DGM for convection-diffusion equation

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Lecture 4

Model problem

- model scalar convection-diffusion equation,
- Let $\Omega \subset \mathbb{R}^d$, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, $Q_T \equiv \Omega \times (0, T)$, we seek $u : Q_T \to \mathbb{R}$ such that

Scalar convection-diffusion problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) - \varepsilon \, \Delta u &= g && \text{in } Q_T, && (1)\\
u &= u_D && \text{on } \partial\Omega_D, \ t \in (0, T),\\
\nabla(u) \cdot \boldsymbol{n} &= g_N && \text{on } \partial\Omega_N, \ t \in (0, T),\\
u(x, 0) &= u^0(x) && x \in \Omega,
\end{aligned}
$$

where: $\vec{f} = (f_1, \dots, f_d)$, $f_s \in C^1(\mathbb{R}), s = 1, \dots, d$, $\varepsilon > 0$.
Suitable assumptions on $\vec{f}$, g, u_D, g_N and u^0 guarantee the existence and uniqueness of the weak solution.

Space discretization by DGM

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) - \varepsilon\, \Delta u = g \qquad (2)$$

- let u be a strong (regular) solution,
- we multiply (2) by $v \in H^2(\Omega, \mathcal{T}_h)$,
- integrate over each $K \in \mathcal{T}_h$,
- apply Green's theorem,
- sum over all $K \in \mathcal{T}_h$,
- we include additional terms vanishing for regular solution,
- we obtain the identity

$$\left(\frac{\partial u}{\partial t}(t), v\right) + \varepsilon a_h(u(t), v) + b_h(u(t), v) + \varepsilon J_h^\sigma(u(t), v) = \ell_h(v)\,(t)$$

$$\forall v \in H^2(\Omega, \mathcal{T}_h)\ \forall t \in (0, T), \quad (3)$$

Convective form

Convective term ("finite volume approach"):

$$\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\vec{f}(u)\,v\,\mathrm{d}x$$

$$= -\sum_{K\in\mathcal{T}_h}\int_K \vec{f}(u)\cdot\nabla v\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\vec{f}(u)\cdot\boldsymbol{n}\,v\,\mathrm{d}S$$

Numerical flux

$$\vec{f}(u)\cdot\boldsymbol{n}|_\Gamma \approx H\left(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma\right), \quad \Gamma\in\mathcal{F}_h,$$

$$b_h(u,v) = -\sum_{K\in\mathcal{T}_h}\int_K \vec{f}(u)\cdot\nabla v\,\mathrm{d}x$$

$$+\sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma H\left(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma\right)[v]_\Gamma\,\mathrm{d}S$$

Definition of forms

Diffusive form

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \int_K \nabla u \cdot \nabla v \,\mathrm{d}x - \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \langle \nabla u\rangle \cdot \boldsymbol{n}[v]\mathrm{d}S$$
$$+\theta \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \langle \nabla v\rangle \cdot \boldsymbol{n}[u]\mathrm{d}S,$$

- $\theta = -1$ SIPG formulation,
- $\theta = 1$ NIPG formulation,
- $\theta = 0$ IIPG formulation.

Definition of forms, cont.

Interior and boundary penalty

$$J_h^\sigma(u,v) = \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \sigma[u]\,[v]\,\mathrm{d}S, \qquad \sigma_\Gamma = \frac{C_W}{h_\Gamma},$$

Right-hand-side

$$\ell_h(v)(t) = \int_\Omega g(t)\,v\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h^N} \int_\Gamma g_N(t)\,v\,\mathrm{d}S$$
$$+ \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \left(\theta\nabla v\cdot\boldsymbol{n}\,u_D(t) + \sigma\,u_D(t)\,v\right)\,\mathrm{d}S$$

Semi-discrete variant

- Let $u(t,x) \in C^1(0,T;H^2(\Omega))$ be the weak solution (1), then

$$\left(\frac{\partial u}{\partial t}(t), v\right) + \varepsilon a_h(u(t),v) + b_h(u(t),v) + \varepsilon J_h^\sigma(u(t),v) \\ = \ell_h(v)\,(t), \quad v \in H^2(\Omega,\mathcal{T}_h),\ t \in (0,T), \qquad (4)$$

- (4) makes sense also for $u(t) \in H^2(\Omega,\mathcal{T}_h)$, $t \in (0,T)$.
- since $S_{hp} \subset H^2(\Omega,\mathcal{T}_h)$, identity (4) makes sense for $u,v \in S_{hp}$

Definition

We say that u_h is a DGFE solution iff

$$\begin{aligned} \text{a)} \quad & u_h \in C^1(0, T; S_{hp}), \\ \text{b)} \quad & \left(\frac{\partial u_h(t)}{\partial t}, v_h \right) + b_h(u_h(t), v_h) + \varepsilon\, a_h(u_h(t), v_h) \\ & \quad + \varepsilon\, J_h^\sigma(u_h(t), v_h) = \ell_h(v_h)\,(t) \qquad \forall\, v_h \in S_{hp},\ t \in (0, T) \\ \text{c)} \quad & u_h(0) = \Pi_{hp} u^0, \end{aligned} \tag{5}$$

where $\Pi_{hp} u^0$ is a projection of IC in S_{hp}

- system of ODEs,
- number of equations = $\dim S_{hp}$
- (semi)-implicit ODE solver advantageous,

Error estimate

Main theorem

Let

- $\partial u/\partial t \in L^2(0, T; H^s(\Omega))$ be the exact regular weak solution,
- $u_h \in C^1(0, T; S_{hp})$ be the approximate solution given by (5)
- assumptions on meshes, numerical flux, problem data,
- $e_h \equiv u_h - u$,

Then

$$\max_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} + \varepsilon \int_0^T |||e_h(\vartheta)|||^2 \, \mathrm{d}\vartheta \qquad (6)$$
$$\leq C_2 h^{2\mu-2} \left(|u|^2_{L^2(\cdot, H^\mu(\Omega))} + |\partial u/\partial t|^2_{L^2(\cdot, H^\mu(\Omega))} \right),$$

$\mu = \min(p + 1, s)$.

$$|b_h(u,v) - b_h(\bar{u},v)| \le C|||v||| \left(\|u-\bar{u}\|^2_{L^2(\Omega)} + \sum_{K\in\mathcal{T}_h} h_K \|u-\bar{u}\|^2_{L^2(\partial K)} \right)^{1/2},$$
$$u,\ \bar{u} \in H^1(\Omega,\mathcal{T}_h) \cap L^\infty(\Omega),\ v \in H^1(\Omega,\mathcal{T}_h),\ h \in (0,\bar{h}) \qquad (7)$$

$$|b_h(u_h,v_h) - b_h(\bar{u}_h,v_h)| \le C|||v_h|||\ \|u_h-\bar{u}_h\|_{L^2(\Omega)},\ u_h,\ \bar{u}_h,\ v_h \in S_{hp},$$

Let $\Pi_{hp} : H^s(\Omega) \to S_{hp}$ and $\eta = u - \Pi_{hp}u$, then

$$|b_h(u,v_h) - b_h(\Pi_{hp}u,v_h)| \le CR_b(\eta)|||v_h|||, \quad v_h \in S_{hp},\ h \in (0,\bar{h}), \quad (8)$$

Let $\xi = u_h - \Pi_{hp}u$, then under the above assumptions,

$$|b_h(u,v_h) - b_h(u_h,v_h)| \le C|||v_h||| \left(R_b(\eta) + \|\xi\|_{L^2(\Omega)}\right), \quad v_h \in S_{hp}. \quad (9)$$

where $R_b(\eta) = \left(\sum_{K\in\mathcal{T}_h} \left(\|\eta\|^2_{L^2(K)} + h_K^2|\eta|^2_{H^1(K)}\right)\right)^{1/2}$.

Sketch of the proof of (7)

We assume Lipschitz continuity of $\vec{f}$ and H:
$|\vec{f}(u) - \vec{f}(\bar{u})| \leq L_f |u - \bar{u}| \qquad \forall u, \bar{u} \in \mathbb{R},$
$|H(u_1, u_2, \boldsymbol{n}) - H(v_1, v_2, \boldsymbol{n})| \leq L_H(|u_1 - v_1| + |u_2 - v_2|) \ \ \forall u_1, u_2, v_1, v_2 \in \mathbb{R}$

$$\begin{aligned} & b_h(u, v) - b_h(\bar{u}, v) && (10) \\ = & -\sum_{K \in \mathcal{T}_h} \int_K \left(\vec{f}(u) - \vec{f}(\bar{u})\right) \cdot \nabla v \,\mathrm{d}x && =: \sigma_1 \\ & + \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \left(H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \boldsymbol{n}) - H(\bar{u}_\Gamma^{(L)}, \bar{u}_\Gamma^{(R)}, \boldsymbol{n})\right) [v] \,\mathrm{d}S && =: \sigma_2. \end{aligned}$$

$$|\sigma_1| \leq L_f \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^{d} |u - \bar{u}| \left| \frac{\partial v}{\partial x_s} \right| \mathrm{d}x \leq \sqrt{d} L_f \|u - \bar{u}\|_{L^2(\Omega)} |v|_{H^1(\Omega, \mathcal{T}_h)}.$$

$$|\sigma_2| \leq L_H \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \left(\left| u_\Gamma^{(L)} - \bar{u}_\Gamma^{(L)} \right| + \left| u_\Gamma^{(R)} - \bar{u}_\Gamma^{(R)} \right| \right) |[v]| \,\mathrm{d}S \ldots$$

Sketch of the proof of error estimates (1)

Approximate solution

$$\left(\frac{\partial u_h(t)}{\partial t}, v_h\right) + b_h(u_h(t), v_h) + \varepsilon\, a_h(u_h(t), v_h) + \varepsilon\, J_h^\sigma(u_h(t), v_h) = \ell_h(v_h)$$

Consistency

$$\left(\frac{\partial u(t)}{\partial t}, v_h\right) + b_h(u(t), v_h) + \varepsilon\, a_h(u(t), v_h) + \varepsilon\, J_h^\sigma(u(t), v_h) = \ell_h(v_h)$$

Error $e_h = u_h - u = \xi + \eta$

$\xi = u_h - \Pi_{hp} u \in S_{hp}$,

$\eta = \Pi_{hp} u - u \in H^s(\Omega, \mathcal{T}_h)$: available info from $u \Rightarrow$ aim: "$\xi \leq \eta$"

Putting $A_h := \varepsilon(a_h + J_h^\sigma)$, subtracting the above relations, $v_h := \xi$ gives

$$\left(\frac{\partial \xi}{\partial t}, \xi\right) + \varepsilon A_h(\xi, \xi) = b_h(u, \xi) - b_h(u_h, \xi) - \left(\frac{\partial \eta}{\partial t}, \xi\right) - A_h(\eta, \xi).$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)} + \varepsilon\, C_C |||\xi|||^2 \leq |b_h(u, \xi) - b_h(u_h, \xi)| + \left|\left(\frac{\partial \eta}{\partial t}, \xi\right)\right| + |A_h(\eta, \xi)|.$$

Sketch of the proof of error estimates (2)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)} + \varepsilon\, C_C |||\xi|||^2 \le |b_h(u,\xi) - b_h(u_h,\xi)| + \left|\left(\frac{\partial\eta}{\partial t},\xi\right)\right| + |A_h(\eta,\xi)|.$$

Particular error estimates

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2} + 2\varepsilon C_C |||\xi|||^2 \le C\left((R_b(\eta) + \|\xi\|_{L^2})|||\xi||| + \varepsilon R_a(\eta)|||\xi||| + \|\partial_t\eta\|_{L^2}\|\xi\|_{L^2}\right)$$

where $R_b(\eta) = \left(\sum_{K\in\mathcal{T}_h}(\|\eta\|^2_{L^2(K)} + h_K^2|\eta|^2_{H^1(K)})\right)^{1/2} = O(h^\mu)$,

$R_a(\eta) = \left(\sum_{K\in\mathcal{T}_h}(|\eta|^2_{H^1(K)} + h_K^2|\eta|^2_{H^2(K)} + h_K^{-2}\|\eta\|^2_{L^2(K)})\right)^{1/2} = O(h^{\mu-1})$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2} + 2\varepsilon C_C |||\xi|||^2 \le C\left(R_b(\eta) + \varepsilon R_a(\eta) + \|\xi\|_{L^2}\right)|||\xi||| + C\|\partial_t\eta\|_{L^2}\|\xi\|_{L^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2} + 2\varepsilon C_C |||\xi|||^2$$

$$\le \frac{C}{\varepsilon}\left(R_b(\eta) + \varepsilon R_a(\eta) + \|\xi\|_{L^2}\right)^2 + \varepsilon C_C|||\xi|||^2 + C\|\partial_t\eta\|^2_{L^2} + C\|\xi\|^2_{L^2}$$

Sketch of the proof of error estimates (3)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_{L^2}^2+\varepsilon|||\xi|||^2 &\leq \frac{C}{\varepsilon}\left(R_b(\eta)+\varepsilon R_a(\eta)+\|\xi\|_{L^2}\right)^2+C\|\partial_t\eta\|_{L^2}^2+C\,\|\xi\|_{L^2}^2\\ &\leq \frac{C}{\varepsilon}\left(R_b(\eta)+\varepsilon R_a(\eta)\right)^2+\frac{C}{\varepsilon}\|\xi\|_{L^2}^2+C\|\partial_t\eta\|_{L^2}^2+C\,\|\xi\|_{L^2}^2\\ &\leq C\left(1+\frac{1}{\varepsilon}\right)\|\xi\|_{L^2}^2+CR_Q(\eta)\end{aligned}$$

integration over $\int_0^t \,\mathrm{d}\vartheta$ gives (we have $\xi(0)=u_h(0)-\Pi_{hp}u^0=0$)

$$\begin{aligned}&\|\xi(t)\|_{L^2(\Omega)}^2+\varepsilon\int_0^t |||\xi(\vartheta)|||^2\,\mathrm{d}\vartheta\\ &\qquad\leq C\left(1+\frac{1}{\varepsilon}\right)\int_0^t\|\xi(\vartheta)\|_{L^2(\Omega)}^2\,\mathrm{d}\vartheta+\int_0^t R_Q(\eta(\vartheta))\,\mathrm{d}\vartheta.\end{aligned}$$

Gronwall's Lemma

$$\|\xi(t)\|_{L^2(\Omega)}^2+\varepsilon C_C\int_0^t|||\xi(\vartheta)|||^2\mathrm{d}\vartheta\leq R(\eta,\varepsilon)c_1(\exp(1/\varepsilon)),\qquad (11)$$

Sketch of the proof of error estimates (4)

$$\|\xi(t)\|_{L^2(\Omega)}^2 + \varepsilon C_C \int_0^t |||\xi(\vartheta)|||^2 \mathrm{d}\vartheta \le R(\eta,\varepsilon) c_1(\exp(1/\varepsilon)),$$

since $e_h = \xi + \eta$

$$\|e_h(t)\|_{L^2(\Omega)}^2 + \varepsilon C_C \int_0^t |||e_h(\vartheta)|||^2 \mathrm{d}\vartheta \le \tilde{R}(\eta,\varepsilon) c_1(\exp(1/\varepsilon)),$$

Approximation properties

- $\mu = \min(p+1, s), \quad u \in C([0,T], H^s(\Omega)), \quad u_h \in S_{hp}$
- $R_a(\eta) = O(h^{\mu-1}),\ R_b(\eta) = O(h^{\mu}) \quad \Rightarrow \quad \tilde{R}(\eta,\varepsilon) = O(h^{\mu-1})$

Finally,

$$\|e_h(t)\|_{L^2(\Omega)}^2 + \varepsilon C_C \int_0^t |||e_h(\vartheta)|||^2 \mathrm{d}\vartheta \le c_1(\exp(1/\varepsilon)) O(h^{\mu-1})$$

Summary of the IPG methods

SIPG

- optimal order of convergence – duality arguments,
- C_W chosen carefully.

NIPG

- sub-optimal order of convergence,
- C_W arbitrary.

IIPG

- sub-optimal order of convergence,
- C_W chosen carefully,
- simpler implementation.

Numerical examples

Linear convection-diffusion equation

$$\frac{\partial u}{\partial x_1} - \varepsilon \Delta u = 1 \quad \text{in} \quad \Omega = (0,1) \times (0,1), \qquad (12)$$

$$u = 0 \quad \text{on} \quad \partial\Omega,$$

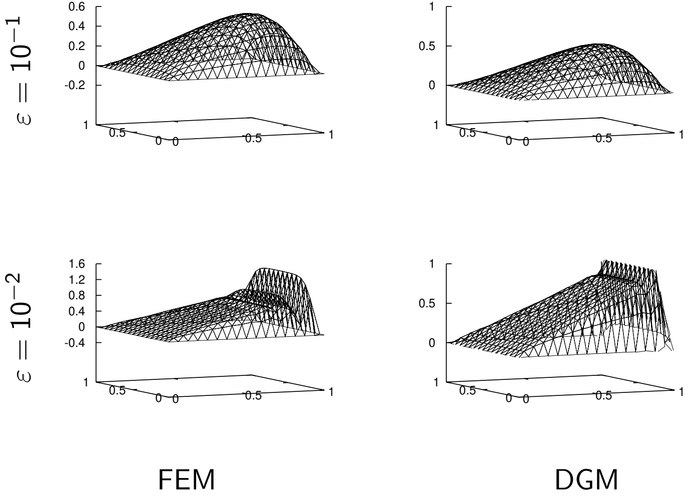
where $\varepsilon > 0$

steep boundary layer

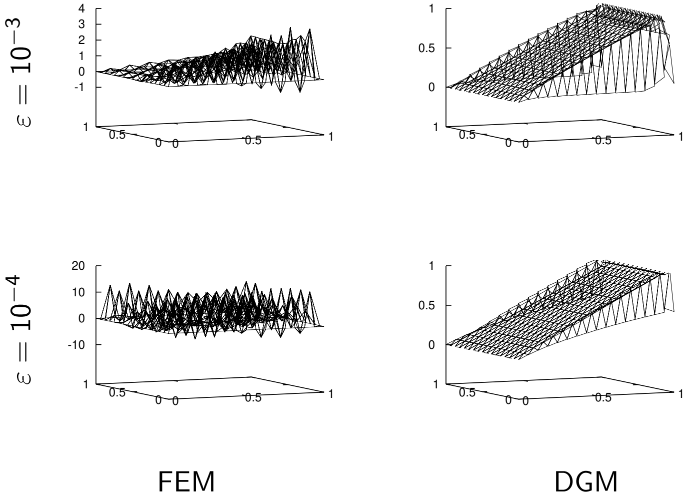
Numerical methods

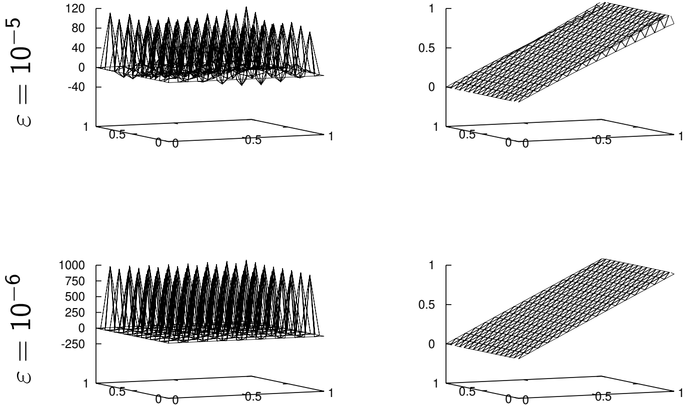
- conforming FEM
- IIPG variant of DGM
- $\varepsilon = 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4},\ 10^{-5}$ and 10^{-6}

Numerical results: cases $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-2}$



Numerical results: cases $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$




$\varepsilon = 10^{-5}$
120
80
40
0
-40
1
0.5
0
0
0.5
1
0.5
0
1
0.5
0
0
0.5
1
$\varepsilon = 10^{-6}$
1000
750
500
250
0
-250
1
0.5
0
0
0.5
1
1
0.5
0
1
0.5
0
0
0.5
1

FEM DGM