux differentiability spaces. There are several classes of Banach spaces defined via differentiability of convex functions. Two of them are now called *Asplund spaces* and *weak Asplund spaces*. They were defined by E. Asplund [**As68**] who called them strong differentiability spaces and weak differentiability spaces. Let us recall the definitions.

A Banach space X is called Asplund if each convex continuous function defined on an open convex subset $D \subset X$ is Fréchet differentiable at points of a dense G_δ subset of D.

Similarly, a Banach space X is called weak Asplund if each convex continuous function defined on an open convex subset $D \subset X$ is Gâteaux differentiable at points of a dense G_δ subset of D.

These two classes are quite different, although their definitions look similar. The reason is a big difference between Fréchet differentiability and Gâteaux differentiability. The class of Asplund spaces has many nice stability properties and many characterizations. We recall some characterizations in the following theorem.

THEOREM 1. *Let X be a Banach space. The following assertions are equivalent.*

(i) *X is an Asplund space.*
(ii) *Each equivalent norm on X is Fréchet-differentiable at least at one point.*
(iii) *For each separable subspace $Y \subset X$ its dual Y^* is separable.*
(iv) *X^* has the Radon-Nikodým property.*
(v) *Each bounded nonempty subset of X^* admits weak* slices of arbitrarily small norm-diameter.*
(vi) *Each bounded nonempty subset of X^* admits nonempty relatively weak* open subsets of arbitrarily small norm-diameter.*

The equivalence of the conditions (i), (ii), (iii), (v) and (vi) is proved for example in [**DeGoZi93**, Section I.5]. The equivalence of (i) and (iv) is proved in [**St78**].

Let us comment some of the above equivalent conditions and related properties of Asplund spaces. First, the set of points of Fréchet differentiability of a convex continuous function is automatically G_δ. Therefore, it does not matter whether in the definition of Asplund spaces we require Fréchet differentiability at points of a dense G_δ set, at points of a dense set or just at at least one point. The equivalence of (i) and (ii) is a strengthening of this observation.

The characterization by the condition (iii) implies, for example, that Asplund spaces are stable to taking subspaces, quotients, several kinds of products and that Asplundness is a "three-space property".

The assertion (v) says that the unit ball (B_{X^*}, w^*) is *fragmented* by the norm.

The situation of weak Asplund spaces and Gâteaux differentiability is more complicated. If X is separable, $D \subset X$ an open convex set and f a convex continuous function defined on D, the set G_f of points of Gâteaux differentiability of f is a dense G_δ subset of D. Thus, in particular, separable spaces are weak Asplund. Anyway, if X is not separable, the set G_f need not be G_δ. In fact this set need not be even Borel, even in case X is Asplund (see [**HoŠmZa98**]).

These facts make natural the following definition: A Banach space X is a *Gâteaux differentiability space*, shortly *GDS*, if each convex continuous function defined on an open convex subset $D \subset X$ is Gâteaux differentiable at points of a dense subset of D. This class was introduced by Larman and Phelps [**LaPh79**]. They established several properties of this class and formulated several open problems. We recall two of these questions which are relevant for this thesis.

Question 1.

(1) [**LaPh79**, Problem 1] *Is every GDS necessarily weak Asplund?*
(2) [**LaPh79**, Problem 5] *Are GDS or weak Asplund spaces preserved by finite products? Is a subspace of one of these spaces again such a space?*

The second question indicates that only a little is known about structure and stability of the classes of weak Asplund spaces and GDS. In fact, it is known that these classes are preserved by quotients. The only progress in stability since the paper [**LaPh79**] was the result of Fabian (reproduced in [**Ph93**, Proposition 6.5]) saying that $X \times \mathbb{R}$ is GDS whenever X is GDS and, later, the result of Cheng and Fabian [**ChFa01**] asserting that $X \times Y$ is GDS if X is GDS and Y is separable.

1.2.3. Subclasses of weak Asplund spaces. The lack of nice characterizations of weak Asplund spaces led to introducing and studying large subclasses with nice properties. The largest subclasses are the classes of the Stegall type. The definition of this class is inspired by the characterization of Gâteaux differentiability using subdifferential (the definition and properties can be found for example in [**Ph93**, Chapters 1 and 2]): Let X be a Banach space, $D \subset X$ an open convex set and $f : D \to \mathbb{R}$ a continuous convex function. By the *subdifferential* of f at $x \in D$ we mean the set of all $x^* \in X^*$ such that $f(x+h) \geq f(x) + x^*(h)$ for all $h \in X$ of sufficiently small norm. This set is denoted by $\partial f(x)$.

The set $\partial f(x)$ is always a nonempty convex weak* compact set. Moreover, f is Gâteaux differentiable at x if and only if $\partial f(x)$ is a singleton (and this unique point is then the Gâteaux derivative). Further, the set-valued mapping $x \mapsto \partial f(x)$ is upper-semicontinuous from the norm topology to the weak* topology. (I.e., the set $\{x \in D : \partial f(x) \subset U\}$ is norm-open for each weak* open set $U \subset X^*$.)

Hence, it is clear that Gâteaux differentiability of convex continuous functions is related to singlevaluedness of upper-semicontinuous multifunctions. This leads to the definition of Stegall's class. This class was defined by Stegall [**St83**] using another terminology. We follow [**Fa97**, Chapter 3].

A topological space T is said to be in *Stegall's class* if for each Baire topological space B and each upper-semicontinuous nonempty-compact-valued mapping φ from B to subsets of T, which is minimal with respect to inclusion, there is at least one point $b \in B$ such that $\varphi(b)$ is a singleton. Let us remark that the set of such points is then automatically residual (by the Banach localization principle).

It is not hard to show that X is weak Asplund as soon as (X^*, w^*) (equivalently, (B_{X^*}, w^*)) belongs to Stegall's class, see [**Fa97**, Theorem 3.2.2]. This class of Banach spaces is denoted by $\widetilde{\mathcal{S}}$ in [**Fa97**] and is preserved by taking subspaces and finite products (and even more, see [**Fa97**, Theorem 3.2.3]).

In a sense more concrete subclass of $\widetilde{\mathcal{S}}$ is the class $\widetilde{\mathcal{F}}$ of Banach spaces whose dual is weak* fragmented by some metric. Recall that a topological space T is *fragmented* by a metric ρ if each nonempty subset of T admits nonempty relatively open subsets of arbitrarily small ρ-diameter. Then (in the notation of [**Fa97**, Chapter 5]) $\widetilde{\mathcal{F}}$ is the class of those Banach spaces X such that (X^*, w^*) (equivalently (B_{X^*}, w^*)) is fragmented by some metric.

Fragmentability was introduced by Jayne and Rogers [**JaRo85**] and many properties of this notion were established by Ribarska [**Ri87**]. Each topological space which is fragmented by some metric belongs to Stegall's class, so in particular $\widetilde{\mathcal{F}} \subset \widetilde{\mathcal{S}}$ (see [**Fa97**, Theorem 5.1.11]).

So, we have four classes of Banach spaces which are ordered by inclusion:

$$\widetilde{\mathcal{F}} \subset \widetilde{\mathcal{S}} \subset \text{weak Asplund spaces} \subset \text{GDS}$$

It was a long-standing open problem whether these inclusions are proper. The question on the last inclusion was formulated in the above quoted paper [**LaPh79**], the other questions were mentioned for example in [**Fa97**].

There were many results saying that certain classes of Banach spaces are weak Asplund. Already Asplund [**As68**] proved that X is weak Asplund provided it admits an equivalent norm with strictly convex dual norm. So separable and, more generally, weakly compactly generated Banach spaces are weak Asplund [**DeGoZi93**, Theorems II.2.4 and II.7.3 and Corollary VI.5.3]. The same holds also for weakly countably determined spaces by [**Me87**].

A deep extension of the mentioned Asplund's result is a result of Preiss [**PrPhNa90**, Section 4.2] saying that X is weak Asplund provided it admits an equivalent norm which is Gâteaux differentiable at all points except origin.

Subsequently it was proved that all these spaces in fact belong to the class $\widetilde{\mathcal{F}}$. Ribarska [**Ri92**] showed that any Banach space with an equivalent Gâteaux smooth norm belongs to the class $\widetilde{\mathcal{F}}$. This was extended by Fosgerau [**Fo92**] to Banach spaces which admit a Lipschitz Gâteaux differentiable function with a nonempty bounded support.

Another subclass of weak Asplund spaces is that of Asplund generated Banach spaces and their subspaces, see [**Fa97**, Section 1.3]. These spaces also belong even to the class $\widetilde{\mathcal{F}}$. (This follows, for example, from [**Fa97**, Theorem 5.2.4] and Theorem 1 above.)

So, if we sum up these results – weak Asplund spaces are quite a large class of Banach spaces. Further, all Banach spaces which were known to be weak Asplund belonged in fact to the class $\widetilde{\mathcal{F}}$. There were also two examples of Banach spaces such that it was known that they are not weak Asplund but it was not clear whether they are GDS. One of them is an example of Talagrand [**Ta79**], the second one is due to Argyros and Mercourakis [**ArMe93**]. This describes the situation in 1996. Since then the above mentioned classes have been distinguished. Several key steps to this result are in the papers included in Chapter 2 of this thesis. Let us give a brief summary of this process.

1.2.4. Summary of the Chapter 2 of this thesis. Let us first recall Talagrand's example from [**Ta79**]. Let

$$K = ((0,1] \times \{0\}) \cup ([0,1) \times \{1\}),$$

i.e., K is made from the product $[0,1] \times \{0,1\}$ by deleting the points $(0,0)$ and $(1,1)$. Equip K with the lexicographic order, i.e., $(x,i) < (y,j)$ if and only if either $x < y$ or $x = y$ and $i < j$. If we then endow K with the order topology, it becomes a compact Hausdorff space. Then $C(K)$ is a Banach space and the points of Gâteaux differentiability of the supremum norm form a dense set which is of first category. So, $C(K)$ is not weak Asplund. Anyway, it is not clear whether it is GDS. This question is still open.

In the paper [**Ka99a**], which forms **Section 2.1** of this thesis, modification of Talagrand's example were studied. More precisely, let A be an arbitrary subset of $(0,1)$. We set

$$K_A = ((0,1] \times \{0\}) \cup ((A \cup \{0\}) \times \{1\})$$

and equip this set with the lexicographic order and the order topology. Then K_A is a compact Hausdorff space. In fact, these spaces are well known in topology as they are compact separable linearly ordered spaces. Such spaces were characterized by Ostaszewski [**Os74**]. They have a concrete description similar to the above formula for K_A.

In [**Ka99a**] it was proved that the space K_A is fragmented by some metric if and only if A is countable. In this case K_A is even metrizable. It was further shown that, under certain additional axioms beyond the standard ones, there are uncountable sets A for which K_A belongs to Stegall's class. This shows that Stegall's class of compact spaces can be strictly larger than the class of compact spaces fragmented by some metric.

This result had two disadvantages. The first one is the use of additional axioms. This obstacle have not yet been overcome. The second one is that it is a result on compact spaces, not on Banach spaces. More precisely, it was not clear whether the Banach space $C(K_A)$ belongs to $\widetilde{\mathcal{S}}$ for some uncountable A. Let us note that it is still an open problem whether $C(K) \in \widetilde{\mathcal{S}}$ whenever K is in Stegall's class. (An analogous result on fragmentability is valid, see [**Ri87**].)

The second obstacle was overcome by Kenderov, Moors and Sciffer [**KeMoSc01**]. They proved that there are (under the same additional axioms) uncountable sets A such that $C(K_A) \in \widetilde{\mathcal{S}}$. This means that the class $\widetilde{\mathcal{S}}$ can be strictly larger than the class $\widetilde{\mathcal{F}}$. In particular, there can be weak Asplund spaces which are not in the class $\widetilde{\mathcal{F}}$.

In the paper [**Ka02b**], which form **Section 2.2** of this thesis, it was shown that there can be weak Asplund spaces which do not belong to the class $\widetilde{\mathcal{S}}$. It was done using the results of [**KeMoSc01**] and an improvement of [**Ka99a**]. Namely, it was proved that $C(K_A) \in \widetilde{\mathcal{S}}$ if and only if K_A is in Stegall's class. Moreover, the same equivalence holds for "Stegall's class with respect to Baire spaces of weight at most equal to κ" for any cardinal κ. The definition of this class is the same as that of Stegall's class, with the difference that we consider only Baire spaces of weight at most κ. Finally, it is proved that there is (under some additional axioms) a set $A \subset (0,1)$ of cardinality $\aleph_1$, such that K_A is in Stegall's class with respect to Baire spaces of weight $\aleph_1$ but not in Stegall's class itself. Then $C(K_A) \notin \widetilde{\mathcal{S}}$. Anyway, $C(K_A)$ is weak Asplund as it has weight $\aleph_1$ and its dual belongs to Stegall's class with respect to Baire spaces of weight $\aleph_1$.

A further result of the paper [**Ka02b**] says that, under Martin's axiom and negation of continuum hypothesis, there is a weak Asplund space of the form $C(K_A)$ which does not belong to the class $\widetilde{\mathcal{F}}$. The advantage of this result is the use of a simpler axiom than in [**Ka99a**] and [**KeMoSc01**]. However, under these assumptions it cannot be determined whether the respective space $C(K_A)$ belongs to the class $\widetilde{\mathcal{S}}$.

The up-to-now final result on distinguishing the classes $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{S}}$ and weak Asplund space is contained in the paper [**KaKu05**] which forms **Section 2.3** of this thesis. This paper was inspired by the fact that under some additional axioms we have $\widetilde{\mathcal{F}} \subsetneq \widetilde{\mathcal{S}}$ (by [**KeMoSc01**] and [**Ka99a**]) and under another additional axioms there is a weak Asplund space not belonging to $\widetilde{\mathcal{S}}$ (by [**Ka02b**]). Moreover, the two sets of axioms used in the quoted papers contradict each other. This obstacle was solved in [**KaKu05**] by constructing a model of the set theory in which the three classes are different. The model is constructed using in advanced way the method of forcing. That's why it is co-authored by K. Kunen, who is an expert on forcing.

The question whether the classes $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{S}}$ and weak Asplund spaces can be distinguished without using additional axioms is still open.

The spaces K_A were further used to distinguish weak Asplund spaces and GDS. This began by the manuscript [**Ka97**] which forms **Section 2.2** of the present thesis, where *weakly Stegall* spaces were defined. Namely, a topological space T is weakly Stegall if for each complete metric space M and each upper-semicontinuous nonempty-compact-valued mapping φ from M to subsets of T, which is minimal with respect to inclusion, there is at least one point $m \in M$ such that $\varphi(m)$ is a singleton. Let us remark that the set of such points is then automatically everywhere of second category (by the Banach localization principle). This definition was motivated by an easy observation that a Banach space X is GDS as soon as (X^*, w^*) (equivalently, (B_{X^*}, w^*)) is weakly Stegall. In the mentioned manuscript basic properties of this class were established and a characterization of weakly Stegall spaces among spaces K_A was given. However, it was shown by an example that this class is not preserved by products, so it was not clear whether there is some new set A for which the dual $(C(K_A)^*, w^*)$ is weakly Stegall.

The theory of weakly Stegall space was later developed by Moors and Somasundaram in [**MoSo03**]. They proved in particular that $(C(K_A)^*, w^*)$ is weakly Stegall if $C(K_A)$ is the weakly Asplund space not belonging to $\widetilde{\mathcal{S}}$ constructed in [**Ka02b**]. Finally, in [**MoSo06**] they proved that there is a set $A \subset (0,1)$ such that $C(K_A)$ is not weak Asplund but $(C(K_A)^*, w^*)$ is weakly Stegall. In particular, $C(K_A)$ is GDS. Moreover, no additional axioms are needed. This solves the longstanding problem from [**LaPh79**]. Anyway, it is still not known whether Talagrand's example $C(K_{(0,1)})$ is GDS.

1.3. Decompositions of nonseparable spaces

1.3.1. Projectional resolutions of the identity on nonseparable spaces. A Banach space X is *separable* if it admits a countable dense subset. Separable spaces have many nice geometrical properties. For example, they admit equivalent norms which are simultaneously strictly convex and Gâteaux differentiable outside the origin (and much more, see [**DeGoZi93**, Theorem II.7.1]). Further, a Banach space X is separable if and only if the dual ball (B_{X^*}, w^*) is metrizable (see [**HaHaZi96**, Proposition 62 and Exercise 3.48]). In particular, the weak* topology of bounded sets in X^* can be described using sequences. Finally, the space $C(K)$ (where K is a compact Hausdorff space) is separable if and only if K is metrizable (see [**HaHaZi96**, Exercise 3.47]).

Nonseparable spaces need not have such nice properties. So, a large part of investigation of nonseparable Banach spaces is devoted to the study of decompositions of nonseparable spaces to smaller pieces, in fact to separable ones. This is done namely using families of projections.

One type of such families which now may be called classical is a *projectional resolution of the identity.* We recall the definition. Let X be a nonseparable Banach space and let κ denotes its density, i.e. the smallest cardinality of a dense subset. A projectional resolution of the identity (shortly *PRI*) on X is an indexed family $(P_\alpha : \omega \leq \alpha \leq \kappa)$ of linear projections on X satisfying the following conditions:

(1) $P_\omega = 0$, $P_\kappa = \mathrm{Id}_X$;
(2) $\|P_\alpha\| = 1$ for $\alpha \in (\omega, \kappa]$;
(3) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ whenever $\omega \leq \alpha \leq \beta \leq \kappa$;
(4) $\operatorname{dens} P_\alpha X \leq \operatorname{card} \alpha$;
(5) $P_\mu X = \overline{\bigcup_{\alpha<\mu} P_\alpha X}$ for $\mu \in (\omega, \kappa]$ limit.

A deep result of Amir and Lindenstrauss [**AmLi68**] says that each nonseparable *weakly compactly generated* Banach space admits a projectional resolution of the identity. Recall that a Banach space is weakly compactly generated if it contains a generating weakly compact subset. This class contains, for example, separable spaces, reflexive spaces and spaces $L_1(\mu)$ for any σ-finite measure μ. The result of [**AmLi68**] was later extended to larger classes of Banach spaces. Vašák [**Vaš81**] proved the existence of a PRI in *weakly countably determined* Banach spaces which are called by some authors *Vašák spaces.*

Further results, due to Valdivia [**Val88**], proved the same for *weakly Lindelöf determined spaces.* The name for this class comes from [**ArMe93**]. Various definitions were used for this class which turned to be equivalent (see [**ArMe93, ArMeNe88, Val88**]). We will use the following definition (different from the one used by Valdivia). A Banach space X is called weakly Lindelöf determined (shortly *WLD*) if X admits a linearly dense set $M \subset X$ such that for each $x^* \in X^*$ there are only countably many $x \in M$ with $x^*(x) \neq 0$.

The fact that results on WLD spaces extend Vašák's results is not obvious, but can be proved using PRI's. It was done by Mercourakis who showed in [**Me87**] that any weakly countably determined space is WLD.

In a sense final results in this direction were obtained by Valdivia. He proved in [**Val90**] that the space $C(K)$ has a PRI if K is a *Valdivia compact space* and in [**Val91**] that a Banach space X has a PRI if the dual unit ball is a special type of a Valdivia compact space. He did not use the name Valdivia compact space, but called it compact space of class $\mathcal{A}$. The name Valdivia for this class was introduced by Deville and Godefroy in [**DeGo93**].

Let us give the definition of Valdivia compact spaces. A compact K is called Valdivia if there is a homeomorphic embedding $h : K \to \mathbb{R}^\Gamma$ such that $h^{-1}(\Sigma(\Gamma))$ is dense in K, where $\Sigma(\Gamma)$ is the Σ-product of real lines, i.e.

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is countable}\}.$$

A set which can be expressed as $h^{-1}(\Sigma(\Gamma))$ as above is called a Σ-*subset* of K. Now we can explain the mentioned result of [**Val91**]. It is proved there that X admits a PRI provided the dual unit ball (equipped with the weak* topology) has a dense absolutely convex Σ-subset.

Compact spaces which are homeomorphic to a subset of $\Sigma(\Gamma)$ for a set Γ are called *Corson*. They are related to WLD spaces – namely a Banach space is WLD if and only if the dual unit ball is Corson in the weak* topology.

1.3.2. Applications of projectional resolutions. Probably the main application of the existence of a PRI is proving some properties of Banach spaces using transfinite induction. There are several theorems of the following type:

Suppose that X is a Banach space having a PRI $(P_\alpha : \omega \leq \alpha \leq \kappa)$. If $(P_{\alpha+1} - P_\alpha)X$ have property $\mathcal{P}$ for each $\alpha \in [\omega, \kappa)$, then X has property $\mathcal{P}$ as well.

Statements of this kind can be found for example in [**Fa97**, Section 6.2] and [**Zi84**].

So, the whole proof goes as follows: We have a class $\mathcal{C}$ of Banach spaces and we want to prove that each space from this class has a property $\mathcal{P}$. Suppose that the property $\mathcal{P}$ satisfies the statement from the previous paragraph and that all separable spaces have property $\mathcal{P}$. If each nonseparable $X \in \mathcal{C}$ has a PRI $(P_\alpha : \omega \leq \alpha \leq \kappa)$ such that $(P_{\alpha+1} - P_\alpha)X$ belongs to $\mathcal{C}$ for each $\alpha \in [\omega, \kappa)$, then we can conclude (using transfinite induction on the density of X) that each space from $\mathcal{C}$ does really have the property $\mathcal{P}$.

From this scheme it is clear that it is important to have a PRI such that the spaces $(P_{\alpha+1} - P_\alpha)X$ belong to the same class. This is true for all the above mentioned classes of spaces for which the existence of a PRI was proved. It also follows that mere existence of a PRI in a space X does not say much. There is one exception – if the density of X is $\aleph_1$, the smallest uncountable cardinal, then the ranges of projections are separable.

1.3.3. Projectional skeletons and Valdivia compact spaces. The described disadvantages led to studying strengthenings of the notion of a PRI. One of the strengthenings, which was originally developed as a method of constructing of a PRI is the notion of a *projectional generator*. This notion was introduced by Orihuela and Valdivia [**OrVa90**]. It is explained also in [**Fa97**, Chapter 6]. We will not define this notion here. A more natural and at least formally less restrictive strengthening is the notion of a *projectional skeleton* defined and used by Kubiś [**Ku09**]. A projectional skeleton on a Banach space X is a family $(P_s : s \in \Gamma)$ of bounded linear projections on X indexed by a directed set Γ satisfying the following conditions:

(1) $X = \bigcup_{s\in\Gamma} P_s(X)$ and $P_s(X)$ is separable for each $s \in \Gamma$;
(2) $\forall s,t \in \Gamma : s \le t \Rightarrow P_sP_t = P_tP_s = P_s$;
(3) if $(s_n)_{n\in\mathbb{N}}$ is an increasing sequence in Γ, then the it has a supremum $t \in \Gamma$ and $P_t(X) = \overline{\bigcup_{n\in\mathbb{N}} P_{s_n}(X)}$.

If X has a 1-projectional skeleton, i.e. a projectional skeleton formed by projections of norm one, then there is a PRI $(P_\alpha : \omega \le \alpha \le \kappa)$ such that each of the spaces $(P_{\alpha+1} - P_\alpha)X$ has again a 1-projectional skeleton. This shows that the notion of a projectional skeleton is the right one.

Moreover, there is a relationship of projectional skeletons and Valdivia compact spaces, summed up in the following theorem which follows from the results of [**Ku09**] and [**Ka00a**].

THEOREM 2. *Let X be a Banach space. The following assertions are equivalent.*

(1) *The dual ball (B_{X^*}, w^*) has an absolutely convex dense Σ-subset.*
(2) *There is a linearly dense set $M \subset X$ such that*

$$S = \{x^* \in X^* : \{x \in M : x^*(x) \neq 0\} \text{ is countable}\}$$

is 1-norming (i.e., $S \cap B_{X^*}$ is weak dense in B_{X^*}).*

(3) *X admits a commutative 1-projectional skeleton.*

Commutativity in the assertion (3) means that $P_sP_t = P_tP_s$ for all $s,t \in \Gamma$ (using the above notation). Let us remark that the condition (2) in the definition of a projectional skeleton requires commutativity only for comparable $s,t \in \Gamma$.

The spaces satisfying one of the equivalent conditions of the above theorem are called 1-*Plichko* spaces. Spaces isomorphic to 1-Plichko spaces are called *Plichko*. The reason for this name is the fact that this class was first studied by A. Plichko in [**Pl82**] (using another equivalent definition).

It is worth to mention that this theorem has also a topological counterpart. It is proved in [**KuMi06**, Theorem 6.1] that a compact space is Valdivia if and only if it admits a *commutative retractional skeleton*. We will not give the exact definition here. Let us just remark that it is a family of retractions with metrizable ranges with similar properties to those of a projectional skeleton. (Exact definition can be found in Section 3.4 below.)

The mentioned results show that Valdivia compact spaces and 1-Plichko spaces are natural classes closely related to decompositions of nonseparable Banach spaces to separable ones. A detailed study of these classes is the content of the Chapter 3 of this thesis.

1.3.4. Summary of Chapter 3 of this thesis. Chapter 3 contains four papers with results on Valdivia compact spaces, 1-Plichko and Plichko Banach spaces. A brief description of these papers follows.

Section 3.1 contains the paper [**Ka00c**]. This paper surveys results on Valdivia compact spaces, 1-Plichko and Plichko Banach spaces, many of which have been obtained earlier by the author. It contains also several original results.

The first section of this paper contains a historical introduction, basic definitions and elementary facts on Valdivia compacta. This is done by elaborating auxiliary results from papers [**Ka99b, Ka99d**].

The second section contains a characterization of Valdivia compact spaces and 1-Plichko spaces using a weak topology. This follows the paper [**Ka00a**].

The third section deals with topological properties of Valdivia compact spaces and contains (among others) the result of [**Ka99b**] saying that any non-Corson compact spaces can be continuously mapped onto a non-Valdivia compact space and the result of [**Ka99c**] on embedding a copy of $[0, \omega_1]$ into non-Corson continuous images of Valdivia compact spaces.

The fourth section deals with 1-Plichko and Plichko spaces. It contains basic fact on PRI's and Markushevich bases in these spaces, the result of [**Ka00b**] saying that any non-WLD Banach space can be renormed to have non-Valdivia dual unit ball. This result was later used in [**Ka03a**] to prove that a Banach space X is WLD if (and only if) each nonseparable space isomorphic to a complemented subspace of X admits a PRI. This section further deals with products, subspaces and quotients of 1-Plichko spaces.

The fifth section deals with 1-Plichko $C(K)$ spaces. It contains results on the relationship of the Valdivia property of K and the 1-Plichko property of $C(K)$ from [**Ka00a**], the results from [**Ka99d**] on non-1-Plichko subspaces of a large class of non-WLD $C(K)$ spaces.

The last section collects various examples of Valdivia compact spaces and 1-Plichko Banach spaces.

Section 3.2 is formed by the paper [**Ka02a**]. The main result there is that the space $C([0, \omega_2])$ of continuous functions on the ordinal segment $[0, \omega_2]$ is not Plichko. It is rather easy to see that $C([0, \omega_1])$ is 1-Plichko and $C([0, \omega_2])$ is not 1-Plichko. But the proof that it is not even Plichko, i.e. not isomorphic to a 1-Plichko space, required a new technic. It was partially inspired by a preprint of a later published paper [**AlPl06**].

Section 3.3 consists of the paper [**Ka03b**]. It deals with the class of continuous images of Valdivia compacta and with the class of subspaces of 1-Plichko spaces. It contains also properties of Corson countably compact spaces (i.e., countably compact spaces contained in $\Sigma(\Gamma)$ for a set Γ) and their continuous images. It turns out that these classes have some even nicer properties than Valdivia compact spaces and 1-Plichko spaces, including stability properties and duality.

The final section, **Section 3.4** contains the paper [**Ka08**]. It collects examples of Valdivia compact spaces, their continuous images and 1-Plichko Banach spaces which naturally appear in various branches of mathematics.

It starts by topological constructions which preserve Valdivia compact spaces (products, spaces of probabilities, hyperspaces, Aleksandroff duplicates). Further, nice structure and special properties of linearly ordered Valdivia compact spaces are established.

Another structures closely related to Valdivia compact spaces are compact groups. Any compact group is a continuous image of a Valdivia compact space and, moreover, the space of continuous functions is 1-Plichko. In the paper it is proved for Abelian groups and for groups of weight at most $\aleph_1$; it was later observed by A. Plichko that it holds for all groups.

Natural classes of 1-Plichko spaces include arbitrary L^1 spaces (including hence duals to $C(K)$ space and many spaces of measures), order-continuous Banach lattices (it is an unpublished result of A. Plichko) and preduals of semifinite von Neumann algebras (including hence duals to type I C^* algebras).

1.4. List of articles (sections)

The thesis is formed by the eight articles contained in the following list. Seven of them were published in international journal. One of them is an unpublished manuscript which is including for the sake of completeness, as it was later used and quoted by some authors as explained above. Bolted numbers denote the number of sections, the symbol IF denotes the value of the impact factor of the corresponding journal. (I use the value of the impact factor in the year preceding the year of the publication, according to the common convention.) The list of citations of each article is typed in a small font.

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