Discontinuous Galerkin method

V. Dolejší

Charles University Prague, Faculty of Mathematics and Physics

Lecture 1

DGM for the elliptic problems

Laplace problem

Find a function $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega, \tag{1a}$$

$$u = u_D \quad \text{on } \partial\Omega_D, \tag{1b}$$

$$\boldsymbol{n} \cdot \nabla u = g_N \quad \text{on } \partial\Omega_N, \tag{1c}$$

where f, u_D and g_N are given functions.

Weak solution

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x + \int_{\partial\Omega_N} g_N\, v \,\mathrm{d}S \quad \forall v \in V. \tag{2}$$

$$V = \{v \in H^1(\Omega);\ v|_{\partial\Omega_D} = 0\}.$$

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Weak solution

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f \, v \, \mathrm{d}x + \int_{\partial\Omega_N} g_N \, v \, \mathrm{d}S \quad \forall\, v \in V. \tag{2}$$

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Notation

- $\mathcal{T}_h$ triangulations
- triangles $K \in \mathcal{T}_h$
  - $h_K = \mathrm{diam}(K) =$ diameter of K, $h = \max_{K\in\mathcal{T}_h} h_K$,
  - $\rho_K =$ radius of the largest d-dimensional ball in K
  - $|K| =$ d-dimensional Lebesgue measure of K,
- $\mathcal{F}_h$ we denote the system of all faces of all elements $K \in \mathcal{T}_h$
  - $\mathcal{F}_h^B$ – boundary faces
  - $\mathcal{F}_h^D$ – Dirichlet faces
  - $\mathcal{F}_h^N$ – Neumann faces
  - $\mathcal{F}_h^I$ – interior faces
  - $\mathcal{F}_h^{ID} = \mathcal{F}_h^I \cup \mathcal{F}_h^D$ – interior and Dirichlet faces
- $\boldsymbol{n}_\Gamma$ – unit normal to $\Gamma \in \mathcal{F}_h$

Functional spaces

Broken Sobolev spaces

$$H^k(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega); v|_K \in H^k(K)\ \forall\ K \in \mathcal{T}_h\}, \qquad (3)$$

Norm and seminorm

$\|v\|^2_{H^k(\Omega,\mathcal{T}_h)} = \sum_{K\in\mathcal{T}_h} \|v\|^2_{H^k(K)}, \quad |v|^2_{H^k(\Omega,\mathcal{T}_h)} = \sum_{K\in\mathcal{T}_h} |v|^2_{H^k(K)}$

Jump and mean values

- $\langle v\rangle_\Gamma$ = mean value of $v \in H^1(\Omega, \mathcal{T}_h)$ on $\Gamma \in \mathcal{F}_h^I$
- $[v]_\Gamma$ = jump of $v \in H^1(\Omega, \mathcal{T}_h)$ on $\Gamma \in \mathcal{F}_h^I$
- $v = \langle v\rangle_\Gamma = [v]_\Gamma =$ for $v \in H^1(\Omega, \mathcal{T}_h)$ on $\Gamma \in \mathcal{F}_h^B$

Finite dimensional subspace

$$S_{hp} = \{v \in L^2(\Omega); v|_K \in P_p(K)\ \forall K \in \mathcal{T}_h\}, \qquad (4)$$

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Assumptions

- $\{\mathcal{T}_h\}_{h\in(0,\bar{h})}$ triangulations: shape-regular: $\exists C_R$ such that

$$h_K/\rho_K \leq C_R \quad \forall\, K \in \mathcal{T}_h\ \forall\, h \in (0,\bar{h}). \tag{5}$$

- the quantity h_Γ, $\Gamma \in \mathcal{F}_h$ satisfy the equivalence condition with h_K: $\exists\ C_T, C_G > 0$ such that

$$C_T h_K \leq h_\Gamma \leq C_G h_K, \quad \forall\, K \in \mathcal{T}_h,\ \forall\, \Gamma \in \mathcal{F}_h,\ \Gamma \subset \partial K,\ \forall\, h \in (0,\bar{h}). \tag{6}$$

DGM forms

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, \mathrm{d}x \qquad (7)$$
$$- \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \left(\boldsymbol{n} \cdot \langle \nabla u \rangle [v] + \Theta \boldsymbol{n} \cdot \langle \nabla v \rangle [u] \right) \mathrm{d}S,$$

where $\Theta = 1$ (SIPG), $\Theta = -1$ (NIPG) or $\Theta = 0$ (IIPG).

$$J_h^\sigma(u,v) = \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \sigma [u]\,[v] \, \mathrm{d}S, \qquad u, v \in H^1(\Omega, \mathcal{T}_h). \qquad (8)$$

$$\ell_h(v) = \int_\Omega f\, v \, \mathrm{d}x + \int_{\partial\Omega_N} g_N\, v \, \mathrm{d}S - \sum_{\Gamma\in\mathcal{F}_h^{D}} \int_\Gamma (\Theta \boldsymbol{n} \cdot \nabla v + \sigma v)\, u_D \, \mathrm{d}S.$$

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Approximate solution by DGM

Definition

$u_h \in S_{hp}$ is approximate solution by DGM if

$$A_h(u_h, v_h) = \ell_h(v_h) \quad \forall\, v_h \in S_{hp}. \tag{9}$$

Consistency

if $u \in H^2(\Omega)$ is the weak solution of the Laplace problem then

$$A_h(u, v) = \ell_h(v) \quad \forall\, v \in H^2(\Omega, \mathcal{T}_h), \tag{10}$$

Galerkin orthogonality of the error

let $e_h = u_h - u$ be the error then

$$A_h(e_h, v) = 0 \quad \forall\, v \in S_{hp}. \tag{11}$$

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Practical realization

$$A_h(u_h, v_h) = \ell_h(v_h) \quad \forall\, v_h \in S_{hp}.$$

- let $\{\varphi_i,\ i = 1, \dots, N_h\}$ be a basis of the space S_{hp},
- $u_h(x) = \sum_{j=1}^{N_h} u^j \varphi_j(x)$, $u^j \in \mathbb{R}$, $j = 1, \dots, N_h$ unknowns
- (9) is equivalent to

$$\sum_{j=1}^{N_h} A_h(\varphi_j, \varphi_i) u^j = \ell_h(\varphi_j), \quad j = 1, \dots, N_h. \qquad (12)$$

- matrix form

$$\mathbb{A} U = L,$$

where $\mathbb{A} = (a_{ij})_{i,j=1}^{N_h} = (A_h(\varphi_j, \varphi_i))_{i,j=1}^{N_h}$, $U = (u^j)_{j=1}^{N_h}$ and $L = (\ell_h(\varphi_j))_{j=1}^{N_h}$,
- existence and uniqueness of the solution?

Tools of numerical analysis

Multiplicative trace inequality

There exists a constant $C_M > 0$ such that

$$\|v\|^2_{L^2(\partial K)} \leq C_M \left(\|v\|_{L^2(K)} \, |v|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right), \quad v \in H^1(K). \tag{13}$$

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Approximation properties: $\Pi_{hp} : H^s(\Omega, \mathcal{T}_h) \to S_{hp}$

There exists a constant $C_A > 0$ such that

$$|\Pi_{hp} v - v|_{H^q(\Omega,\mathcal{T}_h)} \leq C_A h^{\mu-q} |v|_{H^\mu(\Omega,\mathcal{T}_h)}, \quad v \in H^s(\Omega, \mathcal{T}_h), \tag{15}$$

where $\mu = \min(p+1, s)$.

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DG-Norm

$$|||u||| = \left(|u|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(u,u) \right)^{1/2}, \tag{16}$$

where $J_h^\sigma(u,v) = \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \sigma[u]\,[v]\,\mathrm{d}S$,

$$\sigma|_\Gamma = \sigma_\Gamma = \frac{C_W}{h_\Gamma}, \quad \Gamma \in \mathcal{F}_h^{ID}, \tag{17}$$

and $C_W > 0$ is the *penalization constant.*

$$\|v\|^2_{1,\sigma} = |||v|||^2 + \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \sigma^{-1}(\boldsymbol{n}\cdot\langle\nabla v\rangle)^2\,\mathrm{d}S \tag{18}$$

$$= |v|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(v,v) + \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \sigma^{-1}(\boldsymbol{n}\cdot\langle\nabla v\rangle)^2\mathrm{d}S.$$

Energy norms

DG-Norm

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Continuity of diffusion bilinear forms

Continuity of a_h

$$|a_h(u,v)| \le \|u\|_{1,\sigma}\,\|v\|_{1,\sigma} \quad \forall\, u,v \in H^2(\Omega,\mathcal{T}_h), \qquad (19)$$

$$|J_h^\sigma(u,v)| \le J_h^\sigma(u,u)^{1/2} J_h^\sigma(v,v)^{1/2} \quad \forall\, u,v \in H^1(\Omega,\mathcal{T}_h), \qquad (20)$$

Continuity of A_h

$$|A_h(u,v)| \le 2\|u\|_{1,\sigma}\|v\|_{1,\sigma} \quad \forall\, u,v \in H^2(\Omega,\mathcal{T}_h). \qquad (21)$$

Coercivity

$$A_h(v_h, v_h) \geq C_C |||v_h|||^2 \quad \forall\, v_h \in S_{hp}, \tag{22}$$

with

$$\begin{aligned} C_C &= 1 && \text{for } A_h = A_h^{\mathrm{n},\sigma} && \text{if } C_W > 0, \\ C_C &= 1/2 && \text{for } A_h = A_h^{\mathrm{s},\sigma} && \text{if } C_W \geq 4 C_G C_M (1 + C_I), \\ C_C &= 1/2 && \text{for } A_h = A_h^{\mathrm{i},\sigma} && \text{if } C_W \geq C_G C_M (1 + C_I). \end{aligned}$$

DG solution

$$u_h \in S_{hp} \quad \text{such that} \quad A_h(u_h, v_h) = \ell_h(v_h) \quad \forall\, v_h \in S_{hp}. \tag{23}$$

Existence of the solution

If C_W is sufficiently large then there exists unique DG solution.

A priori error estimates

DG norm

Let $u \in H^s(\Omega), \geq 2,\ C_W$ sufficiently large and $\mathcal{T}_h$ satisfies the assumptions then

$$|||u - u_h||| \leq C_1 h^{\mu-1} |u|_{H^\mu(\Omega)}, \quad h \in (0, \bar{h}), \tag{24}$$

where $\mu = \min(p+1, s)$

Broken Poincaré inequality

$$\|v_h\|_{L^2(\Omega)} \leq C|||v_h||| \qquad \forall v_h \in H^1(\Omega, \mathcal{T}_h) \tag{25}$$

L^2-error estimate – only SIPG!

Let the dual problem has regular solution ($z \in H^2(\Omega)$)

$$\|u - u_h\|_{L^2(\Omega)} \leq C_3 h^{\mu} |u|_{H^\mu(\Omega)}, \tag{26}$$

where $\mu = \min\{p+1, s\}$.

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Numerical experiment (1)

Problem with regular solution

Find a function $u : \Omega = (0,1) \times (0,1) \to \mathbb{R}$ such that

$$-\Delta u = 8\pi^2 \sin(2\pi x_1)\sin(2\pi x_2) \qquad \text{in } \Omega, \tag{27}$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

Exact solution

$$u = \sin(2\pi x_1)\sin(2\pi x_2), \quad (x_1, x_2) \in \Omega, \tag{28}$$

obviously, $u \in C^\infty(\overline{\Omega})$.

Evaluation of experimental order of convergence (EOC)

Assumption

$$\|e_h\| = Ch^{\mathrm{EOC}}, \qquad (29)$$

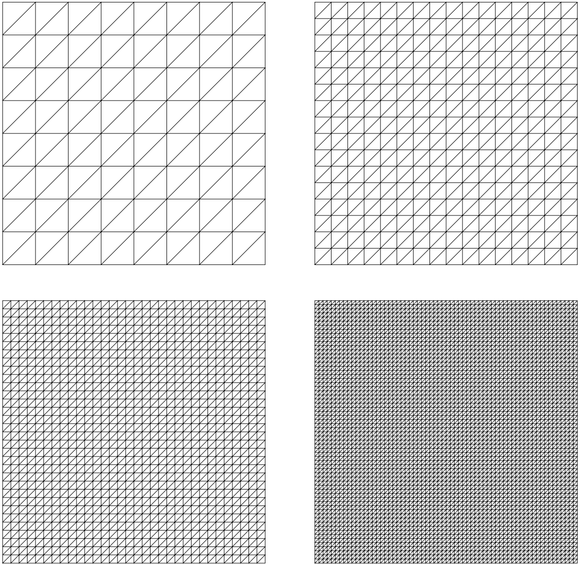
where $C > 0$ is a constant, $h = \max_{K\in\mathcal{T}_h} h_K$, $\mathrm{EOC} \in \mathbb{R}$ is the experimental order of convergence.

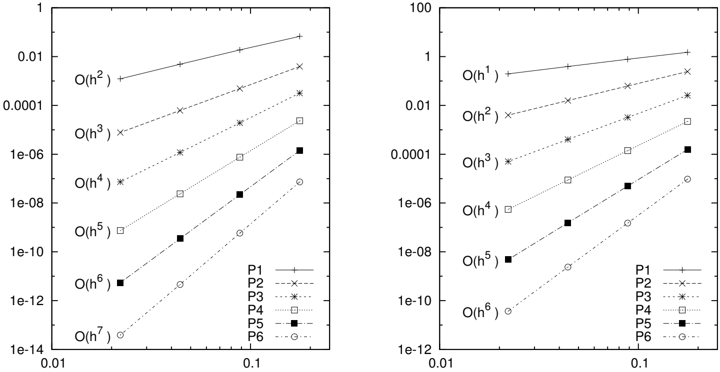
Setting of EOC

- computation on two triangulations with $h_1 < h_2$
- the corresponding errors e_{h_1}, e_{h_2}
- from (29) we derive

$$\mathrm{EOC} = \frac{\log(\|e_{h_1}\|/\|e_{h_2}\|)}{\log(h_1/h_2)}. \qquad (30)$$

Sequence of triangulations





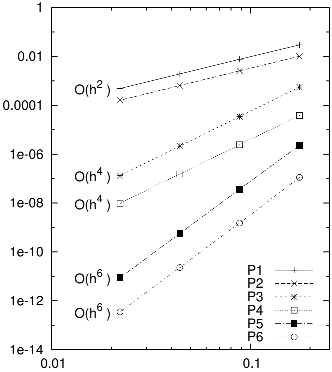
L^2-norm

DG-norm

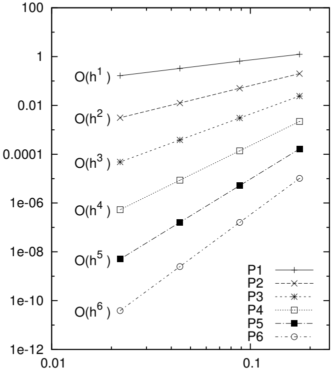
$$\|e_h\|_{L^2} = O(h^{p+1})$$

$$|||e_h||| = O(h^p)$$

Convergence, regular solution NIPG



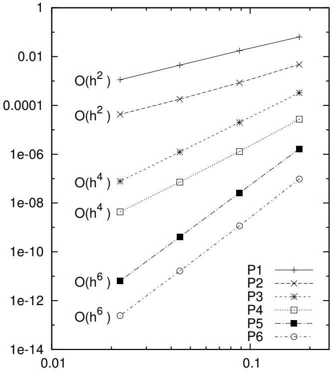
L^2-norm



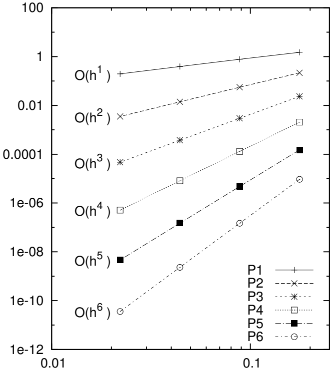
DG-norm

$$\|e_h\|_{L^2} = O(h^p) \qquad |||e_h||| = O(h^p)$$

Convergence, regular solution IIPG



L^2-norm



DG-norm

$$\|e_h\|_{L^2} = O(h^p)$$

$$|||e_h||| = O(h^p)$$

Numerical experiment (2)

Problem with singular solution

Find a function $u : \Omega = (0,1) \times (0,1) \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= g \quad \text{in } \ \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{31}$$

Exact solution

$$u(x_1, x_2) = 2r^\alpha x_1 x_2 (1 - x_1)(1 - x_2) \tag{32}$$

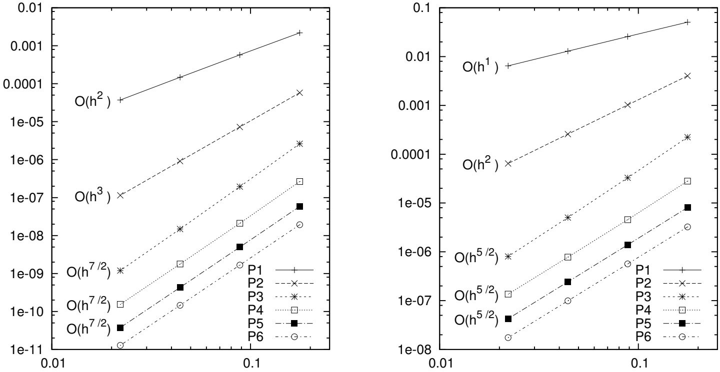
where r is the polar coordinate and $\alpha \in \mathbb{R}$.

Solution regularity

$$u \in H^\beta(\Omega) \quad \forall\, \beta \in (0,\, \alpha + 3), \tag{33}$$

where $H^\beta(\Omega)$ is the Sobolev–Slobodetskii space.

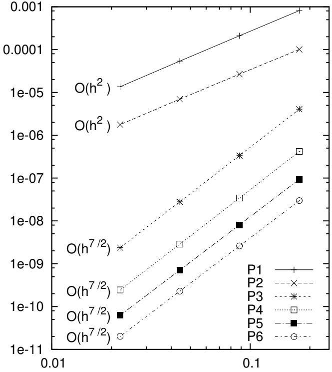
Singular solution, $\alpha = 0.5$, $u \in H^{7/2}(\Omega)$, SIPG



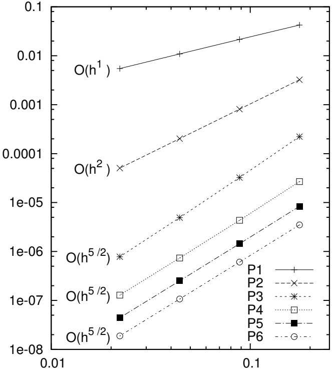
L^2-norm DG-norm

$$\|e_h\|_{L^2} = O(h^{\min(p+1,\,7/2)}) \qquad |||e_h||| = O(h^{\min(p+1,\,7/2)-1})$$

Singular solution, $\alpha = 0.5$, $u \in H^{7/2}(\Omega)$, NIPG



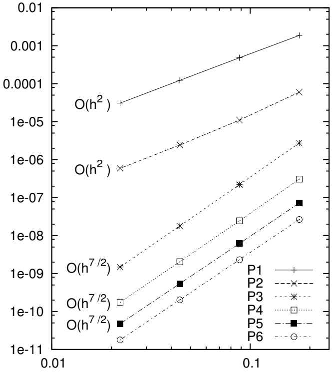
L^2-norm



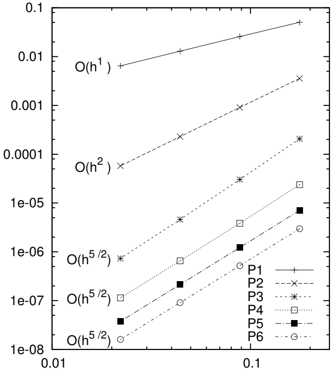
DG-norm

$$\|e_h\|_{L^2} = O(h^{\min(p+1,\,7/2)-1})$$

$$|||e_h||| = O(h^{\min(p+1,\,7/2)-1})$$



L^2-norm

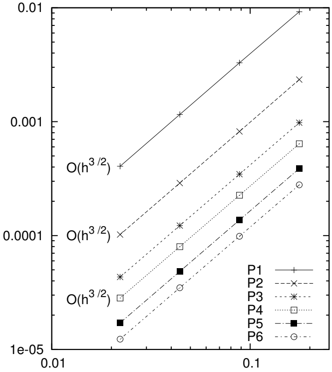


DG-norm

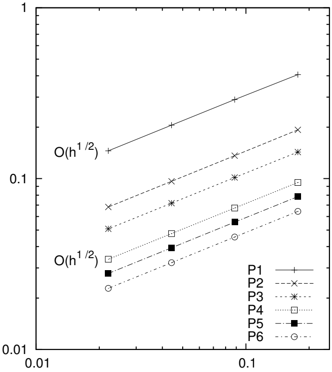
$$\|e_h\|_{L^2} = O(h^{\min(p+1,\,7/2)-1})$$

$$|||e_h||| = O(h^{\min(p+1,\,7/2)-1})$$

Singular solution, $\alpha = -1.5$, $u \in H^{3/2}(\Omega)$, SIPG



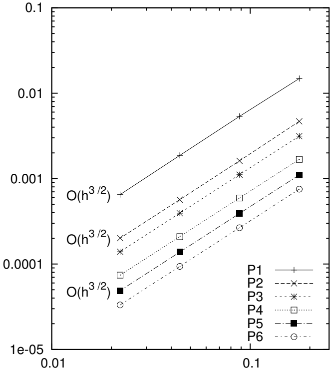
L^2-norm



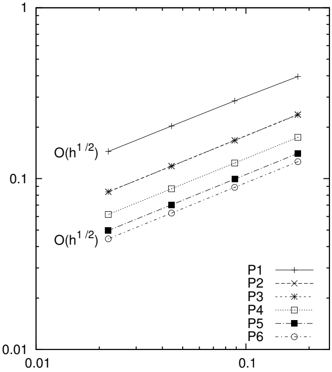
DG-norm

$$\|e_h\|_{L^2} = O(h^{\min(p+1,\,3/2)})$$

$$|||e_h||| = O(h^{\min(p+1,\,3/2)-1})$$



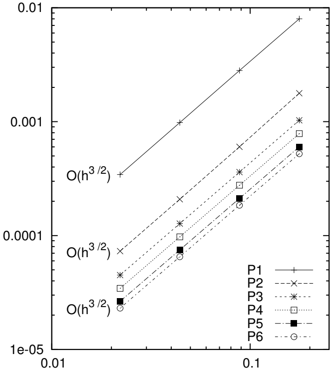
L^2-norm



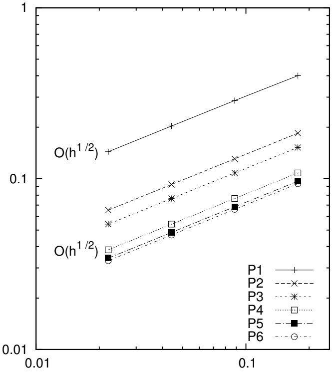
DG-norm

$$\|e_h\|_{L^2} = O(h^{\min(p+1,\,3/2)-1})$$

$$|||e_h||| = O(h^{\min(p+1,\,3/2)-1})$$



L^2-norm



DG-norm

$$\|e_h\|_{L^2} = O(h^{\min(p+1,3/2)-1})$$

$$|||e_h||| = O(h^{\min(p+1,3/2)-1})$$

Summary of numerical experiments

$u \in H^s(\Omega)$, $u_h \in S_{hp}$

SIPG

- $\|e_h\|_{L^2} = O(h^{\min(p+1,\, s)})$
- $|||e_h||| = O(h^{\min(p+1,\, s)-1})$
- full agreement with theory, optimal orders of convergence

NIPG and IIPG

- $\|e_h\|_{L^2} = \begin{cases} O(h^{\min(p+1,\, s)}) & p \text{ odd} \\ O(h^{\min(p,\, s)}) & p \text{ even} \end{cases}$
- $|||e_h||| = O(h^{\min(p+1,\, s)-1})$
- agreement with theory in $|||\cdot|||$, optimal orders of convergence
- optimal order in $\|\cdot\|_{L^2}$ only for p odd (not true in general)